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Density of the Numerators or Denominators of a Continued Fraction

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DENSITY OF THE NUMERATORS OR DENOMINATORS OF A CONTINUED FRACTION

by

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Dedicated to my wife Setareh, my daughter Ghazal, and my brother-in-law Dr. Masoud Nemati, for their love, patience and unending support.

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ABSTRACT

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Let $A = \{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers. There are two related sequences P_n and Q_n obtained from A by taking partial convergents out of the number $[0;\ a_1,\ a_2,\ldots,a_n,\ldots]$, where P_n and Q_n are the numerators and denominators of the finite continued fraction $[0;\ a_1,\ a_2,\ldots,a_n]$.

Let $P(n)$ be the largest positive integer k , such that $P_k \leq n$. The sequence $Q(n)$ is defined similarly.

- A known result of Barnes' Theorem states that $P(n) = o(n)$ and $Q(n) = o(n)$.
- In this paper we improve this result as $P(n) = O(\log \mathrm{n})$ and $Q(n) = O(\log \mathrm{n})$, where it follows that $P(n) = o(n^{\varepsilon})$ and $Q(n) = o(n^{\varepsilon})$ for any $\varepsilon > 0$.

Chapter 1

Introduction

Let $A = \{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers. We can naturally compose an irrational number in the interval $(0,\ 1)$ from A by expansion of simple continued fraction $[0;\ a_1,\ a_2, \ldots, a_n, \ldots]$. Denoting the n[th] convergent of this fraction by $\frac{P_n}{Q_n} = [0;\ a_1,\ a_2, \ldots, a_n]$, the P_n and Q_n are uniquely determined coprime numbers. Thus we obtain two infinite sequences of positive integers from A, the sequence of numerators $P = \{P_n\}_{n=1}^{\infty}$ and the sequence of denominators $Q = \{Q_n\}_{n=1}^{\infty}$. For an arbitrary sequence $\{b_n\}$ let $B(n)$ denotes the number of terms of the sequence not exceeding n. The natural density of B is defined to be $\delta(B) = \lim_{n \to \infty} \frac{B(n)}{n}$ if it exists. It has been proved by Barnes [1] that the natural density of the sequence $P = \{P_n\}_{n=1}^{\infty}$, $\delta(P) = 0$ for any given sequence A. The equality $\delta(P) = 0$ can be rewritten equivalently as $P(n) = o(n)$, which gives an estimation of the magnitude of $P(n)$. We will prove that $P(n) = O(\log n)$ and similarly $Q(n) = O(\log n)$ for any given sequence A.

A simple corollary to this result is that $P(n) = o\left(n^{\varepsilon}\right)$ and $Q(n) = o\left(n^{\varepsilon}\right)$ for any $\varepsilon > 0$.

Chapter 2

A Glance at Sequences of Real Numbers

2.1 Sequence

In this thesis a sequence is a function f from the set of natural numbers into a set of real numbers. Usually a sequence is denoted by $a_1,\ a_2,\ a_3,\ldots,a_n,\ldots$ instead of $f(1), f(2),\ldots, f(n),\ldots$ for historical reasons. Each $a_n = f(n)$ in the sequence is called a term. Therefore, a_n is called the nth term. The sequence $a_1,\ a_2,\ a_3,\ldots,a_n,\ldots$ is also denoted by $\{a_n\}$.

For example:

The set of numbers $1,\ 4,\ 9,\ldots$ is a sequence with the nth term given by

$a_n = n^2$, where $n \geq 1$.

2.2 Subsequence

If $\{a_n\}$ is a sequence of real numbers and if $\{n_k\}$ is a strictly increasing sequence of natural numbers $(n_{k+1} > n_k$ for all $n)$, then the sequence $\{a_{n_k}\}$ is called a subsequence of $\{a_n\}$.

For example,

if $\{a_n\} = \{n^2\} = \{1,\ 4,\ 9, \ldots\}$, and $\{n_k\} = \{1,\ 3,\ 5, \ldots\}$, then $\{a_{n_k}\} = \{1,\ 9,\ 25, \ldots\}$.

It is clear that $\{a_n\}$ has many other subsequences including itself. However, the sequence $\{b_n\} = \{1,\ 25,\ 9, \ldots\}$ is not a subsequence of $\{a_n\}$ since the order of terms is not preserved.

2.3 Bounded Sequences

The sequence $\{a_n\}$ is bounded above if there exists a real number B such that $a_n \le$ B for all n.

For example, if $\{a_n\} = \left\{1,\ \frac{1}{2},\ \frac{1}{3},\ \ldots\right\}$, then $a_n \le 1$ for all n, thus $\{a_n\}$ is bounded above. It is clear that for any number b less than 1, $a_n > b$ for some n (1 is the least upper bound). The sequence $\{a_n\}$ is bounded below if there exists a real number C such that $a_n \ge$ C for all n.

For example, if $\{a_n\} = \left\{1,\ \frac{1}{2},\ \frac{1}{3}, \ldots\right\}$ then $a_n > 0$ for all n. Thus $\{a_n\}$ is bounded below. It is clear that for any number b greater than 0, $a_n < b$ for some n (0 is the greatest lower bound, see definition 2.4 below).

Definition 2.4 Greatest lower bound

A real number C is said to be the greatest lower bound of $\{a_n\}$, if C is a lower bound of $\{a_n\}$ and no real number greater than C is a lower bound of $\{a_n\}$.

The greatest lower bound of $\{a_n\}$ is denoted by $\inf\{a_n\}$ (infimum of $\{a_n\}$) or glb $\{a_n\}$. Thus $C = \inf\{a_n\}$ if and only if it satisfies the following:

(1) $a_n \geq C$ for all n.

(2) For each $\varepsilon > 0$ there exists an integer n such that $a_n < C + \varepsilon$. The least upper bound is defined similarly.

If sequence $\{a_n\}$ is bounded above and below, we say it is bounded.

For example, the sequence $\{a_n\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$ is a bounded sequence, while the sequence $\{b_n\} = \{1, 2, 3, \ldots\}$ is bounded below but not bounded above and the sequence $\{c_n\} = \{0, -1, -2, \ldots\}$ is bounded above but not bounded below. The sequence $\{d_n\} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}$ is unbounded.

2.5 Limit of a Sequence

A sequence $\{a_n\}$ is said to have limit L if given any $\varepsilon > o$, there is a positive integer N such that $|a_n - \mathrm{L}| < \varepsilon$ whenever $n \geq \mathrm{N}$. If a sequence has limit L, we say that the sequence converges to L and we write $\lim a_n = \mathrm{L}$.

For example, if $\{a_n\} = \left\{\frac{1}{n}\right\} = \left\{1, \frac{1}{2}, \frac{1}{3}, \ldots\right\}$, then $\lim a_n = 0$.

2.6 Limit Inferior

Let $\{a_n\}$ be a bounded sequence. A subsequential limit of $\{a_n\}$ is any real number that is the limit of some subsequence of $\{a_n\}$. If S is the set of all subsequential limits of $\{a_n\}$, then we define the limit inferior of $\{a_n\}$ to be infimum of S denoted by lim inf a_n.

We wish to generalize the notion of the limit inferior to the unbounded sequences. We must consider two cases:

Case 1: Suppose that $\{a_n\}$ is unbounded below then we define lim inf $a_n = -\infty$.

Case 2: Suppose that $\{a_n\}$ is bounded below but not bounded above. If some subsequences converge to finite numbers, we define lim inf a_n to be the infimum of the set of subsequential limits. If no subsequence converges to a finite number, we must have lim $a_n = +\infty$, so we define lim inf $a_n = +\infty$.

For example, if

$\{a_n\} = \left\{1,\ \frac{1}{2},\ \frac{1}{3}, \ldots\right\}$, then lim inf $\{a_n\} = 0$.

$\{b_n\} = \{-1,\ -2,\ -3,\ -4,\ -5, \ldots\}$, then lim inf $\{b_n\} = -\infty$.

$\{c_n\} = \left\{-1,\ 1,\ -2,\ \frac{1}{2},\ -3,\ \frac{1}{3}, \ldots\right\}$, then lim inf $\{c_n\} = -\infty$.

$\{d_n\} = \{1,\ 2,\ 3, \ldots\}$, then lim inf $\{d_n\} = +\infty$.

2.7 Density of a Sequence of Positive Integers

The number of positive integers in a set A that are less than or equal to x (x is a real number not necessarily belonging to the set A) is denoted by $A(x)$.

For a positive increasing sequence of integers $\{a_n\}$, we have, $A(a_i) = i$.

For example, if

$A_1 = \{1,\ 2,\ 3, \ldots, k\}$ and $x = 6$, then $A_1(6) = 6$.

$A_2 = \{2,\ 4,\ 6, \ldots, 2k\}$ and $x = 6$, then $A_2(6) = 3$.

$A_3 = \{3,\ 6,\ 9, \ldots, 3k\}$ and $x = 6$, then $A_3(6) = 2$.

Consider the ratio $\dfrac{A(n)}{n}$:

$\dfrac{A_1(6)}{6} = 1$, $\dfrac{A_2(6)}{6} = \dfrac{1}{2}$, and $\dfrac{A_3(6)}{6} = \dfrac{1}{3}$, and in general $\dfrac{A_1(n)}{n} > \dfrac{A_2(n)}{n} > \dfrac{A_3(n)}{n}$.

Roughly speaking, we can say that, A_1 is more dense compared to A_2 and A_3 while A_2 is more dense compared to A_3.

A precise definition of the density is given below:

Definition 2.8 [9] The density of a set A (not necessarily positive integers) is defined by $\delta_1(A) = \liminf \dfrac{A(n)}{n}$. In case the sequence $\dfrac{A(n)}{n}$ has a limit, we say that A has a natural density denoted by $\delta(A)$, $\delta(A) = \lim\limits_{n\to\infty} \dfrac{A(n)}{n}$.

Obviously if $A(n)$ is bounded, then $\delta(A) = 0$.

It is easily seen that if a natural density $\delta(A)$ of a set A exists, then the density $\delta_1(A)$ exists and $\delta_1(A) = \delta(A)$.

Chapter 3

Continued Fractions

In this chapter we study finite and infinite continued fractions.

3.1 [3] Finite Continued Fractions

A finite continued fraction has the form

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cfrac{1}{\ddots \; a_{n-1} + \cfrac{1}{a_n}}}}}$$

where a_i are positive real numbers, except a_0, which could be any real number. The terms $a_0,\ a_1,\ a_2,\ a_3, \ldots,\ a_n$ are the partial denominators of this continued fraction. A continued fraction is called simple if a_i are all positive integers for $i \geq 1$.

Theorem 3.2 [3] Any rational number can be written as a finite simple continued fraction.

Proof : Let $\frac{a}{b}$ be any rational number where $b > 0$. Then by using the Euclidean division algorithm, we can write:

$$\begin{array}{ll} a = ba_0 + r_1 & 0 < r_1 < b, \\ b = r_1 a_1 + r_2 & 0 < r_2 < r_1, \\ r_1 = r_2 a_2 + r_3 & 0 < r_3 < r_2, \\ \vdots \quad \vdots \quad \vdots & \quad \vdots \\ r_{n-2} = r_{n-1}\, a_{n-1} + r_n & 0 < r_n < r_{n-1}, \\ r_{n-1} = r_n\, a_n . & \end{array}$$

Since r_k is greater than r_{k+1}, then a_k is positive for any k with $0 < k \le n$.

We rewrite the above equations as follows:

$$\frac{a}{b} = a_0 + \frac{r_1}{b} = a_0 + \frac{1}{\frac{b}{r_1}},$$

$$\frac{b}{r_1} = a_1 + \frac{r_2}{r_1} = a_1 + \frac{1}{\frac{r_1}{r_2}},$$

$$\frac{r_1}{r_2} = a_2 + \frac{r_3}{r_2} = a_2 + \frac{1}{\frac{r_2}{r_3}},$$

$$\vdots$$

$$\frac{r_{n-1}}{r_n} = a_n .$$

Eliminating $\frac{b}{r_1}$ from the first equation by using the second equation, we get

$\frac{a}{b} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{\cfrac{r_1}{r_2}}}$. Now eliminating $\frac{r_1}{r_2}$ from the second equation by using the third

equation, we get $\frac{a}{b} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{r_2}{r_3}}}$. If we continue this process we can then write

$\frac{a}{b}$ as a finite simple continued fraction.

The following notation is used to write $\frac{a}{b}$ as a finite simple continued fraction

$\frac{a}{b} = [a_0,\ a_1,\ a_2, \ldots,\ a_n]$. Notice that the representation of a rational number as a

simple continued fraction is not unique, as the following example shows.

<u>Example 3.3</u> Express $\frac{16}{57}$ as a finite simple continued fraction.

$$\frac{16}{57}=\frac{1}{\frac{57}{16}}=\frac{1}{3+\frac{9}{16}}=\frac{1}{3+\frac{1}{\frac{16}{9}}}=\frac{1}{3+\frac{1}{1+\frac{7}{9}}}=\frac{1}{3+\frac{1}{1+\frac{1}{\frac{9}{7}}}}=\frac{1}{3+\frac{1}{1+\frac{1}{1+\frac{2}{7}}}}=\frac{1}{3+\frac{1}{1+\frac{1}{1+\frac{1}{\frac{7}{2}}}}}$$

$$=\frac{1}{3+\frac{1}{1+\frac{1}{1+\frac{1}{3+\frac{1}{2}}}}}=\frac{1}{3+\frac{1}{1+\frac{1}{1+\frac{1}{3+\frac{1}{1+\frac{1}{1}}}}}}.$$

or $\frac{16}{57}=[0,\ 3,\ 1,\ 1,\ 3,\ 2]=[0,\ 3,\ 1,\ 1,\ 3,\ 1,\ 1]$ where $a_0=0$.

<u>Example 3.4</u> Express $\frac{187}{57}$ as a finite simple continued fraction.

We can write $\frac{187}{57}=3+\frac{16}{57}$ but from example 3.4 we know $\frac{16}{57}=[0,\ 3,\ 1,\ 1,\ 3,\ 1,\ 1]$,

thus $\frac{187}{57}=[3,\ 3,\ 1,\ 1,\ 3,\ 1,\ 1]$.

<u>Example 3.5</u> Express $-\frac{19}{51}$ as a finite simple continued fraction.

$-\frac{19}{51}=-1+\frac{32}{51}$ where $\frac{32}{51}=[1,\ 1,\ 1,\ 2,\ 6]$, then $-\frac{19}{51}=[-1,\ 1,\ 1,\ 1,\ 2,\ 6]$ where $a_0=-1$.

<u>Definition 3.6</u> The continued fraction made from $[a_0,\ a_1,\ a_2,\ldots,\ a_n]$ by cutting off the expansion after k[th] partial quotient a_k is called the k[th] convergent of the given continued fraction, and is denoted by $c_k=[a_0,\ a_1,\ a_2,\ldots,\ a_k]\quad 1\le k\le n$.

<u>Example 3.7</u> For $\frac{16}{57}=[0,\ 3,\ 1,\ 1,\ 3,\ 2]$, find c_n where $n=0,\ 1,\ 2,\ldots,\ 5$.

$c_0=0,\qquad c_1=0+\frac{1}{3}=\frac{1}{3}=[0,\ 3],$

$$c_2=0+\cfrac{1}{3+\cfrac{1}{1}}=\frac{1}{4}=[0,\ 3,\ 1],$$

$$c_3=0+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{1}}}=\frac{2}{7}=[0,\ 3,\ 1,\ 1],$$

$$c_4=0+\cfrac{1}{3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3}}}}=\frac{7}{25}=[0,\ 3,\ 1,\ 1,\ 3],$$

$$c_5 = 0 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{3 + \cfrac{1}{2}}}}} = \frac{16}{57} = [0,\ 3,\ 1,\ 1,\ 3,\ 2].$$

Now let us compare c_k with $\frac{16}{57}$ for $k = 1,\ 2,\ 3, \ldots,\ 5$.

$$c_1 = \frac{1}{3} = \frac{19}{57} > \frac{16}{57},$$

$$c_2 = \frac{1}{4} = \frac{14.25}{57} < \frac{16}{57},$$

$$c_3 = \frac{2}{7} = \frac{114}{399} > \frac{16}{57} = \frac{112}{399},$$

$$c_4 = \frac{7}{25} = \frac{15.96}{57} < \frac{16}{57},$$

$$c_5 = \frac{16}{57}.$$

Notice that $\left|\frac{1}{3} - \frac{16}{57}\right| = \frac{3}{57}$, $\left|\frac{1}{4} - \frac{16}{57}\right| = \frac{1.75}{57}$, $\left|\frac{2}{7} - \frac{16}{57}\right| = \frac{0.285}{57}$, and $\left|\frac{7}{25} - \frac{16}{57}\right| = \frac{0.04}{57}$.

We see that except for c_5, which is equal to $\frac{16}{57}$, these c_k's are alternately less than or greater than $\frac{16}{57}$. Each convergent is closer to $\frac{16}{57}$ than the previous one.

Now let us define P_k and Q_k as follows for $k = 0,\ 1,\ 2, \ldots, n$

$$P_0 = a_0, \qquad Q_0 = 1.$$

$$P_1 = a_1 a_0 + 1, \qquad Q_1 = a_1.$$

$$P_k = a_k P_{k-1} + P_{k-2}, \qquad Q_k = a_k Q_{k-1} + Q_{k-2} \quad \text{for} \quad k \geq 2.$$

Observe that, $\frac{P_0}{Q_0} = \frac{a_0}{1} = a_0 = c_0$,

$\frac{P_1}{Q_1} = \frac{a_1 a_0 + 1}{a_1} = a_0 + \frac{1}{a_1} = c_1$ and

$$\frac{P_2}{Q_2} = \frac{a_2 P_1 + P_0}{a_2 Q_1 + Q_0} = \frac{a_2(a_1 a_0 + 1) + a_0}{a_2 a_1 + 1} = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2}} = c_2.$$

In general we have the following.

<u>Theorem 3.8</u> [3] The k[th] convergent of the simple continued fraction $[a_0,\ a_1, \ldots, a_n]$ has the value $c_k = \frac{a_k P_{k-1} + P_{k-2}}{a_k Q_{k-1} + Q_{k-2}} = \frac{P_k}{Q_k}$.

Proof by Induction:

We know the theorem is true for $k = 0,\ 1$ and 2. Let us assume it is true for $k = m$,

$c_m = \frac{a_m P_{m-1} + P_{m-2}}{a_m Q_{m-1} + Q_{m-2}}$, we must show it is true for $k = m + 1$.

If we replace a_m by $a_m + \dfrac{1}{a_{m+1}}$ in c_m, then we have

$$c_{m+1} = [a_0,\ a_1, \ldots, a_{m-1},\ a_m,\ a_{m+1}\ ] = \left[a_0,\ a_1, \ldots, a_{m-1},\ a_m + \frac{1}{a_{m+1}}\right] =$$

$$\frac{\left(a_m + \dfrac{1}{a_{m+1}}\right)P_{m-1} + P_{m-2}}{\left(a_m + \dfrac{1}{a_{m+1}}\right)Q_{m-1} + Q_{m-2}},$$

or $c_{m+1} = \dfrac{a_{m+1}(a_m P_{m-1} + P_{m-2}) + P_{m-1}}{a_{m+1}(a_m Q_{m-1} + Q_{m-2}) + Q_{m-1}} = \dfrac{a_{m+1}P_m + P_{m-1}}{a_{m+1}Q_m + Q_{m-1}}$. The proof is complete.

Now let us see how this works for $\dfrac{16}{57} = [0,\ 3,\ 1,\ 1,\ 3,\ 2] = [a_0,\ a_1,\ a_2,\ a_3,\ a_4,\ a_5]$.

$P_0 = a_0 = 0$, $\quad Q_0 = 1$,

$P_1 = a_1 a_0 + 1 = 1$, $\quad Q_1 = a_1 = 3$,

$P_2 = a_2 P_1 + P_0 = 1$, $\quad Q_2 = a_2 Q_1 + Q_o = 1(3) + 1 = 4$,

$P_3 = a_3 P_2 + P_1 = 1(1) + 1 = 2$, $\quad Q_3 = a_3 Q_2 + Q_1 = 1(4) + 3 = 7$,

$P_4 = a_4 P_3 + P_2 = 3(2) + 1 = 7$, $\quad Q_4 = a_4 Q_3 + Q_2 = 3(7) + 4 = 25$,

$P_5 = a_5 P_4 + P_3 = 2(7) + 2 = 16$, $\quad Q_5 = a_5 Q_4 + Q_3 = 2(25) + 7 = 57$,

then $c_0 = \dfrac{P_0}{Q_0} = 0$, $\quad c_1 = \dfrac{P_1}{Q_1} = \dfrac{1}{3}$, $\quad c_2 = \dfrac{P_2}{Q_2} = \dfrac{1}{4}$,

$c_3 = \dfrac{P_3}{Q_3} = \dfrac{2}{7}$, $\quad c_4 = \dfrac{P_4}{Q_4} = \dfrac{7}{25}$, $\quad c_5 = \dfrac{P_5}{Q_5} = \dfrac{16}{57}$.

Theorem 3.9 [3] If $c_k = \frac{P_k}{Q_k}$ is the k[th] convergent of the simple continued fraction

$[a_0,\ a_1,\ a_2, \ldots, a_n]$, then $P_k Q_{k-1} - Q_k P_{k-1} = (-1)^{k-1} \quad (1 \le k \le n)$.

Proof by Induction: For $k = 1$ we have

$P_1 Q_0 - Q_1 P_0 = (a_1 a_0 + 1)\, 1 - a_1(a_0) = 1 = (-1)^0 = 1$.

Now assume it is true for $k = m$ which means that $P_m Q_{m-1} - Q_m P_{m-1} = (-1)^m$. Then

we must show it is true for $k = m+1$, which is $P_{m+1} Q_m - Q_{m+1} P_m = (-1)^{m+1}$. We can

write

$$(a_{m+1} P_m + P_{m-1}) Q_m - (a_{m+1} Q_m + Q_{m-1}) P_m = -(P_m Q_{m-1} - Q_m P_{m-1}) = -1\,(-1)^m = (-1)^{m+1}.$$

The proof is complete.

Corollary 3.10 [3] For $1 \le k \le n$ P_k and Q_k are relatively prime.

Proof: If $d = \gcd\,(P_k, Q_k)$ (gcd stands for greatest common divisor),

then d must divide $(-1)^{k-1}$ and since $d > 0$ then $d = 1$, which means that P_k and Q_k are relatively prime. Notice that we use the fact that if d divides a and b, then d divides $ax + by$ where x and y are arbitrary integers.

Corollary 3.11 [3] For $1 \le k \le n$, $Q_{k-1} \le Q_k$ and for $k > 1$, $Q_{k-1} < Q_k$

Proof by Induction: When $k = 1$, $Q_0 \le Q_1$ is true because $1 \le a_1$.

Suppose for $k = m$, $Q_{m-1} \le Q_m$. Then we must show that for $k = m+1$, we have

$Q_m \le Q_{m+1}$. It is easily seen that $Q_{m+1} = a_{m+1} Q_m + Q_{m-1} > a_{m+1} Q_m \ge 1 Q_m = Q_m$.

By the same argument we can show $P_{k-1} < P_k$.

Theorem 3.12 [3]

(1) If $c_k = \dfrac{P_k}{Q_k}$, then the convergents with even subscripts form a strictly increasing sequence, that is, $c_0 < c_2 < c_4 < \dots$.

(2) If $c_k = \dfrac{P_k}{Q_k}$, then the convergents with odd subscripts form a strictly decreasing sequence, that is, $c_1 > c_3 > c_5 > \dots$.

(3) If $c_k = \dfrac{P_k}{Q_k}$, then every convergent with an odd subscript is greater than every convergent with an even subscript.

Proof : With the aid of Theorem 3.9 we find that

$$c_{k+2} - c_k = (c_{k+2} - c_{k+1}) + (c_{k+1} - c_k) = \left(\frac{P_{k+2}}{Q_{k+2}} - \frac{P_{k+1}}{Q_{k+1}}\right) + \left(\frac{P_{k+1}}{Q_{k+1}} - \frac{P_k}{Q_k}\right)$$

$$= \frac{P_{k+2} \cdot Q_{k+1} - P_{k+1} Q_{k+2}}{Q_{k+2} \cdot Q_{k+1}} + \frac{P_{k+1} \cdot Q_k - P_{k+1} Q_{k+1}}{Q_{k+1} \cdot Q_k}$$

$$= \frac{(-1)^{k+1}}{Q_{k+2} \cdot Q_{k+1}} + \frac{(-1)^k}{Q_{k+1} \cdot Q_k} = \frac{(-1)^k (Q_{k+2} - Q_k)}{Q_k Q_{k+1} Q_{k+2}}.$$

Recalling that $Q_n > 0$ for all $n \geq 0$ and that $Q_{k+2} - Q_k > 0$, it is evident that $c_{k+2} - c_k$ has the same algebraic sign as does $(-1)^k$. If k is an even integer say $k = 2i$, then $c_{2i+2} - c_{2i} > 0$, hence $c_{2i+2} > c_{2i}$ for all i. Thus $c_0 < c_2 < c_4 < \dots$.

Similarly if k is an odd integer say $k = 2i - 1$, hence $c_{2i+1} < c_{2i-1}$ for all i.

Thus $c_1 > c_3 > c_5 > \dots$.

It remains only to show that any odd-numbered convergent c_{2r-1} is greater than any even-numbered convergent c_{2s}.

Since $P_k Q_{k-1} - Q_k P_{k-1} = (-1)^{k-1}$, upon dividing both sides of the equation by $Q_k Q_{k-1}$ we obtain

$$\frac{P_k}{Q_k} - \frac{P_{k-1}}{Q_{k-1}} = \frac{(-1)^{k-1}}{Q_k Q_{k-1}},$$

$$c_k - c_{k-1} = \frac{(-1)^{k-1}}{Q_k Q_{k-1}}.$$

This means that $c_{2i} < c_{2i-1}$.

The effect of tying the various inequalities together is that

$c_{2s} < c_{2s+2r} < c_{2s+2r-1} < c_{2r-1}$, as desired.

Let us check these for $\frac{16}{57} = [0,\ 3,\ 1,\ 1,\ 3,\ 2]$.

$$c_0 = 0, \quad c_1 = \frac{1}{3} = 0.3333, \quad c_2 = \frac{1}{4} = 0.2500, \quad c_3 = \frac{2}{7} = 0.2875,$$

$c_4 = \frac{7}{25} = 0.2800, \quad c_5 = \frac{16}{57} = 0.2807$ and we see that

$$0.0000 < 0.2500 < 0.2800 < 0.2807 < 0.2857 < 0.3333.$$

3.13 Infinite Continued Fractions

If $a_0,\ a_1,\ a_2, \cdots$ is an infinite sequence of integers, all positive except perhaps for a_0,

then the expression
$$a_0 + \cfrac{1}{a_1 + \cfrac{}{a_2 + \ddots}}$$
denoted by $[a_0,\ a_1,\ a_2, \cdots]$ is called an infinite simple continued fraction. We define the value of $[a_0,\ a_1,\ a_2, \ldots]$ as

$$\lim_{n\to\infty} c_n = \lim_{n\to\infty} [a_0,\ a_1,\ a_2, \ldots, a_n].$$

We will prove that this limit exists and it is always an irrational number, but before doing this, we will state and prove the following facts.

Lemma 3.14 A convergent sequence $\{a_n\}$ is bounded.

Proof: Since $\{a_n\}$ is convergent then $\lim_{n\to\infty} a_n = \mathrm{L}$. We must show that there exists a positive number B such that $|a_n| < \mathrm{B}$ for all n.

Since $|a_n| = |a_n - \mathrm{L} + \mathrm{L}| \le |a_n - \mathrm{L}| + |\mathrm{L}|$ (by triangular inequality) and for a fixed $\varepsilon > 0$ we can find N such that $|a_n - \mathrm{L}| < \varepsilon$ for all $n > N$, hence $|a_n| < \varepsilon + |\mathrm{L}|$ for all $n > N$.

It follows that $|a_n| < B$ for all n, where $B = \max\{a_1, \ldots, a_N, \varepsilon + |L|\}$.

The converse of Lemma 3.14 is not true; not every bounded sequence is necessarily convergent. For example, $\{a_n\} = \{0, 1, 0, 1, \ldots\}$ is bounded but does not converge.

Theorem 3.15 [2] If $\{a_n\}$ is a monotone increasing sequence, then $\{a_n\}$ converges if and only if it is bounded.

Proof: We saw in Lemma 3.14 that a convergent sequence is bounded. Now if we prove that a bounded monotone increasing sequence is convergent, then the proof of this theorem is complete.

Suppose $\{a_n\}$ is a bounded monotonic increasing sequence, then it has the least upper bound, say L and $a_n \le \mathrm{L}$ for all n. But, for any $\varepsilon > 0$ $\mathrm{L} - \varepsilon$ is not an upper bound of $\{a_n\}$ and there exists N such that $a_N \ge L - \varepsilon$ and for $n \ge N$ $\mathrm{L} - \varepsilon < a_N < a_n \le \mathrm{L}$.

It follows that $|a_n - \mathrm{L}| < \varepsilon$ for $n \ge N$. This means that $\{a_n\}$ converges to L.

Corollary 3.16 If $\{a_n\}$ is a monotone decreasing sequence, then $\{a_n\}$ converges if, and only if, it is bounded.

Proof : It is similar to that of Theorem 3.15.

Theorem 3.17 [3] The value of any infinite continued fraction exists and is an irrational number.

Proof : We want to show that $\lim_{n\to\infty} c_n = \lim_{n\to\infty} [a_{0,}\ a_1,\ a_2, \ldots, a_n]$ exists and it is always an irrational number.

We saw $c_0 < c_2 < c_4 < \ldots < c_{2n} < \ldots < c_{2n+1} < \ldots < c_5 < c_3 < c_1$.

Since $\{c_{2n}\}$ is monotically increasing bounded by c_1 it will converge to a limit, say l_1, which is greater than each c_{2n}. Similarly the monotically decreasing sequence $\{c_{2n+1}\}$ is bounded below by c_0 and it converges to a limit, say l_2, which is less than each c_{2n+1}.

Now we show $l_1 = l_2$.

As we point out that $l_1 > c_{2n}$, $l_2 < c_{n+1}$,

hence, $l_2 - l_1 < c_{2n+1} - c_{2n} = \dfrac{P_{2n+1}}{Q_{2n+1}} - \dfrac{P_{2n}}{Q_{2n}}$

$$l_2 - l_1 < \frac{P_{2n+1}Q_{2n} - Q_{2n+1}P_{2n}}{Q_{2n+1}Q_{2n}} = \frac{(-1)^2}{Q_{2n+1}Q_{2n}} = \frac{1}{Q_{2n+1}Q_{2n}}.$$

Then $|l_2 - l_1| < \dfrac{1}{Q_{2n}Q_{2n+1}} < \dfrac{1}{Q_{2n}^2}$ because $Q_{2n+1} > Q_{2n}$.

Since Q_{2n} increases without bound, for any given $\varepsilon > 0$ there is an N such that for $n \geq N$, $|l_2 - l_1| < \varepsilon$, this means that $l_2 = l_1$. Thus c_n has a limit, say L and the value of $[a_0,\ a_1,\ a_2, \ldots] = \lim_{n \to \infty} [a_1,\ a_1,\ a_2, \ldots, a_n] = L$.

Now we will show that the value of any infinite continued fraction is an irrational number.

Suppose $L = [a_0,\ a_1,\ a_2, \ldots] = \lim_{n \to \infty} [a_1,\ a_1,\ a_2, \ldots, a_n] = \lim_{n \to \infty} c_n$ and $c_{2n} < L < c_{2n+1}$.

Then $0 < |L - c_{2n}| < |c_{2n+1} - c_{2n}| = \left| \frac{P_{2n+1}}{Q_{2n+1}} - \frac{P_{2n}}{Q_{2n}} \right| = \frac{1}{Q_{2n} Q_{2n+1}}$.

Suppose L is a rational number, say $L = \frac{a}{b}$.

Then $0 < \left| \frac{a}{b} - \frac{P_n}{Q_n} \right| < \frac{1}{Q_n Q_{n+1}}$ or $0 < |aQ_n - bP_n| < \frac{b}{Q_{n+1}}$.

Since Q_n increases without a bound, then $\frac{b}{Q_{n+1}} < 1$ for sufficiently large n or $0 < |aQ_n - bP_n| < 1$. However, $a,\ b,\ P_n$ and Q_n are integers and the above inequality implies that there is a positive integer between 0 and 1, which is impossible. Therefore, L is an irrational number.

Chapter 4

The Fibonacci Sequence

The Fibonacci Sequence $u_1,\ u_2,\ u_3, \ldots, u_n$ is defined by

$u_1 = 1 \quad u_2 = 1$ and $u_n = u_{n-1} + u_{n-2}$ for $n \geq 3$. From this definition we see that

$\{u_n\} = \{1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21, \ldots\}$.

However, we need more information about this sequence before we proceed.

Theorem 4.1 [3] Successive Fibonacci numbers are relatively prime or

$d = \gcd(u_n,\ u_{n+1}) = 1$.

Proof : Suppose that the integer $d \geq 1$ divides both u_n and u_{n+1}. Then d divides

$u_{n+1} - u_n = u_{n-1}$.

If we continue in this manner, we conclude that d divides $u_1 = 1$. Thus $d = 1$ and the theorem is proved.

Problem 4.2 Consider $\frac{u_{n+1}}{u_n}$ where u_n and u_{n+1} are two successive Fibonacci numbers.

We wish to determine $\lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n}$. Consider the continued fraction $x = [1,\ 1,\ 1, \ldots]$. It is easily seen that for this particular continued fraction $P_n = u_{n+2}$, $Q_n = u_{n+1}$ $n \geq 0$ (see the definition of P_k and Q_k before Theorem 3.8 on page 14).

Thus, $\lim\limits_{n\to\infty} \frac{P_n}{Q_n} = \lim\limits_{n\to\infty} \frac{u_{n+2}}{u_{n+1}} = \lim\limits_{n\to\infty} \frac{u_{n+1}}{u_n} = x$.

Now we find the value of x.

$$x = \lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lim_{n\to\infty} \frac{u_n + u_{n-1}}{u_n} = \lim_{n\to\infty} \left(1 + \frac{u_{n-1}}{u_n}\right) = \lim_{n\to\infty} \left(1 + \frac{1}{\frac{u_n}{u_{n-1}}}\right).$$

Hence $x = 1 + \dfrac{1}{\lim\limits_{n\to\infty} \frac{u_n}{u_{n-1}}} = 1 + \dfrac{1}{x}$.

Then $x^2 - x - 1 = 0$, and the positive root is $x = \dfrac{1+\sqrt{5}}{2}$.

Therefore $[1,\ 1,\ 1, \ldots] = \dfrac{1+\sqrt{5}}{2}$.

Problem 4.3 [3] Let $\alpha = \frac{1}{2}\left(1+\sqrt{5}\right)$. Show the n[th] fibonacci number u_n is given by

$$u_n = \frac{\alpha^n - (-\alpha)^{-n}}{\sqrt{5}}.$$

Proof by Induction: For $n = 1$ we have to show $u_1 = \dfrac{\alpha - (-\alpha)^{-1}}{\sqrt{5}} = \dfrac{\alpha + \frac{1}{\alpha}}{\sqrt{5}}$.

Since $\alpha + \dfrac{1}{\alpha} = \dfrac{1+\sqrt{5}}{2} + \dfrac{2}{1+\sqrt{5}} = \dfrac{1+5+2\sqrt{5}+4}{2\left(1+\sqrt{5}\right)} = \dfrac{10+2\sqrt{5}}{2\left(1+\sqrt{5}\right)} = \dfrac{5+\sqrt{5}}{1+\sqrt{5}}$,

therefore, $u_1 = \dfrac{\frac{5+\sqrt{5}}{1+\sqrt{5}}}{\sqrt{5}} = \dfrac{5+\sqrt{5}}{5+\sqrt{5}} = 1$ which is true.

Suppose it is true for $n = k$. We then have, $u_k = \dfrac{\alpha^k - (-\alpha)^k}{\sqrt{5}}$.

Now we must show it is true for $n = k+1$. That is to say $u_{k+1} = \dfrac{\alpha^{k+1} - (-\alpha)^{-(k+1)}}{\sqrt{5}}$.

Since $u_{k+1} = u_k + u_{k-1}$, we now have

$$u_{k+1} = \frac{\alpha^k - (-\alpha)^{-k}}{\sqrt{5}} + \frac{\alpha^{k-1} - (-\alpha)^{-(k-1)}}{\sqrt{5}} = \frac{1}{\sqrt{5}}\left[\alpha^k\left(1+\alpha^{-1}\right) - (-\alpha)^{-k}(1-\alpha)\right]$$, but

$$1-\alpha = 1 - \frac{1+\sqrt{5}}{2} = \frac{1-\sqrt{5}}{2} = \frac{\left(1-\sqrt{5}\right)\left(1+\sqrt{5}\right)}{2\left(1+\sqrt{5}\right)} = \frac{-4}{2\left(1+\sqrt{5}\right)} = -\frac{2}{1+\sqrt{5}} = -\frac{1}{\alpha} = (-\alpha)^{-1}$$

$1-\alpha = (-\alpha)^{-1}$ or $1 + \dfrac{1}{\alpha} = \alpha$.

Therefore, $u_{k+1} = \frac{1}{\sqrt{5}}\left[\alpha^k(\alpha) - (-\alpha)^{-k}(-\alpha)^{-1}\right] = \frac{1}{\sqrt{5}}\left[\alpha^{k+1} - (-\alpha)^{-(k+1)}\right]$

and the proof is complete.

Chapter 5

Barnes' Theorem

Let $[a_0,\ a_1, \ldots, a_n, \ldots]$ be a simple continued fraction, and its approximants be $\frac{P_n}{Q_n}$. Then the sequence $\{P_n\}$ of numerators of the approximants is well defined. It has an interesting property which was proved by Barnes in 1973.

<u>Theorem 5.1</u> [1] (Barnes' Theorem) The sequence of numerators of approximants to every simple continued fraction has the natural density of zero.

Proof : Before proving Barnes' Theorem we first will prove the following lemma.

<u>Lemma 5.2</u> Suppose $\{b_n\}$ is an increasing positive sequence and if $\sum_{n=1}^{\infty} \frac{1}{b_n}$ converges, then $\lim_{n\to\infty} \frac{n}{b_n} = 0$.

Proof: It is easily seen that $\left\{\frac{1}{b_n}\right\}$ is a decreasing sequence because $\{b_n\}$ is an increasing positive sequence. Since $\sum_{n=1}^{\infty} \frac{1}{b_n}$ converges, then for any given $\varepsilon > 0$ there exists a positive integer N such that for $n > m > N$ $0 < \frac{1}{b_m} + \frac{1}{b_{m+1}} + \frac{1}{b_{m+2}} + \cdots + \frac{1}{b_n} < \frac{\varepsilon}{2}$ or

$(n-m)\frac{1}{b_n} < \frac{1}{b_m} + \frac{1}{b_{m+1}} + \frac{1}{b_{m+2}} + \cdots + \frac{1}{b_n} < \frac{\varepsilon}{2}$ or $\frac{n}{b_n} < \frac{\frac{\varepsilon}{2}n}{n-m}$, taking $n = 2m$,

$\frac{n}{b_n} < \frac{\frac{\varepsilon}{2}(2m)}{m} = \varepsilon$. Thus $\lim_{n\to\infty} \frac{n}{b_n} = 0$.

Notice that the converse is not true.

This means that if $\lim_{n\to\infty} \frac{n}{b_n} = 0$ it is not necessarily true that $\sum_{n=1}^{\infty} \frac{1}{b_n}$ converges.

Look at the following example:

If $b_n = n\log n$, then $\lim_{n\to\infty} \frac{n}{b_n} = \lim_{n\to\infty} \frac{n}{n\log n} = \lim_{n\to\infty} \frac{1}{\log n} = 0$, but $\sum_{n=2}^{\infty} \frac{1}{n\log n}$, does not converge by the integral test.

Now we can prove Barnes' Theorem.

If $c_n = \frac{P_n}{Q_n}$, then we want to show that $P = \{P_n\}_{n=1}^{\infty}$ has the natural density of zero.

We first show by induction that $P_n \geq u_n$, where u_n is the n[th] term of the Fibonacci Sequence.

Note that $P_1 = a_1 \geq 1 = u_1$ and we assume it is true for $n \leq k$. Therefore $P_n \geq u_n$. Now we must show it is true for $n = k+1$ or $P_{k+1} \geq u_{k+1}$. But, we know from the definition on page 14 that $P_{k+1} = a_k P_k + P_{k-1} \geq P_k + P_{k-1} \geq u_k + u_{k-1} = u_{k+1}$ as desired.

But, we know $\lim \frac{u_n}{u_{n+1}} = \frac{2}{1+\sqrt{5}} < 1$. Then $\sum_{n=1}^{\infty} \frac{1}{u_n}$ converges by ratio test.

Since $\frac{1}{P_n} \leq \frac{1}{u_n}$, then by comparison test $\sum_{n=1}^{\infty} \frac{1}{P_n}$ converges. And $\lim \frac{\mathrm{n}}{\mathrm{P_n}} = 0$ by lemma 5.2.

<u>Proposition 5.3</u>

If $\{b_n\}$ is a positive increasing sequence with $\lim_{n\to\infty} \frac{n}{b_n} = 0$, then $\lim_{n\to\infty} \frac{B(n)}{n} = 0$.

Proof: Let ε be an arbitrary positive number. There exists a positive integer N such that $\frac{k}{b_k} < \varepsilon$ for every $k > N$. Let $M = b_N$. Note that $B(M) = N$.

We prove that for $m \geq M$, $\frac{B(m)}{m} < \varepsilon$. Let $B(m) = k$. Thus $a_1,\ a_2, \ldots, a_k \leq m$ and $\frac{B(m)}{m} = \frac{k}{m} \leq \frac{k}{a_k} < \varepsilon$, since $k = B(m) \geq B(N) = N$, the proof is complete. By this proposition then, $\delta(P) = 0$ and the proof of Barnes' Theorem is complete.

Using a similar argument, we can prove that for the sequence $\{Q_n\}$, in the denominator of the approximants of a simple continued fraction that, $\delta(Q) = 0$.

Definition 5.4 [2] Big O and little o notation.

We write $f(x) = o(g(x))$

(with the understanding that x is near some point of interest x_0, possibly ∞), if

$\lim_{x \to x_0} \frac{f(x)}{g(x)} = 0$, and $f(x) = O(g(x))$, if $|f(x)| \le C|g(x)|$ (for some $C > 0$) in some neighborhood of x_0.

Using little o notation we can write the result of Barnes' Theorem in a concise form. Let $[a_0, a_1, \ldots, a_n, \ldots]$ be a simple continued fraction. Let its approximants be $\frac{P_n}{Q_n}$. Then $P(n) = o(n)$ and $Q(n) = o(n)$.

Chapter 6

Improvement of Barnes' Theorem

<u>Theorem 6.1</u> Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers.

There are two related sequences P_n and Q_n obtained by taking numerators and denominators of the partial convergent out of the continued fraction number $[0,\ a_1,\ a_2,\dots,a_n,\dots]$.

Then $P(n) = O(\log n)$ and $Q(n) = O(\log n)$.

Proof: If $c_n = \dfrac{P_n}{Q_n} = [0,\ a_1,\ a_2,\dots,a_n]$ then we can write

$$P_0 = 0, \qquad P_1 = 1, \qquad P_n = a_n P_{n-1} + P_{n-2},$$

$$Q_0 = 1, \qquad Q_1 = a_1, \qquad Q_n = a_n Q_{n-1} + Q_{n-2}.$$

As we know $P_n \geq u_n$ and $Q_n \geq u_n$.

Therefore, $P(n) \leq U(n)$ and $Q(n) \leq U(n)$ for any positive integers.

In Chapter 4 we have shown that $u_n = \dfrac{\alpha^n - (-\alpha)^{-n}}{\sqrt{5}}$ where $\alpha = \dfrac{1+\sqrt{5}}{2}$ and u_n is the n^th^ term of the Fibonacci Sequence.

Now let n be a positive integer and let $U(n) = k$.

We then have $u_k = \frac{\alpha^k - (-\alpha)^{-k}}{\sqrt{5}} \le n$, $\alpha^k - (-\alpha)^{-k} \le \sqrt{5}\, n$,

$$\alpha^k - \frac{1}{(-\alpha)^k} \le \sqrt{5}\, n, \qquad \alpha^k + \frac{(-1)}{(-1)^k \alpha^k} \le \sqrt{5}\, n,$$

$$\alpha^{2k} - \sqrt{5}\, n\alpha^k + (-1)^{1-k} \le 0.$$

Thus α^k is between the roots of the quadratic equation $x^2 - \sqrt{5}\, nx + (-1)^{1-k} = 0$.

specifically $\alpha^k \le \frac{\sqrt{5}\, n + \sqrt{5n^2 - 4(-1)^{1-k}}}{2}$ and we can write,

$$k \le \log_\alpha \left[\frac{\sqrt{5}\, n + \sqrt{5n^2 - 4(-1)^{1-k}}}{2} \right] \le \log_\alpha \sqrt{5}\,(n+1) = \frac{\log\left[\sqrt{5}\,(n+1)\right]}{\log\alpha}.$$

Using big O notation we can write

$U(n) = O(\log n)$ and $P(n) = O(\log n)$.

By the same argument we have $Q(n) = O(\log n)$.

Notice that $\lim\limits_{n\to\infty} \frac{\log n}{n^\varepsilon} = 0$ for any given $\varepsilon > 0$ because $\lim\limits_{n\to\infty} \frac{\log n}{n^\varepsilon} = \lim\limits_{n\to\infty} \frac{1}{\varepsilon n^\varepsilon} = 0$ by L'Hopital's Rule.

Therefore, Collary 6.2 follows:

Corollary 6.2 Let ε be any given positive number.

Then $P(n) = o\left(n^\varepsilon\right)$ and $Q(n) = o\left(n^\varepsilon\right)$.

Chapter 7

A Conjecture for Further Study

Let $A^{(1)} = \left\{a_1^{(1)},\ a_2^{(1)}, \ldots,\ a_n^{(1)}, \ldots\right\}$ and $A^{(2)} = \left\{a_1^{(2)},\ a_2^{(2)}, \ldots,\ a_n^{(2)}, \ldots\right\}$ be two increasing sequences of natural numbers. We can now construct a new sequence $A^{(1)} \oplus A^{(2)}$ (where $\oplus$ is conventional symbol) as follows.

Let $b_1 = \min\left\{a_1^{(1)},\ a_1^{(2)},\ a_2^{(1)},\ a_2^{(2)}, \ldots,\ a_n^{(1)},\ a_n^{(2)}, \ldots\right\}$

and $b_2 = \min\left\{\left\{a_1^{(1)},\ a_1^{(2)},\ a_2^{(1)},\ a_2^{(2)}, \ldots,\ a_n^{(1)},\ a_n^{(2)}, \ldots\right\} \setminus \{b_1\}\right\}$ (where \ means the difference of two sets).

If b_n is well defined, let

$$b_{m+1} = \min\left\{\left\{a_1^{(1)},\ a_1^{(2)},\ a_2^{(1)},\ a_2^{(2)}, \ldots,\ a_n^{(1)},\ a_n^{(2)}, \ldots\right\} \setminus \{b_1,\ b_2, \ldots, b_m\}\right\}.$$

The sequence $\{b_1,\ b_2, \ldots, b_n, \ldots\}$ is well defined.

Let $A^{(1)} \oplus A^{(2)} = \{b_1,\ b_2, \ldots, b_n, \ldots\}$.

It is now easily seen that $A^{(1)} \oplus A^{(2)}$ is also an increasing sequence of natural numbers.

Similarly if $A^{(i)}(i = 1, 2, \ldots)$ are increasing sequences of natural numbers, we can further define an increasing sequence of $\oplus_{i=1}^{\infty} A^{(i)}$. This definition is based on "Archimdedes Axiom": every set of natural numbers has a minimum. Now consider two sequences of natural numbers A_1 and A_2. From A_1, A_2 we can have two sequences $P_n(A_1)$ and $P_n(A_2)$ of numerators of the continued fraction corresponding to the sequences A_1, A_2. Since $P_n(A_1)$, $P_n(A_2)$ are two increasing sequences of natural numbers, we can then have $P_n(A_1) \oplus P_n(A_2)$. It is quite easy to prove that $\delta(P_n(A_1) \oplus P_n(A_2)) = O(\log \mathrm{n})$ because, $O(\log \mathrm{n}) + O(\log \mathrm{n}) = O(\log n)$.

But, if there are countably infinitely many sequences A_1, $A_2, \ldots, A_n, \ldots$, the density of the sequence $\oplus_{i=1}^{\infty} P_n(A_i)$ is still unknown. We make the following conjecture by observation of a few examples:

Conjecture Let A_1, $A_2, \ldots, A_n, \ldots$, be countably infinitely many sequences. Then $\delta\left(\oplus_{i=1}^{\infty} P_n(A_i)\right) = O(\log \mathrm{n})$.

Bibliography

1. Barnes, C. W. The Density of the Approximants to a Simple Continued Fraction, Journal of Academy of Science of North Carolina, 1973 p.61.

2. Bartle, Robert G. The Elements of Real Analysis, Second Edition, John Wiley & Sons, Inc., New York 1972.

3. Burton, David M. Elementary Number Theory, Second Edition, Wm C. Brown Publishers, Dubuque, IA, 1989.

4. Feferman, Solomon The Numbers System, Addison-Wesley Publishing Company, Inc., 1964.

5. Fridy, J. A. Introductory Analysis, the Theory of Calculus, Harcort Brace Jovanovich, Inc., Orlando, FL, 1987.

6. Keng, Hua Loo Introduction to Number Theory, Springer-Verlag, New York, 1982.

7. Koblitz, Neal A Course in Number Theory and Cryptography, Springer-Verlag, New York, 1987.

8. Leveque, W.J. Studies in Numbers Theory, Volume 6, The Mathematical Association of America, 1969.

9. Niven & Zuckerman An Introduction to the Theory of Numbers, Third Edition, John Wiley & Sons, Inc., New York 1972.

10. Rosen, Kenneth H. Elementary Number Theory and Its Applications, Third Edition, AT&T Bell Laboratories, Murrary Hills, New Jersey, 1993.

11. Stewart, B. M. Theory of Numbers, Second Edition, The Macmillan Company, New York, 1968.