Tournaments and a Fibonacci Link

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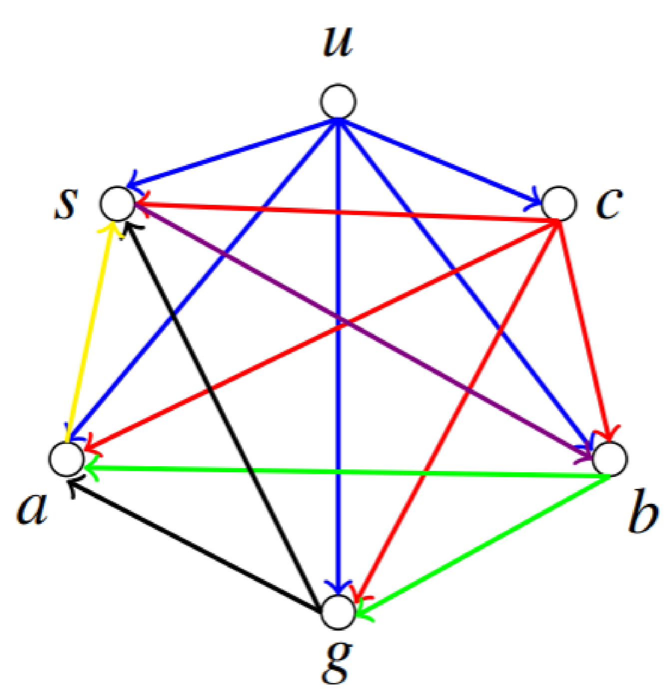
Tournaments & the Dream Team

Tournaments are a type of directed graph with a variety of applications, including infrastructure design and athletic competitions. *Round-robin* tournaments are frequently used to determine playoff seeding in professional sporting events, including the Olympics. In 1992, the U.S. Men's basketball team defeated each of their opponents in the preliminary round before going on to eventually win the gold medal.



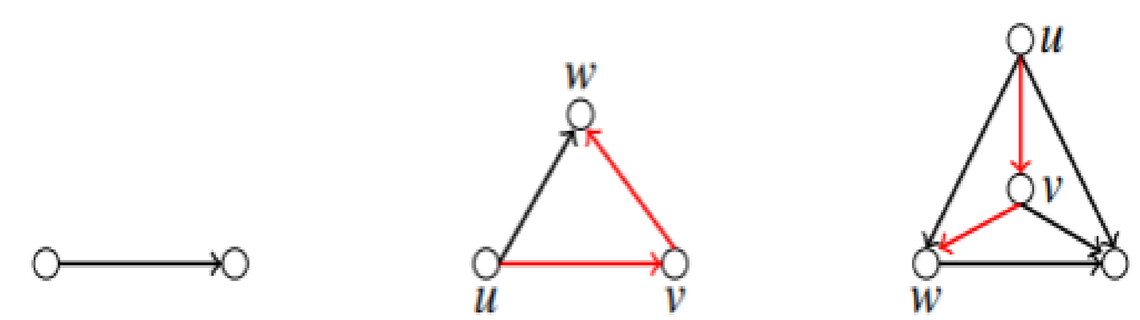
In this tournament, each vertex represents one of the six teams in Group A (u for United States, c for China, etc.) and each of the colored arcs represents the wins for each respective team.

Properties of Tournaments

Tournaments have a variety of intriguing properties, such as transitivity and connectivity, that make them a fascinating focal point in the realm of graph theory.

Transitivity

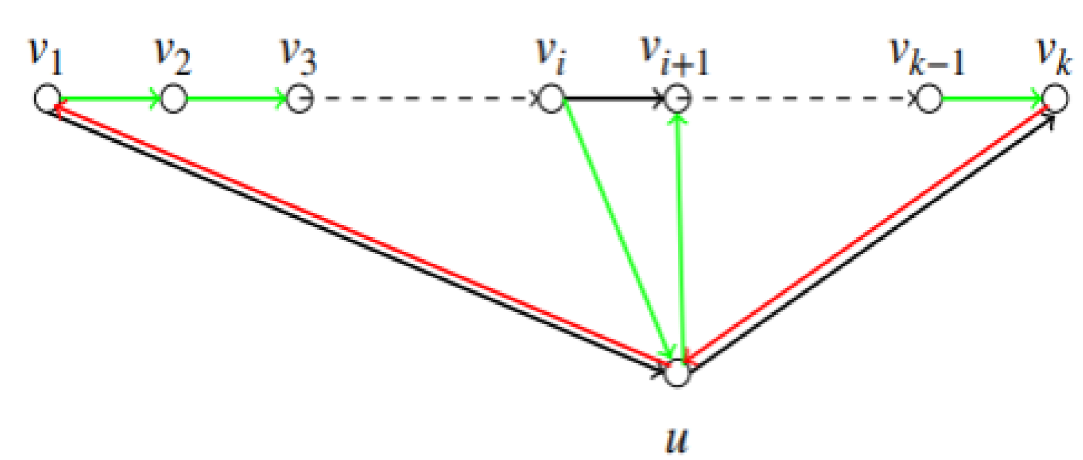
Transitive tournaments have a unique property: For every three vertices u, v, and w, if u beats (or dominates) v, and v beats w, then u beats w.



The UCF Paradox

In 2017, the University of Central Florida football team claimed themselves national champions after defeating Auburn University, who had previously beaten the University of Alabama (the team that went on to eventually be crowned the official national champions). UCF made this claim by utilizing selective transitivity, a paradox presented by a logical fallacy; that is, by assuming that the tournament they played in was transitive, they believed they could call themselves the champions despite not having competed for the championship.

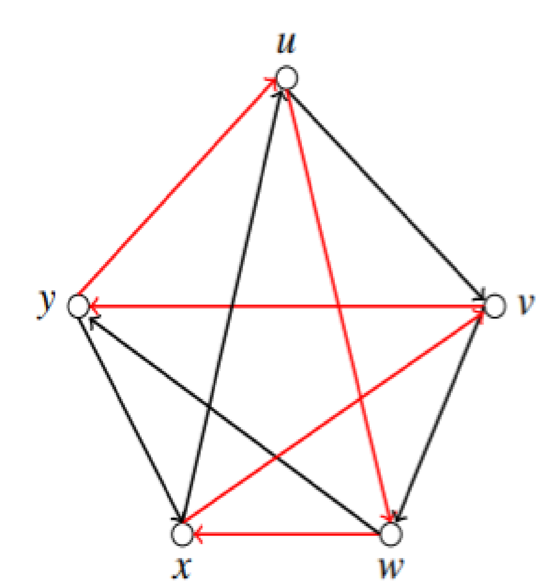
Hamiltonian Paths

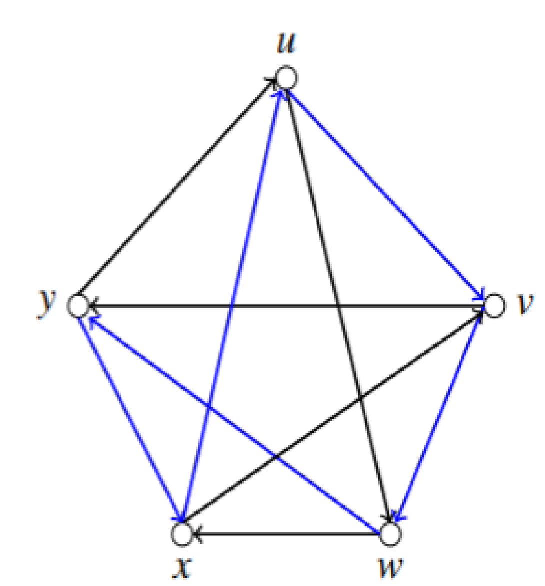


Further, every tournament contains a *Hamiltonian path*, a spanning walk in which no vertices repeat (illustrated in green above).

Hamiltonian Cycles

Strong connectivity is an aspect of digraphs that has particular relevance in tournaments. In short, a tournament is strongly connected if there exist both an a-to-b path and a b-to-a path. A *Hamiltonian cycle* is a closed spanning path, meaning that the first and last vertices of the path are the same vertex. Strongly connected tournaments necessarily contain Hamiltonian cycles. Below are two examples of Hamiltonian cycles, in red and blue.





Fibonacci & Hamiltonian Cycles

R.J. Douglas discovered an identity relating the number of unique Hamiltonian cycles in tournaments of order $n \geq 4$ to the Fibonacci numbers.

Douglas's identity states that the number of unique Hamiltonian cycles, $T(n)$, in a tournament of order n is the $(2n - 6)$th Fibonacci number. We demonstrate this property for $n = 5$(right), in which case the $(2n - 6)$th Fibonacci is the 4th Fibonacci number, which is 3. This process, though increasingly tedious as the order rises, holds similarly for all $n > 4$.

$$T(n) = 1 + \sum_{k=1}^{n-3} \sum_{p=0}^{min(k-1,n-k-3)} 2^{n-k-p-4}$$

$$\times \left[2\binom{n-k-3}{p}\binom{k-1}{p+1} + \binom{n-k-4}{p}\binom{k-1}{p} \right]$$

where $\binom{a}{b} = 0$ whenever $b > a$.

For $T(5)$, we have

$$T(5) = 1 + \sum_{k=1}^{2} \sum_{p=0}^{min(k-1,5-k-3)} 2^{5-k-0-4}$$

$$\times \left[2\binom{5-k-3}{0}\binom{k-1}{1} + \binom{5-k-4}{0}\binom{k-1}{0} \right]$$

If $k = 1$, $min(1-1, 5-1-3) = min(0,1) = 0$, and if $k = 2$, $min(2-1, 5-2-3) = min(1,0) = 0$. Therefore, $p = 0$ for any k here.

References

1. G. Chartrand and P. Zhang, *A First Course in Graph Theory*, 2012, Dover Publications.
2. M.R. Garey, On enumerating tournaments that admit exactly one Hamiltonian circuit, *J. Combin. Theory B* 13 (1972), 266-269.
3. J. Gross, J. Yellen, and P. Zhang, *Handbook of Graph Theory*, Second Edition, 2014, CRC Press.