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What Is Higher Mathematics? Why Is It So Hard to Interpret? What Can Be Done?

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Cover Page Footnote

The author wishes to express his appreciation to Professors Fat Lam and James Nickerson of Gallaudet University for welcoming him into their classrooms. I also wish to thank Linda Ginis of the New York School for the Deaf and Gary Waite of the American School for the Deaf for their assistance in discovering what kinds of mathematics students at those schools were learning during the 1850s. Finally, I extend a special note of appreciation to Dr. Sherry Shaw for her thoughtful suggestions and for her patience.

Interpreters often comment on the difficulties associated with interpreting lectures on higher mathematics, and the more advanced the mathematics, the greater the difficulties. Mathematical language can be difficult to interpret because its vocabulary and grammar are so far removed from conversational speech, but unfamiliar vocabulary and grammar do not alone account for the often opaque nature of mathematical discourse. Even the subject matter can be problematic in the sense that it may not be clear what a particular branch of mathematics, e.g. topology, is "about." Why would anyone study it? What is its relationship to the physical world? What does it *mean*?

This article has a three-fold purpose. First, it identifies what makes higher mathematics, the study of mathematics per se, unique among all branches of knowledge. It is hoped that a better understanding of the nature of mathematics will enable interpreters to place their efforts in a more comprehensive conceptual framework. Second, it describes attempts to express the language of higher mathematics in American Sign Language (ASL). In particular, this article provides an overview of the historical and current state of mathematical ASL. It compares mathematical ASL to mathematical English, and it identifies some noteworthy efforts by interpreters and interpreting groups to improve the interpretation of mathematical language. This article also examines how other linguistic groups have resolved the problem of expressing higher mathematics. Finally, this article suggests ways that a more expressive form of mathematical ASL might be created, one that would enable interpreters to provide a higher level of service for deaf students of higher mathematics and for deaf mathematics professionals, thereby enabling these two groups to participate more fully in the broader mathematical community.

This subject is new in that no literature exists on the expression of higher mathematics in ASL. As will be seen, the problems encountered while interpreting higher mathematics are qualitatively and quantitatively different from those encountered when interpreting elementary mathematics or applied mathematics. It is hoped that this paper, which is concerned solely with the expression of higher mathematics in ASL, will be the first of several to explore this important topic.

Higher mathematics is different from applied mathematics, the branch of mathematics used to gain insight into problems arising in science and engineering. From a linguistic point of view, applied mathematics is easier to express in ASL because it is grounded in the physical and computational sciences. Key ideas in applied mathematics are often expressed via analogies with physical phenomena. This is rarely the case for higher mathematics, which often involves ideas not grounded in the physical, computational, or social sciences. Solving the problems associated with expressing higher mathematics in ASL would resolve most difficulties associated with expressing applied mathematics in ASL, but the converse does not hold true. The term *pure mathematics* is sometimes used as a synonym for higher mathematics. Students do not usually take their first course in higher mathematics until they have completed the entire calculus sequence.

The Nature of Mathematics

From an interpreter's point of view, a particularly useful theory of the nature of mathematics was advanced early in the last century by a group of German scholars, the most prominent of which was David Hilbert (1862-1943), one of the founders of modern mathematical thought. Hilbert's conception of mathematics is especially useful to interpreters because it explains why mathematical vocabulary has become so extensive and so specialized and why

mathematical discourse has become so opaque that, to the nonspecialist, it often seems that mathematicians are speaking an entirely separate language (Franks, 2009).

To understand what makes higher mathematics different from other subjects, a good place to begin is with the game of chess. We often identify the game of chess with a chessboard and a collection of chess pieces, but neither the board nor the pieces are necessary to play chess. One can play chess using a virtual board and virtual pieces, and some people play chess without any sort of board and with no pieces at all. These people play chess “in their heads.” In a similar sort of way, computers, which now play at the highest level, also have no need of chess paraphernalia, virtual or otherwise. At its most fundamental level, chess is a set of terms, e.g. “pawn,” “king,” “queen,” and “bishop,” and a set of rules that govern how each piece (term) may be “moved.” The names of the chess pieces are undefined. In particular, one could easily rename every chess piece and the game would be played exactly as before. Rules are more fundamental; they place restrictions on moves. For example, “pawns may not move backwards,” and “bishops may only move along diagonals.” Within this system of undefined terms and rules, players are free to discover many different sequences of moves. A great many sequences are possible, but many other sequences are prohibited. To play chess, one must remain within the rules. Change the rules and one is no longer playing chess. The rules determine the game.

Axioms are to mathematics as rules are to chess. Axioms are sentences that describe relationships among terms. Each collection or set of axioms describes a mathematical discipline, and over the years mathematicians have created many different sets of axioms. Axioms determine the subject. The set of axioms that the Greek mathematician Euclid (fl. 300 B.C.E.) listed in his text *Elements* defines a geometry now known as *Euclidean geometry*. The terms that appear in Euclid’s axioms — e.g. “point” and “line” — help the mathematician visualize the subject in the same way that chess pieces and a chessboard help the chess player visualize the game, but the visualizations are not to be confused with the subject itself. Euclidean geometry is about the logical relationships that exist among terms such as “point,” “line,” and “plane”; it is not about the terms themselves. This distinction is critical to understanding higher mathematics as it is taught and used today. Hilbert put it this way: “One must at all times be able to replace, ‘points, lines, planes’ by ‘tables, chairs, beer mugs’” (Boyer & Merzbach, 1991, p. 611).

To see the effect of a change in axioms, consider the work of the Russian mathematician Nikolai Lobachevski (1793-1856) and the Hungarian mathematician János Bolyai (1802-1860). They independently developed sets of axioms that were logically equivalent to each other but different from Euclid’s. Lobachevski and Bolyai’s non-Euclidean axioms characterize what is now known as hyperbolic geometry. Although Lobachevski and Bolyai changed only one of Euclid’s axioms, the resulting geometry is strikingly different from that described by Euclid. In Euclid’s geometry, for example, the sum of the measures of the interior angles of a triangle always equals 180°. In hyperbolic geometry, the sum of the interior angles of a triangle is always less than 180°. Neither statement is wrong, nor does one conflict with the other. Each statement is a logical consequence of its own set of axioms. Just as the rules of chess define what it means to play chess, different sets of axioms define different mathematical disciplines. Change the axioms and one changes the subject. Not every set of sentences qualifies as a set of axioms. The requirements for a collection of sentences to form a set of axioms are, however, well understood. The interested reader can find a nontechnical discussion of the logical criteria to which a set of axioms must conform in Tabak (2011). What is important here is that many sets of axioms have been created, and from these sets of axioms many different mathematical disciplines have evolved.

It is vital to understand that there is no requirement that a set of axioms reflect any aspect of the physical world as it is revealed through observation and experimentation. To the mathematician interested in higher mathematics, such connections are irrelevant. In fact, many of the most important objects of study, e.g. infinite sets, have no known basis in nature (e.g. amounts of mass, momentum, and energy in the universe are all finite). An interpreter who wants to create an iconic mathematical sign—that is, a sign whose form reveals a correspondence between the mathematical term and its meaning—may be disappointed because often, as Hilbert indicated, mathematical terms have no intrinsic meaning. The abstract nature of modern mathematical language arises from the fact that mathematics has no standard for truth other than logic and no subject other than itself. For a somewhat more rigorous introduction to these ideas the reader is referred to Wilder (2000).

To summarize, one can visualize a mathematical discipline, such as Euclidean geometry or hyperbolic geometry, as a sort of organizational chart. Each node on the chart represents a term. Some of the terms are undefined; other terms are defined via the primitive (undefined) terms. The central feature of the chart is the set of lines that connect the nodes. The lines specify relationships that exist among nodes. In a mathematical context, each line represents a theorem, a statement deduced from the axioms that reveals logical relationships among the terms. While the lines are the central aspect of the chart, one must give names to the nodes in order to describe the chart. Mathematicians have found it necessary to create an extensive and often abstruse vocabulary to describe their work product. Mastering that vocabulary is one of the goals of a mathematics education. It is also, as will be shown, the principal barrier to successfully interpreting mathematical language.

Why Is It So Hard to Sign?

The Beginnings of Mathematical Sign Language

In the 1860s, when he was advocating for a college for deaf students, E. M. Gallaudet (1983), wrote that the American School for the Deaf (ASD) and the New York School for the Deaf (NYSD) were the only two schools out of the nation's then-24 schools for the Deaf offering coursework adequate to prepare their students for a rigorous program of college-level study. We can, therefore, be confident that prior to the establishment of the Columbia Institution, ASD and NYSD were the only schools for deaf students offering mathematics courses beyond simple arithmetic. Information about their programs can be obtained from the annual reports filed by these schools. The first annual report at ASD to mention that the students were studying what was then called "higher math" is dated 1852. Algebra was the subject. The annual report of 1854 indicates that geometry and surveying were also taught at the school (surveying is important because it requires knowledge of trigonometry). The annual report of 1860 also mentions algebra and geometry. More specific information about the courses at ASD seems to have been lost. (G. Waite, ASD archivist, personal communication, February 13, 2012).

Records at NYSD are more complete. In 1852, students in the newly-formed "higher class" at NYSD studied algebra (Thirty-Fourth Annual Report, 1853). The text was *Elementary Algebra: Embracing the First Principles of Science* (Davies, 1845). Topics included polynomial arithmetic and transformation of first- and second-degree equations. Davies's text is a rigorous introduction to the subject, emphasizing formal calculations, and is largely devoid of practical applications. By way of example, an 1855 test problem at NYSD instructed students to "extract

the square root of 1089" and to explain the process (Thirty-Sixth Annual Report, 1855). No information is given as to how this mathematics was signed. That the class would have been taught in sign is, however, beyond doubt. Clarke School for the Deaf, the nation's first oral school for deaf students, was not founded until 1867, and other methods such as total communication and signed English were not developed until the 20th century. The only language of instruction at American schools for deaf students during the time in question was called the "natural language of signs," which, as the movies made by the National Association of the Deaf during the early years of the 20th century indicate, would now be called ASL. In short, there were no alternatives to what we now call ASL as the language of instruction.

For many years, the Columbia Institution, later Gallaudet College and then University, filed detailed annual reports with Congress describing the college's programs and sometimes even listing the textbooks used in the courses. These reports make it clear that there were no substantive changes in the mathematics program during the 19th century. In particular, all students were required to take three years of math. The required courses, along with the textbooks, are listed in some of the annual reports, and although the textbooks changed a few times during this period, the basic coursework did not. More detailed descriptions of the coursework, including exam questions, can be found in the *Announcements*, the name of the college catalog, which was published yearly. By way of example, during the 1872-73 school year, the courses for the first two years of study, along with the required texts (editions unidentified), are as follows (Sixteenth Annual Report, 1873):

1. Algebra. Loomis, E. (1846). *A Treatise on Algebra*. New York, NY: Harper & Brothers.
2. Geometry. Loomis, E. (1850). *Elements of Geometry and Conic Sections*. New York, NY: Harper & Brothers.
3. Conic Sections. Loomis, E. (1850). *Elements of Geometry and Conic Sections*. New York, NY: Harper & Brothers. ("Conic sections" refers to the set of curves consisting of parabolas, hyperbolas, and ellipses. Spherical trigonometry, interest in which has since lapsed, is the study of triangles on spheres. Its principal use is in navigation.)
4. Plane and Spherical Trigonometry and Surveying. Loomis, E. (1848). *Elements of Plane and Spherical Trigonometry, with Their Applications to Mensuration, Surveying and Navigation*. New York, NY: Harper & Brothers.

The freshman algebra course began with the study of second-degree equations (Announcement, 1883). Loomis's text, and so presumably the course, emphasized calculations and applications. Similarly, judging from the texts, both Geometry and Conic Sections emphasized classical Greek geometry, updated somewhat to include some applications and a modest amount of algebra. The trigonometry text, and so presumably the course as well, emphasized applications.

These courses were not interpreted because the language of instruction at the Columbia Institution was the natural language of signs. It would be helpful to know how the courses were signed, but that information appears to be lost. Here are some sample exam questions from this period (Announcement, 1880).

1. Geometry: "Construct a plane triangle, having [been] given the perimeter and the angles of the triangle." (p. 32)

2. Conic Sections: “Prove that perpendiculars drawn from the foci upon a tangent to the ellipse meet the tangent in the circumference of a circle whose diameter is the major axis.” (p. 32)
3. Trigonometry: “Given two sides of a plane triangle 201 and 140, and the angle opposite the latter, 36° 44′. Find all of the other parts.” (p. 33)

The second of these problems, which seems the hardest of the three to express in ASL, could be illustrated with a picture. Depending on the reader’s background, these math problems may be hard to solve, but they are not so abstract that they cannot be conveyed using existing signs and possibly some diagrams. Compare those three problems with the following three theorems, which are taken from texts often used in (upper level) undergraduate math courses today:

1. “A space X is homeomorphic to an open subset of a compact Hausdorff space if and only if X is locally compact Hausdorff” (Munkres, 1975, p. 186).
2. “A finite integral domain is a field” (Herstein, 1975, p. 127).
3. “A function F is an indefinite integral if and only if it is absolutely continuous” (Royden, 1968, p. 107).

It does not seem likely that these problems can be conveyed via a combination of diagrams and conversational (nontechnical) language, whether that language is ASL or English. Every word of more than four letters appearing in each of the three theorems has a meaning that is specific to mathematics. The content of each theorem is not geometric and consequently not easily summarized in a diagram, and the definitions of the words used to express the theorems are defined in terms of other words that are also specific to mathematics. This is the language of higher mathematics, as the term is understood today.

As indicated in the Congressional Reports and the Announcements, during the 19th century, third-year math classes at Columbia/Gallaudet emphasized applications. During the 1870s and early 1880s, students studied astronomy. Judging from the astronomy text (Loomis, 1869) then in use (Announcement, 1880), the course provided students with ample opportunity to use their newly-acquired skills in algebra and geometry. During the 1880s, the department dropped astronomy in favor of classical mechanics. From the point of view of the interpreter, courses in astronomy and mechanics may be easier to produce in ASL than courses in higher mathematics. Physics problems are generally expressed in terms of forces, positions, distances, and other related physical concepts, all of which are reasonably concrete notions that can be expressed without too much specialized vocabulary. The following exam questions, which were taken from the *Announcements*, are representative of Columbia/Gallaudet’s mechanics course and were chosen because they are expressed without specialized mathematical notation:

1. “What is the centre of gravity of a triangular pyramid? Give the demonstration” (Announcement, 1880, p. 35) (“Give the demonstration” means “Prove your answer”).
2. “Suppose at the instant that a body begins to fall from D, another body is projected upward from B, with a velocity which would carry it to A; it is required to find the point where they would meet” (Announcement, 1883, p. 40).
3. The length of an inclined plane is 5 feet, and the height is 3 feet. Find into what two parts a weight of 104 pounds must be divided, so that one part hanging over the top of the plane may balance the other part resting on the plane (Announcement, 1883, p. 40).

The preceding questions illustrate that the specialized vocabulary required to express classical mechanics in ASL is smaller than the specialized vocabulary required to express higher mathematics in ASL, or even the more elementary mathematics questions cited above. In addition to the required courses, two electives were offered throughout the 19th century: Analytic Geometry, which was a second-year elective, and Calculus, which was a third-year elective. During the 1891-92 school year, *The Elements of Analytical Geometry and of the Differential and Integral Calculus* (Loomis, 1851) was used as the text for both courses (Announcement, 1892). The written mathematical language used at Columbia/Gallaudet during the 19th century is still the type of language with which most academic interpreters are familiar today. The algebra was so-called College Algebra, the geometry was Euclidean, and essentially identical physics problems can be found in many introductory mechanics texts today. All of the mathematical signed language required to express these subjects was (apparently) developed prior to 1880.

Finally, it is worthwhile to take note of the method of teaching employed by the faculty at Columbia. A description of the method by which students learned geometry stated: "[Geometric] demonstrations are occasionally made in writing, but the usual course is for the student to draw a diagram, and to give the proof by means of signs and the manual alphabet, pointing out each angle [and] line…as it is needed in the argument" (Nineteenth Annual Report, 1876, p. 736). This approach to teaching clearly required both faculty and students to develop a "mathematical extension" of ASL sufficient to express Euclidean geometry. Given this description of how geometry was learned and the test questions listed in the *Announcements,* it is possible that mathematical ASL, at least as it applied to Euclidean geometry, was as well developed in 1876 as it is today, perhaps better.

Current Use of ASL for Higher Mathematics

To appreciate how mathematics is expressed in ASL today, one must consider two cases: (1) the situation at Gallaudet University and (2) the situation everywhere else. As of this writing, at the National Technical Institute for the Deaf (NTID), the only mathematics courses offered in which ASL is the language of instruction are at a remedial level. For college-level classes (calculus and beyond), students enroll in mainstream courses where the teacher is a hearing mathematician with little or no knowledge of ASL, and an interpreter (and possibly other support staff) assists the student. For this reason, NTID is best categorized with other mainstream colleges and universities. It is worth mentioning that the Rochester Institute of Technology, the parent institution of NTID, only offers a degree in applied mathematics; it does not offer a degree (or the necessary coursework for a degree) in higher mathematics.

Since 1900, the biggest change in the way that deaf college and university students encounter mathematics occurred in the latter decades of the 20th century. Academic interpreting services became widely available. In response, many college-bound deaf students enrolled in colleges and universities where the language of instruction was English. California State University, Northridge and NTID were especially important in this regard. The method of learning mathematics at Columbia/Gallaudet (described earlier), wherein students and teachers communicated in the same language, is now something that most deaf college students never experience because, for mainstreamed deaf students, their first language is rarely that of their instructors'. For these deaf students, interpreters have played a critical role in their mathematical education.

Another way of thinking about this change is that during the latter decades of the 20th century, mathematical ASL, to the extent that it exists, became an interpreted language. Today, students wishing to learn higher mathematics often learn mathematical ASL from their interpreters, who possibly have had previous experience interpreting higher mathematics. In this situation interpreters might be looked upon as resident experts on signed mathematical language and as linguistic models in the use of that language.

The situation in which the interpreter becomes a teacher, at least with respect to mathematical ASL, runs counter to a model often used to describe what interpreters do when they interpret. A common model for interpreting dynamics, the so-called "Communication Model" (Stewart, Schein, & Cartwright, 2004), portrays interpreters as neutral conduits for information, changing the language used by one party to the language used by the other while minimizing the effects of the change. But when applied to higher mathematics, the Communication Model fails in three important ways. (There are, of course, other ways of modeling the dynamics of interpreting [Metzger, 1999], but any model in which the interpreter functions as a bridge between two fully developed languages fails to account for what happens when interpreting the language of higher mathematics. The Communication Model is used here solely to simplify the exposition.)

First, a student who enrolls in a course in higher mathematics without knowledge of mathematical ASL, or at least those aspects of mathematical sign pertinent to the course, will need to learn the relevant mathematical language, which is typically one of the objectives of any math course. In this case, the experienced academic interpreter models signed mathematical language for the student, who may be seeing the course-specific, higher level mathematical language for the first time. The interpreter may well be the student's only model for higher level mathematical signing. In such circumstances, it may also be the case that the interpreter will be the only individual with whom the student can use ASL to engage in discussions about the subject. This could happen for two reasons. In the event that only a few deaf students enroll in classes in higher mathematics, peer-to-peer discussions may not be possible due to an absence of peers. Second, in higher math classes the interpreter and the client may develop an extensive ad hoc mathematical vocabulary, one that is understood by the two of them exclusively. Even students at the same institution but with different interpreters may learn different signs for the same key terms. An ad hoc vocabulary is a two-edged sword. It facilitates the short-term exchange of mathematical information between interpreter and student, but over the longer term, it inhibits the exchange of information between the student and a different interpreter or between deaf individuals with different interpreters because the mathematical lexicon, once expressed as a series of ad hoc signs, becomes interpreter-student specific.

The second way that the Communication Model fails to account for what happens when interpreters interpret higher mathematics is that interpreters may not be proficient in mathematical English. When interpreting a course in higher mathematics for the first time, the meaning of certain key statements may be unclear to the interpreter. As a consequence, determining the best way to convey a technical-sounding mathematical statement may not be evident on first hearing. This is, of course, no secret, and some interpreters have sought to pool their experiences in order to improve the quality of their mathematical interpreting. Of particular importance is the work done by groups of interpreters at several educational institutions on the west coast, especially at the University of California, Northridge, as well as the work done at NTID, which has attempted to institutionalize the process. The efforts of those involved in this work have improved the quality of interpreting at their respective institutions, especially in

calculus, linear algebra, and statistics, courses that students in mathematics, engineering, and the sciences are required to take. But the fact that these courses are also required of science and engineering students demonstrates that the emphasis is still on applied mathematics, not higher mathematics.

Third, the scale of ASL deficit is not generally recognized. Having compiled a preliminary list, the author estimates that in the course of pursuing a BA/BS in mathematics, the average mathematics student learns approximately 1,800 subject-specific words. Taking into account that students at different institutions will be exposed to different aspects of mathematics—an emphasis on number theory, for example, versus an emphasis on differential equations—a reasonably complete dictionary of undergraduate mathematics should include about 2,500 words. This is in good agreement with the number of entries in undergraduate English language mathematics dictionaries. Incidentally, there are two types of mathematics dictionaries. The dictionaries described in this paragraph, which are aimed at an undergraduate audience, list mathematical terms together with their definitions. By way of example, the mathematics dictionary edited by James and James (1992) and the dictionary edited by Daintith and Clark (1999) both have approximately 2,000 entries. The dictionary edited by Clapham and Nicholson (2005) contains about 2,900 entries. How large, then, is today's signed undergraduate mathematical vocabulary, and what is the quality of the available signs?

There are a number of online reference works that can be used to gauge the extent of the standard lexicon as it exists today. Some of these sites have been available for years without much change. The *Texas Math Sign Language Dictionary* (2013) lists mathematics signs for grades K-12. The high school lexicon is 72 words long. There is no college lexicon. The *Science Signs Lexicon* (2013) has a "Math for Science" section that contains an idiosyncratic list of about 350 entries. Signed glosses for *convert*, *cost*, and *count* were deemed worthy of inclusion, although these signs are in common use in nonacademic conversation and would be familiar to any native signer. Glosses for the common mathematical terms *polynomial*, *matrix*, and *hyperbola* are absent in this resource although they are included in the Texas dictionary.

A more recent attempt, and one that has attracted some attention, is that of Richard Ladner at the University of Washington. Ladner has established a website to which anyone can contribute signs for math and science. Called *ASL-STEM Forum* (2013), the math section contains, as of this writing, more than 1,000 math terms, but very few of them are accompanied by signed glosses. The section on mathematics is divided into 21 topics, including discrete mathematics, calculus, general math terminology, geometry, cryptology, abstract algebra, and algebraic topology. The treatment is uneven and unsatisfactory. The geometry section contains 31 words, but only 17 are accompanied by signed glosses; the section on abstract algebra contains 228 entries, but only six are accompanied by signed glosses; the statistics section contains eight entries, but only one is accompanied by a signed gloss; and the "field of topology" section contains 155 entries only two of which are accompanied by signed glosses. ASL-STEM Forum is an attempt to crowd source the problem of creating a dictionary of signed mathematics, but so far the results are no better than those in the other online dictionaries already described. Another contribution of note is *American Sign Language for Mathematics—Secondary* (2013), which lists 110 signs described as suitable for "a typical high school mathematics class." Finally, *The Gallaudet Dictionary of American Sign Language* (Valli, Renner, & Swartzel Lott, 2005) is a general-purpose dictionary. With respect to mathematics signs, it has no more, but no less, to offer than most of the online sites already described. To be sure, other dictionaries of math signs have been compiled, but the author could find no dictionary of signed terms used in higher

mathematics, either online or in print, better than the ones just listed.

All of these dictionaries have two properties in common. First, they consist largely or entirely of signs that a mathematically ambitious student would have encountered in high school. Second, all of these sites include some signs that work well in high school but fail to "scale up" to higher-level mathematics discourse. The concept of scaling up is an important one and worth considering in some detail. Signs that are adequate for lower level courses may fail to work in higher math courses. Often, the reason that they are unsuitable for higher mathematical discourse is that they do not permit the signer to express the corresponding English language sentence fluently; they are examples of signs that do not "flow." We illustrate the idea with an example. The exponential function, the sine function, and the cosine function first arise in elementary courses. In these courses, the exponential function is often signed E-X-P, where the hyphens indicate a fingerspelled three-letter gloss. The sine function is represented as S-I-N, and the cosine function is represented by C-O-S. When interpreting an elementary course "math by abbreviation" works well enough because of the way that the corresponding words are used within the course, but these signed abbreviations become problematic in more advanced contexts. Consider, for example, Euler's equation, which is encountered in every undergraduate course in complex variables:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Using the fingerspelled abbreviations just described, Euler's equation might be expressed in the following way:

E-X-P *i* T-H-E-T-A = C-O-S T-H-E-T-A + *i* S-I-N T-H-E-T-A.

Or if one uses a sign for the Greek letter theta (at least one such sign is in use; see ASL/STEM Forum) then Euler's equation could be expressed awkwardly as

E-X-P *i* θ = C-O-S θ + *i* S-I-N θ.

Commonly used signs, as opposed to fingerspelled abbreviations, exist only for the symbols "=" and "+".

Expressing Euler's equation as a long series of fingerspelled words and/or fingerspelled abbreviations can be as exhausting for the student to read as it is slow and awkward to convey. It is worth noting that all of the dictionaries cited above that contain signed glosses for sine and cosine recommend using s-i-n and c-o-s.

It has long been known (Bonvillian, 1983; Odom, Blanton, & McIntyre, 1970; Spencer, Dale, & Klions, 1989) that signs that are simple to execute and that conform to ASL sign-making conventions are easier to learn and remember than complicated gestures that fail to conform to ASL conventions. Avoiding the extensive use of fingerspelled abbreviations is not just an aesthetic goal. Rather, an easy-to-sign lexicon can facilitate mathematical communication. ASL mathematical vocabulary is, therefore, not extensive enough or of sufficiently high quality to express fluently the concepts and techniques found in many branches of higher mathematics.

Deaf students interested in higher mathematics can avoid the difficulties currently associated with interpreted higher mathematics by attending Gallaudet University, where ASL is the language of instruction. This has several advantages. As an example, when a course in higher

mathematics is taught in ASL and the teacher pauses to fingerspell a word (e.g. the topological term *homeomorphism*), the resulting delay in presentation is of necessity experienced by teacher and student alike and tends, therefore, to be transparent to both parties. Compare this to the problem an interpreter encounters when attempting to fingerspell *homeomorphism* intelligibly while at the same time trying to keep up with the speaker. More generally, because mathematicians fluent in ASL have faced the same linguistic challenges as their deaf students with respect to the expression of higher mathematics in ASL, they can be expected to be more sensitive to the issues in involved. This is certainly what the author observed on a visit to Gallaudet University.

What, then, are the characteristics of mathematical language as it is used at Gallaudet? One way of describing math at Gallaudet is to look at the courses required for a bachelor's degree in mathematics. The curriculum has a mild bias toward applications. Gallaudet offers its mathematics majors (applied) courses in numerical analysis, statistics, cryptography, and operations research (Gallaudet University, Department of Mathematics, 2013). By contrast, the subject of set theoretic topology, which is commonly found as an upper level course in many undergraduate mathematics departments, is missing from the curriculum even as an elective, although Gallaudet has faculty members who are quite expert in the subject. (To obtain a background in set theoretic topology, which is a necessity for work in higher mathematics, one must master a highly specialized and very extensive vocabulary.)

To better understand how mathematics is communicated at Gallaudet, the author visited three classes offered within the Gallaudet mathematics department. Two, differential equations and abstract algebra, were courses in higher mathematics, and the third was calculus, which is not a higher math course. As this article is concerned with higher mathematics, all attention is directed toward the differential equations and abstract algebra classes. Senior mathematics professors who are fluent in sign taught these courses. Both classes were presented in a lecture format in which the instructor stood at the front of the class and delivered prepared remarks before a passive but attentive audience. Very little specialized signed mathematical vocabulary was employed during the classes that I observed—far less, for example, than what one would see in an interpreted lecture of the corresponding math class at a university other than Gallaudet. Instead, each lecture was expressed almost entirely in "conversational sign," that is, the signed lexicon used by the faculty was almost entirely nontechnical. By way of example, no attempt was made to express the words *induction*, *coefficient*, *dimension*, or *inverse* in sign despite the fact that these terms were important in the lectures observed that day. Instead, in each class, the words (or symbols representing the words) were written on the board, and when the professor required the corresponding concept, he would point to the place on the board where the word/symbol was written. To be specific, in the class on differential equations, the author counted only eight signs/sign combinations that were subject-specific, e.g. *basis*, *fundamental set*, and *matrix*. With respect to abstract algebra, most of the subject-specific vocabulary that one would expect to find in a lecture on group theory was suppressed, the exception being the terms *kernel*, *isomorphism*, *factor*, and *homomorphism*, which were expressed as K-E-R, I-S-O, F-A-C-T-O-R and H-O-M-O. In both classes, equations were written on the board, teachers pointed to the equations when necessary, and no attempt was made to express the equations in sign. The advantage of such a presentation is that it is unambiguous and transparent. As anyone who has interpreted mathematics lectures knows, mathematicians tend to depend heavily on the board (or on a projector) to communicate with their audiences. The written word/symbol removes ambiguity. The presentations that the author observed at Gallaudet University were, however, far

more reliant on the board than any of the thousands of other mathematics lectures that he has observed elsewhere.

Most lectures in higher mathematics depend in an essential way on an extensive and specialized vocabulary, yet these Gallaudet professors have found a way to convey significant mathematical content in ASL without ambiguity and with a minimum of specialized vocabulary. Given the clarity of the presentation, the author considers the approach used at Gallaudet to be something of a success, but there are significant costs associated with this method that should also be taken into account. One problem is that students do not see their teachers using mathematical language. If professors do not model high-level mathematical ASL, it is difficult see how students can learn to express themselves using mathematical ASL. One can argue that ASL is currently ill suited to express higher mathematics. While this may be true, it is hardly an argument for not creating the necessary mathematical extension of this otherwise vibrant language. Keep in mind that for hearing students, one of the goals of a mathematical education is to become fluent in mathematical language, spoken and written, so that they can more fully participate in the broader mathematical community. The same should be true for deaf undergraduates.

A second problem with this approach is that it tends to isolate its practitioners from the broader mathematical community. The method of communication that the author observed at Gallaudet requires a good deal of experience and skill. The choice of which terms to write versus which terms to sign and the visual presentation of the necessary vocabulary, equations, and theorems on the board yielded an intelligible visual whole, but the skills required to make this type of presentation take time to acquire. It seems unlikely that guest lecturers, hearing or deaf, could adopt it on short notice. Consequently, while this method of lecturing obviates the need for both an interpreter and a large active mathematics vocabulary, its use is almost certainly confined to Gallaudet. It is not a method of communication likely to be encountered in the broader mathematical community. From the interpreter's point of view there appears to be little to learn from the method for communicating higher mathematics that the author observed at Gallaudet. What, then, can be learned from the ways that other linguistic groups communicate higher mathematics?

Mathematical Language and Other Linguistic Groups

Since the 19th century, when the first research was conducted into the theory of sets, a theory on which much of modern mathematics is founded, there have been only four mathematical languages, a term used here to denote languages with a large mathematical literature. They are German, French, English, and Russian. German and French dominated until the 1930s and were then gradually displaced by English and Russian. The importance of Russian diminished with the dissolution of the Soviet Union, and at present, most research into higher mathematics is expressed in English, irrespective of the nationality of the author (Ammon, 2001).

For many years, the German-, French-, English-, and Russian-speaking mathematical communities evolved together on a more-or-less equal footing, exchanging ideas and results. To facilitate communication among the groups, mathematicians published a second type of mathematical dictionary. (The first type has already been described.) This second type of dictionary, and the only type of interest in this section, is used to facilitate reading mathematics in languages other than one's own. These dictionaries are, in concept, similar to most sign

language dictionaries. They also illustrate aspects of mathematical language that are important from the point of view of interpreters. Several such dictionaries exist. For purposes of illustration, I will use *Dictionary of Mathematics in Four Languages* (Eisenreich & Sube, 1982).

Eisenreich and Sube (1982) assume that the reader already knows the meaning of a particular mathematical term in his or her own language. Their dictionary provides only the equivalent word or phrase in each of the other three mathematical languages. This is an important point: The relationship between mathematical terms in each of the four mathematical languages is usually one-to-one, a relationship that reflects the fact that the lexicons were developed in a more or less coordinated fashion. One consequence of this coordination is that translation between languages is fairly straightforward. Each page of the dictionary is printed with four columns, one column for each of the four languages. Each row consists of four mathematical terms, one in English, French, German, and Russian; each entry in a given row corresponds to the same (mathematical) concept. The authors confine their attention to higher mathematics. The dictionary contains more than 23,000 English language entries.

That a dictionary devoted solely to mathematics can contain 23,000 entries and still, as the editors admit in the introduction, fail to be comprehensive reflects an important distinction between the mathematics lexicon and the lexicon of conversational language. Mathematicians create definitions in order to create usage (Edwards & Ward, 2008); lexicographers do not. In addition, as Edwards and Ward (2008) point out, when first created, a mathematical definition has no “truth value”; it is stipulated. By contrast, definitions created by lexicographers reflect usage. In lexicography, either a definition succeeds in reflecting actual usage, in which case the definition is true, or it fails to capture actual usage, in which case the definition is false. The nature of mathematical definitions helps to account for the extent of the lexicon and makes the approach found in Eisenreich and Sube’s (1982) work possible.

By way of illustration, consider the following four mathematical examples in English, German, and French. They were taken from Eisenreich and Sube (1982) (Russian entries omitted):

1. homeomorphism, Homöomorphie, homéomorphie
2. group axioms, Gruppenaxiome, axioms de groupes *or* postulats de groupes
3. midpoint, Intervallmitte, milieu
4. indeterminate system of equations, unbestimmtes Gleichungssystem, systemé indeterminé

A tightly controlled vocabulary coupled with the restricted and specialized nature of mathematical grammar, explains why many mathematicians can read in two or three mathematical languages without having any fluency in a conversational language other than their own.

Given the specialized nature of mathematical language, it is difficult to see how an interpreter can faithfully interpret a mathematics lecture without employing a mathematical ASL vocabulary as rich as the corresponding English mathematical vocabulary. One cannot rely on mathematical synonyms because as a general rule none exist. (Either one knows a sign for “hyperbola” or one does not.) A dictionary of mathematical ASL would, therefore, have the same type of pairing that one finds in Eisenreich and Sube (1982): an extensive one-to-one correspondence between signs and words. This would entail the creation of a large number of signs or the conscious decision to use familiar signs in a highly restricted way when employing them in the interpretation of mathematical speech.

And what of other linguistic groups? For many students whose first language is not a mathematical language, there are additional linguistic and cultural barriers to overcome. In Austin and Howson (1979), one can find a list of some of the more important of these barriers together with a number of interesting, albeit non-ASL, examples. The change from one's first language to mathematical English does, on average, affect educational attainment (Barton & Neville-Barton, 2003). While the effects can be mitigated, the transition to English must be effected. There is no escape from the centrality of mathematical English to mathematical communication and research (Montgomery, 2004). Consequently, as with their deaf counterparts, hearing students of higher mathematics for whom English is a second language and whose first language is not a mathematical one, must learn English, mathematical English, and mathematics simultaneously. An important difference between these hearing students and their deaf counterparts is that the language of instruction in courses of higher mathematics for hearing students, regardless of their first language, is usually (uninterpreted) English (Kaplan, 2001).

Evidently, the situation is more complicated for deaf students, but a version of mathematical English is what Gallaudet mathematicians seem to be creating when they write English language mathematical terms on the board and then point to them as the need arises. They are using a combination of English and ASL as the language of instruction. This approach is, however, not as efficient for deaf students as it may be for hearing students. Writing words and symbols on a board at the front of the classroom and pointing to them (rather than signing them) places a substantial limitation on the number of terms that can be introduced into a lecture. Words that are written on the board are more difficult for audience members to incorporate into any questions or comments that they might make, especially if the list is long and the words are closely spaced. Consequently, the best way to communicate higher mathematics to/from/among deaf individuals whose first language is ASL is to make ASL a mathematical language.

What Can Be Done?

Mathematical language is more than vocabulary. Mathematical grammar is also an important consideration for interpreters, but the difficulties associated with interpreting mathematical grammar are not nearly as great as those posed by the vocabulary. The grammar of mathematics is that of so-called first order logic, the expression of which depends on a small number of sentence "types"—a far narrower range of grammatical constructions than one finds in ordinary spoken English (Ferreirós, 2007).

By way of an (informal) example, consider a theorem used earlier in the text as an illustration of mathematical language. It is repeated here for ease of reference: "A space X is homeomorphic to an open subset of a compact Hausdorff space if and only if X is locally compact Hausdorff." Stripped of its content, this theorem can be represented as a statement about the equivalence of two sentences, "*A* if and only if *B*," where the letter *A* represents the sentence "A space X is homeomorphic to an open subset of a compact Hausdorf space." and the letter *B* represents the sentence "X is locally compact Hausdorf." The logical content of the theorem is that *A* is true provided *B* is true, and *B* is true provided *A* is true. Specifying *A* and *B* requires a good deal of specialized vocabulary, but specifying their logical relationship is simple: Some interpreters, for example, sign the *A*-sentence on one side of the interpreter's signing space then fingerspell I-F-F ("if and only if") at the midline of their signing space and conclude by signing the *B*-sentence on the other side: *A* I-F-F *B*. Most of the logical "skeleton" of mathematical discourse is well suited to similar visual representations.

So few grammatical constructions are in common use in higher mathematics that they could be explicitly listed in a brief paper along with various ways that interpreters have expressed these constructions. That paper is now in preparation. Once one develops an awareness of the grammatical simplicity of mathematical language, the centrality of the vocabulary deficit becomes clearer.

At present, interpreters are the main actors in the drive to make higher mathematics accessible to deaf students and deaf mathematics professionals. Those few mathematicians who are fluent in ASL could do more to raise the level of signed mathematical discourse. To create a "finished product," a mathematical extension of ASL, one requires a sufficiently large group of mathematically sophisticated signers willing to do the hard work of learning to express higher mathematics in sign. Through regular and frequent (signed) mathematical dialog, mathematical vocabulary and grammar would evolve as a matter of necessity. One method of creating such a group is via the so-called Moore method, a method of teaching most famously practiced by University of Texas mathematician, Robert Lee Moore (1882-1974). Moore was a prominent topologist who had enormous success as a teacher and many of his students became distinguished mathematicians. The value of the Moore method has been discussed in detail elsewhere (Coppin, Mahavier, May, & Parker, 2009).

In brief, the Moore method is the Socratic Method applied to mathematics: The teacher asks questions and through those questions controls the direction of the class's progress. The teacher also ensures that the discussion is rigorous. Students learn the subject by reinventing it—proving each theorem and critiquing each other's work in the process. There are no lectures; there are no books; and note-taking is kept to a minimum. Regular and frequent mathematical conversations—together with a great deal of individual effort—are the means by which mathematical knowledge is passed from one generation to the next. In one oft-quoted remark, Moore asserted, "That student is taught the best who is told the least." (Parker, 2005, p. vii)

The Moore method seems to offer the best hope that the requisite signed mathematical grammar will evolve in such a way as to meet the needs of the target population because a necessary condition for the success of the Moore method is that the participants use mathematical language. It is interesting to speculate that the brief description of geometry classes at the Columbia Institution, quoted above, wherein, students "give the proof by means of signs and the manual alphabet, pointing out each angle, line, &c., as it is needed in the argument." may indicate that some variant of the method was used during the 19th century at the Columbia Institution. Gallaudet University seems the best place to implement the Moore method, again illustrating the vital nature of this institution with respect to higher mathematics and deaf students.

Once mathematical ASL has been created, it can be disseminated. Videos of discussions and lectures in which mathematical ASL is the language of discourse should be made available to interpreters. These would be of great help to academic interpreters because they would provide a standardized model—the first such model—for how higher mathematics can be expressed as a non-interpreted signed language. Once a comprehensive mathematical signed language is created, interpreters of higher mathematics can return to the role of functioning as a bridge between two *developed* languages, mathematical English and mathematical ASL.

Finally, it is important to note that linguistic issues are often associated with political ones. Some readers will look with suspicion upon any attempt to add signs to ASL because of worries about the "purity" of the language. From a historical perspective, such skepticism is well justified. Attempts to destroy ASL have been documented elsewhere (Tabak, 2006). In addition,

ASL and the people who depend upon it have not always benefited from the efforts of those well-intentioned individuals who conflate ASL with manual representations of the English language. Some of these manual representations of English have caused confusion and introduced a number of nonsensical gestures into an otherwise elegant language. For three reasons, these concerns are ill founded with respect to mathematical ASL.

First, mathematics cannot at the present time be discussed in ASL with the same freedom and economy of expression with which it can be discussed in the four major mathematical languages. The theorem on compact Hausdorff spaces quoted earlier is a case in point. Creating a mathematical extension of ASL cannot help but make higher mathematics more intelligible to deaf students and more attractive to deaf students as a career choice. Deaf mathematics professionals will benefit by having a language in which they can discuss mathematics and mathematically intensive areas of science and engineering with the same freedom and economy as their hearing counterparts. Second, professionals would create mathematical ASL for professional use. It would have no more impact on the evolution of conversational ASL than mathematical English has had on the evolution of conversational English—which is to say, no impact at all. Third, because mathematical ASL is presently in such a rudimentary state, it is unnecessarily difficult for interpreters to effectively interpret higher mathematics because they must create a mathematical form of ASL. Worse, at present, this activity is carried out repeatedly, once for each interpreter or, at best, interpreter consortium. A fully developed standardized mathematical extension of ASL is a necessary condition for interpreters to effectively interpret lectures in higher mathematics, and effectively interpreted lectures are a necessary precondition for deaf math students and mathematics professionals to fully integrate with the broader mathematical community. It is difficult to see a downside to this goal.

In summary, a mathematical extension of ASL would open up many professional opportunities to talented deaf mathematicians and students. It would allow them to express their ideas with the same spontaneity as their hearing counterparts. It would enable interpreters to more effectively interpret higher mathematics and as a consequence deaf mathematics professionals could become full participants in seminars and lectures. The risk that mathematical ASL would somehow corrupt conversational ASL is vanishingly small. It is hoped that a well-organized, long-term project to address the issues identified in this article is undertaken soon.

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