Block Designs

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Software Compatibility Testing

Consider the following problem: Assume we have downloaded 7 new programs to our computer. We have enough memory to run at most 3 of them at a time. What is the most efficient way to test if each pair of programs is pairwise compatible with each other? Assume the simultaneous operation of any k programs $(k \leq 3)$ takes 1 minute?

Possible solutions:

Test every 3 element subset: $\binom{7}{3} = 35$ minutes

Test each pair of programs: $\binom{7}{2} = 21$ minutes

These solutions are wasteful, as they test the same sets of programs multiple times. Can a waste-free scheme be constructed?

Block Designs

A block design (S, B), or simply B, consists of a set S of v vertices and a collection B of b non-empty subsets of S.

If in a design, each block consists of k vertices, then the design is called uniform or k-uniform.

A block design is called incomplete if there exists a block in B that does not contain all vertices of S; otherwise it is complete.

If each vertex of a design occurs in exactly r blocks, then the design is called regular or r-regular.

$$\mathcal{B} = \begin{array}{c} \{A, B\} \\ \{A, C\} \\ \{A, D\} \\ \{B, C\} \\ \{B, D\} \\ \{C, D\} \end{array}$$

B has $k = 2$ vertices in each block, and each vertex occurs in $r = 3$ blocks.

Balanced Incomplete Block Designs

If in a k-uniform, r-regular incomplete design (S, B), each pair of vertices occurs together in exactly λ blocks, we say that (S, B) is a balanced incomplete block design, or BIBD, of parameters (b, v, r, k, λ).

The computer program problem from the Introduction section can be modelled by the following BIBD of parameters $(7, 7, 3, 3, 1)$, where each program corresponds to one of the vertices in $S = \{A, B, C, D, E, F, G\}$:

- Each program is tested with each other one exactly $\lambda = 1$ time.
- The block size is $k = 3$

$$\mathcal{B} = \begin{array}{c} \{A, B, D\} \\ \{B, C, E\} \\ \{A, F, C\} \\ \{A, E, G\} \\ \{B, F, G\} \\ \{C, D, G\} \\ \{D, E, F\} \end{array}$$

A block design can be represented by an incidence matrix A that has v rows that correspond to each of the vertices and b columns that correspond to each block of the design.

An entry a_{ij} can assume a value of 0 or 1: 1 if the vertex i appears in the block j, otherwise 0.

$$A = \begin{array}{cc} & \begin{array}{ccccccc} B_1 & B_2 & B_3 & B_4 & B_5 & B_6 & B_7 \end{array} \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \\ F \\ G \end{array} & \begin{bmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

Theorem (Fisher's Inequality): if a BIBD exists, then $b \geq v$, that is, every BIBD has at least as many blocks as it has vertices.

Proof idea: Let A be the incidence matrix of a BIBD. Richard Brualdi proves that the determinant of A times its transpose $\mathrm{AA}^{\mathrm{T}} = (r - \lambda)^{v-1}(r + (v - 1)\lambda)$. Since $\lambda < r$, this determinant is nonzero. Thus, AA^{T} is invertible. This implies the rank of AA^T is equal to v. So, the rank of A is at least v, and since A is a v-by-b matrix, we have $b \geq v$.

Triple Systems

Balanced block designs with block size $k = 3$ are called Steiner triple systems.

Steiner triple systems are extremely powerful; they can be constructed easily if v and λ are known:

$$r = \frac{\lambda(v - 1)}{2} \qquad b = \frac{\lambda v(v - 1)}{6}$$

A Steiner triple system with k fixed at 1 is known as a Kirkman triple system.

Kirkman triple systems were made famous by the following problem posed by Thomas Kirkman in *The Lady's and Gentleman's Diary*, a British mathematical journal from the 1850s:

VI. Query; *by the* Rev. Thos. P. Kirkman, *Croft, near Warrington.*
Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily, so that no two shall walk twice abreast.

A Kirkman triple system can easily be constructed to solve this problem with $r = 7$ and $v = 15$:

$\{0, 1, 2\}$	$\{0, 3, 4\}$	$\{0, 5, 6\}$	$\{0, 7, 8\}$
$\{3, 7, 11\}$	$\{1, 7, 9\}$	$\{1, 8, 10\}$	$\{1, 11, 13\}$
$\{4, 9, 14\}$	$\{2, 12, 13\}$	$\{2, 11, 14\}$	$\{2, 4, 5\}$
$\{5, 10, 12\}$	$\{5, 8, 14\}$	$\{3, 9, 13\}$	$\{3, 10, 14\}$
$\{6, 8, 13\}$	$\{6, 10, 11\}$	$\{4, 7, 12\}$	$\{6, 9, 12\}$

$\{0, 9, 10\}$	$\{0, 11, 12\}$	$\{0, 13, 14\}$
$\{1, 12, 14\}$	$\{1, 3, 5\}$	$\{1, 4, 6\}$
$\{2, 3, 6\}$	$\{2, 8, 9\}$	$\{2, 7, 10\}$
$\{4, 8, 11\}$	$\{4, 10, 13\}$	$\{3, 8, 12\}$
$\{5, 7, 13\}$	$\{6, 7, 10\}$	$\{5, 9, 11\}$

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