A Bayesian Meta-Analysis Using the Gibbs Sampler

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A BAYESIAN META-ANALYSIS USING THE GIBBS SAMPLER

by

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A meta-analysis is the combination of results from several similar studies, conducted by different scientists, in order to arrive at a single, overall conclusion. Unlike common experimental procedures, the data used in a meta-analysis happen to be the descriptive statistics from the distinct individual studies.

In this thesis, we will consider two regression studies performed by two scientists. These studies have one common dependent variable, $Y$, and one or more independent common variables, $X$. A regression of $Y$ on $X$ with other independent variables is carried out on both studies. We will estimate the regression coefficients of $X$ metaanalytically. After combining the two studies, we will derive a single regression model. There will be observations that one scientist witnesses and the other does not. The missing observations are considered parameters and are estimated using a method called Gibbs sampling.
Chapter 1

1.1 INTRODUCTION

In recent years, Meta-analysis has become a popular research method used to analyze several topics in education, psychology, and other medical areas. The studies performed in these fields are in some way related to each other. Meta-analysis is to combine the results from several similar studies, conducted by different scientists, in order to arrive at a single, overall conclusion.

Unlike traditional experimental procedures, the data used in Meta-analysis happen to be the summary statistics from the different individual studies. In general, the statistical combination of the outcomes of two or more independent studies is called a Meta-analysis (Ghamsary, 1997). Meta-analysis involves enormous literature reviews and comprehensive examinations of the individual results from the independent experiments.

In this thesis, we will consider two regression studies performed by two scientists. These studies have one common dependent variable, $Y$, and one or more common independent variables, $X$. A regression of $Y$ on $X$ with other independent variables is carried out on both studies. We will estimate the regression coefficients of $X$ meta-analytically. After combining the two studies, we will derive a single regression model. There will be observations that one scientist witnesses and the other does not. The missing observations are considered parameters and are estimated using a method called Gibbs sampling. This will be discussed in more detail later.
1.2 OBJECTIVE

Consider the two scientists that both observe \( Y \), the dependent variable, and \( X \), the independent variable. In addition, scientist 1 observes independent variable \( V \) and not \( W \), while scientist 2 observes independent variable \( W \) and not \( V \). There are two different cases that will be considered in this thesis. They are as follows:

1. Scientist 1 observes \( (Y, X, V) \), scientist 2 observes \( (Y, X, W) \), where \( Y, V, \) and \( W \) are continuous random variables and \( X \) is a binary random variable.

2. Scientist 1 observes \( (Y, X_1, X_2, V) \), scientist 2 observes \( (Y, X_1, X_2, W) \), where \( Y, X_1, V, \) and \( W \) are continuous random variables, and \( X_2 \) is a binary random variable.

We will use a Bayesian Meta-analysis to combine the data, run a regression model of \( Y \) on \( X, V, \) and \( W \), and estimate the regression coefficients with the Gibbs sampler.
Chapter 2

2.1 LITERATURE REVIEW

It is believed by some people that the concept of meta-analysis has been around since the early 1800's (Stigler, 1986). By the 1900's, more and more scientists became interested in combining results from individual studies. It was not until 1976 that Gene Glass gave this topic a name: Meta-analysis.

Glass recommended that the use of “effect sizes” to compare the different studies. He defined effect size to be a standardization of the treatment effect minus the control effect. For more on the classical methods of meta-analysis, see the following useful books: Hunter et al. (1982), Glass et al. (1981), and Hedges and Olkin (1985).

Gilbert, McPeek, and Mosteller (1977) were the first few to suggest that a Bayesian approach was appropriate for meta-analysis. This same approach was used by Raudenbush and Bryk (1992), Morris and Normand (1992), and Louis and Zelterman (1994). DuMouchel (1994) used several hierarchical Bayesian methods to combine the effects.

Lindley and Press (1995) used a linear regression model in their Bayesian meta-analysis. Mahmood Ghamsary’s dissertation was based in part on the techniques of Lindley and Press (1995). Lindley and Press used backward regression to estimate the missing data, while Ghamsary used the Gibbs sampler. This thesis is based on Ghamsary’s methods and procedures.
2.2 BAYESIAN STATISTICS

Bayesian statistics take into account any prior knowledge of the experiment that the scientist has. Our two scientists have observed the data $y_1$ and $y_2$ from $L(y_1, y_2|\tau)$, and both are interested in a common parameter, $\tau$. Bayes' theorem for continuous distributions is the following:

The posterior density for $\tau$ is given by

$$p(\tau|y_1, y_2) = \frac{L(y_1, y_2|\tau)p(\tau)}{\int L(y_1, y_2|\tau)p(\tau)d\tau},$$

(1.1)

where $L$ is the likelihood function and $p(\tau)$ is the prior density of $\tau$.

Thus, posterior density $\propto$ likelihood $\times$ prior density,

$$p(\tau|y_1, y_2) \propto L(y_1, y_2|\tau)p(\tau)$$

(1.2)

or

$$p(\tau|y_1, y_2) \propto L(y_2|\tau)[L(y_1|\tau)p(\tau)],$$

(1.3)

since $y_1$ and $y_2$ are independent.

If we take $L(y_1|\tau)p(\tau)$ as the kernel of the posterior density for the first scientist, then this can be thought of as the prior density for the second scientist (Ghamsary, 1997).

In Mahmood Ghamsary's dissertation on Bayesian Meta-analysis (1997), he states the following on the Bayesian approach to this situation:
... We know that the likelihood function relates the distribution of the sample statistics to one or more unknown parameters.... a parametric statistical model is constructed to relate the parameters from the separate studies to each other.... when we combine the two scientists' data we will have some missing observations. The missing observations are considered as additional parameters. These make sense in a Bayesian formulation, since the parameters are not just fixed unknown constants, but they are random variables. Furthermore in Bayesian computations, the parameters of interest are estimated from a posterior distribution; which is a conditional probability based on the given data. This is why the Bayesian approach is being used to estimate missing observations, since the posteriors of the missing observations are conditioned on other scientists' data.

The missing observations, parameters, are estimated using Gibbs sampling. To understand the Gibbs sampling technique, we will take a look at one of our scientists. Suppose scientist 1 observes $x_i$ but not $v_i$, and let $\{y_i = (x_i, v_i), i=1,..,n\}$. Using the observed data, we want to know more about $\tau$. We can use the following:

$$p(\tau \mid X) = \int p(\tau \mid X, V) p(V \mid X) dV,$$  \hspace{1cm} (1.4)

where $X=(x_1,..,x_n)$ and $V=(v_1,..,v_n)$. Assuming we know the distribution of $V$, we can get samples $v^{(1)}, .., v^{(m)}$ from $p(V \mid X)$. According to Ghamsary (1997), we can approximate the posterior distribution of $\tau$ by

$$p(\tau \mid X) \approx \frac{1}{m} [p(\tau \mid X, v^{(1)}) + ... + p(\tau \mid X, v^{(m)})].$$  \hspace{1cm} (1.5)
2.3 THE GIBBS SAMPLER

Gibbs sampling has become increasingly more popular due to its simplicity and adaptability. The following is according to Ghamsary (1997):

The Gibbs sampler technique easily deals with this missing data problem. However, the objective of our analysis is to estimate the common coefficient(s) meta-analytically through Gibbs sampling. Missing data are regarded as additional parameters, to be updated at each iteration of the Gibbs sampler in the same way as all the other parameters. Thus the Gibbs sampler can be thought of here as imputing a missing value for each missing observation at each iteration, and then updating the original parameters in the usual way, conditioning on the imputed values.

The algorithm for the Gibbs sampler is very straightforward. Tanner (1993) describes the process as follows:

Given the starting point \( (\theta_1^{(0)}, \theta_2^{(0)}, \ldots, \theta_d^{(0)}) \), this algorithm iterates the following loop:

a. Sample \( \theta_1^{(i+1)} \) from \( p(\theta_1 | \theta_2^{(i)}, \ldots, \theta_d^{(i)}, Y) \)

b. Sample \( \theta_2^{(i+1)} \) from \( p(\theta_2 | \theta_1^{(i+1)}, \theta_3^{(i)}, \ldots, \theta_d^{(i)}, Y) \)

\[ \vdots \]

\[ \vdots \]

d. Sample \( \theta_d^{(i+1)} \) from \( p(\theta_d | \theta_1^{(i+1)}, \ldots, \theta_{d-1}^{(i+1)}, Y) \), where \( i=0,1,2,\ldots \) and \( \theta_j^{(i)} \) denotes the \( j^{th} \) parameter on the \( i^{th} \) iteration.
The vectors $\theta^{(0)}, \theta^{(1)}, ..., \theta^{(i)}, ...$ are a realization of a Markov chain, where $\theta^{(i)} = (\theta_1^{(i)}, \theta_2^{(i)}, ...).$ According to Tanner (1993), two results follow from the Gibbs sampler:

1. The joint distribution of $(\theta_1^{(i)}, ..., \theta_d^{(i)})$ converges geometrically to $p(\theta_1, ..., \theta_d | y)$, as $i \to \infty$.

2. $\frac{1}{t} \sum_{i=1}^{t} f(\theta_i) \xrightarrow{a.s.} \int f(\theta) p(\theta | y) d\theta(\theta)$, as $t \to \infty$.

For assessing convergence, refer to page 93 of Ghamsary (1997).
3.1 TWO SCIENTISTS WITH ONE COMMON BINARY VARIABLE

In this segment, we will analyze the situation in which scientist 1 and scientist 2 both observe a continuous variable $Y$ and a binary variable $X$. In addition, scientist 1 observes continuous variable $W$, and scientist 2 observes continuous variable $V$. Scientist 1 observes $(Y, X, W)$ $n_1$ times and runs a conditional regression $Y$ on $(X, W)$. Scientist 2 observes $(Y, X, V)$ $n_2$ times and runs a conditional regression $Y$ on $(X, V)$. The data of size $n_1 + n_2$ $(Y, X, W, V)$ is obtained by combining the two scientists' data. However, scientist 1 did not observe $V$ and scientist 2 did not observe $W$. We will call these missing observations $\tilde{v}$ and $\tilde{w}$, respectively. These missing values are considered parameters that need to be estimated in the Gibbs sampler. Thus, scientist 1 has the data $[(Y_1)_{n_1 \times 1}, (X_1)_{n_1 \times 1}, (W)_{n_1 \times 1}, (\tilde{V})_{n_1 \times 1}]_{n_1 \times 4}$ and scientist 2 has the data $[(Y_2)_{n_2 \times 1}, (X_2)_{n_2 \times 1}, (\tilde{W})_{n_2 \times 1}, (V)_{n_2 \times 1}]_{n_2 \times 4}$. Consider now running a regression on the above data.

Model

We will use Ghamsary’s (1997) Bayesian meta-analysis regression model in which the missing observations are estimated using the Gibbs sampler instead of the backwards regression used by Lindley & Press (1995). The assumption is made that the regressors $v$ and $w$ are jointly normal with mean vector $\mu_{2 \times 1}$ and variance-covariance matrix $\Sigma_{2 \times 2}$. 


It is assumed that the vector of observations can be written as follows:

\[
\begin{pmatrix}
Y_{n,1} \\
Y_{n,2}
\end{pmatrix}
| U, \gamma, \sigma^2 \rangle = U_{n \times 4} Y_{4 \times 1} + \gamma_{n,1},
\]

(3.1)

where \( U, \gamma, \) and error terms \( u_i \) are defined in (3.2).

\[
U_{n \times 4} = \begin{bmatrix}
e_{n,1} & x_1 & w & \bar{y} \\ e_{n,2} & x_2 & \bar{w} & v
\end{bmatrix}, \quad
y_{n,1} = \begin{bmatrix}
y_1 \\
y_2
\end{bmatrix}, \quad
\gamma_{n,1} = \begin{bmatrix}
1 \\
1
\end{bmatrix}, \quad
\gamma_{i,1} = \begin{bmatrix}
\gamma_1 \\
\gamma_2 \\
\gamma_3 \\
\gamma_4
\end{bmatrix},
\]

\( n = n_1 + n_2, \) and \( \gamma_{n,1} \sim N_2(0_{2 \times 1}, \sigma^2 I_2) \)

(3.2)

**Likelihood**

Since all \( u_i \sim N(0, \sigma^2) \), we have that \( \gamma \sim N(\gamma, \sigma^2 I) \) from equation (3.1). Thus, the joint probability density function for \( \gamma \), given \( U, \gamma, \) and \( \sigma^2 \), is

\[
p(\gamma|U, \gamma, \sigma^2) \propto \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2\sigma^2} (\gamma - U\gamma)'(\gamma - U\gamma) \right\},
\]

or if \( A = (U\gamma)'(\gamma - U\gamma) \),

\[
p(\gamma|U, \gamma, \sigma^2) \propto \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{ -\frac{1}{2\sigma^2} A \right\}.
\]

(3.3)
Now, $U$ can be rewritten as the following:

$$U = \begin{pmatrix} 
    e_{n_1} & x_1 & w & 0 \\
    e_{n_2} & x_2 & 0 & v 
\end{pmatrix} + \begin{pmatrix} 
    0 & 0 & 0 & \tilde{v} \\
    0 & 0 & \tilde{w} & 0 
\end{pmatrix} = U_{\text{observed}} + U_{\text{missing}},$$

where $U_{\text{observed}} = \begin{pmatrix} 
    e_{n_1} & x_1 & w & 0 \\
    e_{n_2} & x_2 & 0 & v 
\end{pmatrix}$ is the observed part of the design matrix, and $U_{\text{missing}} = \begin{pmatrix} 
    0 & 0 & 0 & \tilde{v} \\
    0 & 0 & \tilde{w} & 0 
\end{pmatrix}$ is the missing part of the design matrix.

Substituting $U$ into $A$, we obtain

$$A = \left[ y - (U_{\text{obs}} + U_{\text{mis}}) \gamma \right] \left[ y - (U_{\text{obs}} + U_{\text{mis}}) \gamma \right].$$

Rearranging the above product produces the following:

$$A = (y - U_{\text{obs}} \gamma)'(y - U_{\text{obs}} \gamma) + [\gamma' U_{\text{mis}}' U_{\text{mis}} \gamma - 2(y - U_{\text{obs}} \gamma)' U_{\text{mis}} \gamma].$$

Now, let $g(\tilde{v}, \tilde{w}) = [\gamma' U_{\text{mis}}' U_{\text{mis}} \gamma - 2(y - U_{\text{obs}} \gamma)' U_{\text{mis}} \gamma]$ (3.4)

and $A_1 = (y - U_{\text{obs}} \gamma)'(y - U_{\text{obs}} \gamma)$,

so

$$A = (y - U_{\text{obs}} \gamma)'(y - U_{\text{obs}} \gamma) + g(\tilde{v}, \tilde{w}) = A_1 + g(\tilde{v}, \tilde{w}).$$

Note that $g(\tilde{v}, \tilde{w})$ is a quadratic equation in terms of $\tilde{v}$ and $\tilde{w}$.
Also note that

\[
U_{\text{mis}Y} = \begin{pmatrix}
0 & 0 & 0 & \tilde{\nu} \\
0 & 0 & \tilde{\omega} & 0 \\
\hat{\gamma}_1 & \hat{\gamma}_2 & \hat{\gamma}_3 & \hat{\gamma}_4
\end{pmatrix}
= \begin{pmatrix}
\tilde{\nu} \\
\tilde{\omega} \\
\tilde{\gamma}_4
\end{pmatrix},
\]

\[
U_{\text{obs}Y} = \begin{pmatrix}
e_n & x_1 & w & 0 \\
e_n & x_2 & 0 & v \\
\hat{\gamma}_1 & \hat{\gamma}_2 & \hat{\gamma}_3 & \hat{\gamma}_4
\end{pmatrix}
= \begin{pmatrix}
\gamma_1 e_n + x_1 \gamma_2 + w \gamma_3 \\
\gamma_1 e_n + x_2 \gamma_2 + v \gamma_4
\end{pmatrix}.
\]

Thus,

\[
g(\tilde{\nu}, \tilde{\omega}) = \left[ (U_{\text{mis}Y})' U_{\text{mis}Y} - 2(y - U_{\text{obs}Y})' U_{\text{mis}Y} \right],
\]

where \((U_{\text{mis}Y})' (U_{\text{mis}Y}) = \begin{pmatrix}
\tilde{\nu} \\
\tilde{\omega} \\
\tilde{\gamma}_4
\end{pmatrix} \begin{pmatrix}
\tilde{\nu} \\
\tilde{\omega} \\
\tilde{\gamma}_4
\end{pmatrix} = \gamma_4 \tilde{\nu}^2 + \gamma_3 \tilde{\omega}^2 + \gamma_4 \tilde{\gamma}_4^2,
\]

or \((U_{\text{mis}Y})' (U_{\text{mis}Y}) = \sum_{i=1}^{n_1} \gamma_4^2 \tilde{\nu}_i^2 + \sum_{j=1}^{n_2} \gamma_3^2 \tilde{\omega}_j^2.
\]

Let us define the following:

\[
K_1 = y_1 - (\gamma_1 e_n + x_1 \gamma_2 + w \gamma_3) \quad \text{and} \quad K_2 = y_2 - (\gamma_1 e_n + x_2 \gamma_2 + v \gamma_4).
\]

Therefore,

\[
-2(y - U_{\text{obs}Y})' (U_{\text{mis}Y}) = -2(y - U_{\text{obs}Y})' \begin{pmatrix}
\tilde{\nu} \\
\tilde{\omega} \\
\tilde{\gamma}_4
\end{pmatrix}
= -2 \left( y_1 - (\gamma_1 e_n + x_1 \gamma_2 + w \gamma_3) \right) \begin{pmatrix}
\tilde{\nu} \\
\tilde{\omega} \\
\tilde{\gamma}_4
\end{pmatrix} = -2 K_1 \tilde{\nu} - 2 K_2 \tilde{\omega}_3 =
\]

\[
-2 \gamma_4 \tilde{\nu} K_1 - 2 \gamma_3 \tilde{\omega} K_2 = -2 \sum_{i=1}^{n_1} k_i \gamma_4 \tilde{\nu}_i - 2 \sum_{j=1}^{n_2} k_3 \gamma_4 \tilde{\omega}_j,
\]
where \( k_{li} \) is the \( i \)th row of \( K_1 \), and \( k_{lj} \) is the \( j \)th row of \( K_2 \). Therefore, after substituting the above into equation (3.4) we obtain the following:

\[
g(\tilde{v}, \tilde{w}) = \sum_{i=1}^{n_1} \gamma_4 \tilde{v}_i^2 + \sum_{j=1}^{n_2} \gamma_3 \tilde{w}_j^2 - 2 \sum_{i=1}^{n_1} k_{li} \gamma_4 \tilde{v}_i - 2 \sum_{j=1}^{n_2} k_{lj} \gamma_3 \tilde{w}_j \quad (3.5)
\]

Priors

Assuming that \( \gamma, \sigma^2, \tilde{v} \), and \( \tilde{w} \) are independent, we know that

\[
p(\gamma, \sigma^2, \tilde{v}, \tilde{w}) = p(\gamma) p(\sigma^2) p(\tilde{v}) p(\tilde{w}). \quad (3.6)
\]

If we use non-informative, or vague, priors for \( \gamma \) and \( \sigma^2 \), we obtain

\[
p(\gamma) p(\sigma^2) \propto \frac{1}{\sigma^2}. \quad (3.7)
\]

Observe that \( \tilde{v}_i \) are the rows of \( \tilde{v} \) and \( \tilde{w}_j \) are the rows of \( \tilde{w} \). Also, if we assume that \( \tilde{v}_i \sim i.i.d. N(\mu_1, \sigma_1^2) \) and \( \tilde{w}_j \sim i.i.d. N(\mu_2, \sigma_2^2) \) where \( \mu_1, \mu_2, \sigma_1^2, \) and \( \sigma_2^2 \) are known, then let us define \( \Omega = (\mu_1, \mu_2, \sigma_1^2, \sigma_2^2) \). Therefore,

\[
p(\tilde{v}|\mu_1, \sigma_1) = \prod_{i=1}^{n_1} p(\tilde{v}_i|\mu_1, \sigma_1) \propto \frac{1}{\sigma_1^{n_1/2}} \exp\left\{ -\frac{1}{2\sigma_1} \sum_{i=1}^{n_1} (\tilde{v}_i - \mu_1)^2 \right\}
\]

and

\[
p(\tilde{w}|\mu_2, \sigma_2) = \prod_{j=1}^{n_2} p(\tilde{w}_j|\mu_2, \sigma_2) \propto \frac{1}{\sigma_2^{n_2/2}} \exp\left\{ -\frac{1}{2\sigma_2} \sum_{j=1}^{n_2} (\tilde{w}_j - \mu_2)^2 \right\}.
\]

Consequently, replacing parts of equation (3.6) gives us the following:
Joint Posterior

We can use Bayes' theorem to say

\[
p(\gamma, \sigma^2, \tilde{v}, \tilde{w}) \propto \frac{1}{\sigma^2} \times \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n_v} (\tilde{v}_i - \mu_1)^2 \right\} \times \exp \left\{ -\frac{1}{2\sigma^2} \sum_{j=1}^{n_w} (\tilde{w}_j - \mu_2)^2 \right\}. \tag{3.8}
\]

After substituting (3.5) and (3.8) into (3.9), we get the joint posterior

\[
p(\tilde{v}, \tilde{w}, \gamma, \sigma^2 | y, U, \Omega) \propto p(y | \tilde{v}, \tilde{w}, \gamma, \sigma^2, U, \Omega) p(\tilde{v}, \tilde{w}, \gamma, \sigma^2 | \Omega) \tag{3.9}
\]

Equation (3.10) then becomes the following after a little rearranging:

\[
p(\tilde{v}, \tilde{w}, \gamma, \sigma^2 | y, U, \Omega) \propto \frac{1}{(\sigma^2)^{n+1}} \exp \left\{ -\frac{1}{2\sigma^2} \left[ (y - U_{\text{obs}}) (y - U_{\text{obs}}) + g(\tilde{v}, \tilde{w}) \right] \right\} \times \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n_v} (\tilde{v}_i - \mu_1)^2 \right\} \times \exp \left\{ -\frac{1}{2\sigma^2} \sum_{j=1}^{n_w} (\tilde{w}_j - \mu_2)^2 \right\}. \tag{3.11}
\]
Now, let \[ \tilde{A} = \left\{ \frac{-1}{2\sigma_1^2} \sum_{i=1}^{n} (\tilde{v}_i - \mu_1)^2 \right\} + \left\{ \frac{-1}{2\sigma_2^2} \sum_{j=1}^{n} (\tilde{w}_j - \mu_2)^2 \right\} + \frac{-1}{2\sigma^2} g(\tilde{v}, \tilde{w}). \]

This equation has all of the missing data that was in the joint posterior. Substituting \( \tilde{A} \) into (3.11) obtains the following:

\[
p(v, \tilde{w}, y, U, \Omega) \propto \frac{1}{(\sigma^2)^2} \exp \left\{ -\frac{1}{2\sigma^2} \left[ (y - U_{\text{obs}} y) (y - U_{\text{obs}} y) + \tilde{A} \right] \right\} .
\]

Now, \( \tilde{A} \) can be rearranged by combining like terms. Thus,

\[
\tilde{A} = \sum_{i=1}^{n} \left[ \left( \frac{\gamma_i}{\sigma_1^2} + \frac{1}{\sigma_1^2} \right) \tilde{v}_i^2 - 2 \left( \frac{k_i Y_{i4}}{\sigma_2^2} + \frac{\mu_1}{\sigma_1^2} \right) \tilde{v}_i \right] \\
+ \sum_{j=1}^{n} \left[ \left( \frac{\gamma_j}{\sigma_2^2} + \frac{1}{\sigma_2^2} \right) \tilde{w}_j^2 - 2 \left( \frac{k_j Y_{j3}}{\sigma_2^2} + \frac{\mu_2}{\sigma_2^2} \right) \tilde{w}_j \right]. \quad (3.12)
\]

Completing the square gives us

\[
\tilde{A} = \sum_{i=1}^{n} \left[ \frac{(\tilde{v}_i - v^*)^2}{\sigma_1^\ast} \right] + \sum_{j=1}^{n} \left[ \frac{(\tilde{w}_j - w^*)^2}{\sigma_2^\ast} \right] + \text{constant},
\]

where \( v_i^* = (\sigma_1^\ast)^2 \left( \frac{k_i Y_{i4}}{\sigma_2^2} + \frac{\mu_1}{\sigma_1^2} \right) \) and \( \sigma_1^\ast = \left( \frac{\gamma_i}{\sigma_2^2} + \frac{1}{\sigma_1^2} \right)^{-1} \), and
where \( w_j^* = (\sigma_1^*)^2 \left( \frac{k_j/\gamma_3 + \mu_j^2}{\sigma^2} \right) \) and \( \sigma_2^* = \left( \frac{\gamma^2 + \frac{1}{\sigma^2}}{\sigma^2} \right)^{-1} \). Consequently, the joint posterior density is now the following:

\[
p(\tilde{\nu}, \tilde{w}, \gamma, \sigma^2 | y, U, \Omega) \propto \frac{1}{(\sigma^2)^{n/2}} \exp \left\{ -\frac{1}{2\sigma^2} \left[ (y - U_{\text{obs}}) (y - U_{\text{obs}}^*) \right] \right\} \\
\times \exp \left\{ -\frac{1}{2\sigma_1^2} \sum_{i=1}^{n_1} (\tilde{v}_i - v_i^*)^2 \right\} \times \exp \left\{ -\frac{1}{2\sigma_2^2} \sum_{j=1}^{n_2} (\tilde{w}_j - w_j^*)^2 \right\} \quad (3.13)
\]

**Conditional Posteriors**

Because \( y, U, \) and \( \Omega \) are fixed and known, the joint posterior density of our parameters given the latter three elements (i.e. \( p(\tilde{\nu}, \tilde{w}, \gamma, \sigma^2 | y, U, \Omega) \)) is considered a conditional density under Bayesian ideology (Ghamsary, 1997). Hence,

\[
p(\gamma | \tilde{\nu}, \tilde{w}, \sigma^2, y, U, \Omega) = \frac{p(\gamma, \tilde{\nu}, \tilde{w}, \sigma^2 | y, U, \Omega)}{\int_{\gamma} p(\gamma, \tilde{\nu}, \tilde{w}, \sigma^2 | y, U, \Omega) d\gamma}, \text{ which implies}
\]

\[
p(\gamma | \tilde{\nu}, \tilde{w}, \sigma^2, y, U, \Omega) \propto p(\gamma, \tilde{\nu}, \tilde{w}, \sigma^2 | y, U, \Omega) \quad \text{since the denominator is a constant. Conditional densities for each unknown parameter can be figured out by this approach. We will use these conditional densities in the Gibbs sampler to}
\]

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estimate the missing data. Thus, there are \( (n_1 + n_2 + 4 + 1) \) unknowns (i.e. \( \tilde{v}_{n_1}, \tilde{w}_{n_2}, Y_{4x1}, \sigma^2 \)). So the conditional posterior density of \( \tilde{v} \) is given below:

\[
p(\tilde{v} | y, \sigma^2, y, U_{obs}, \Omega) = \frac{p(y, \tilde{v}, \tilde{w}, \sigma^2 | y, U_{obs}, \Omega)}{\int_{\mathbb{R}^n} p(y, \tilde{v}, \tilde{w}, \sigma^2 | y, U_{obs}, \Omega) d\tilde{v}}.
\]

However, since \( \tilde{v} \) and \( \tilde{w} \) are conditionally independent, we get that

\[
p(\tilde{v} | y, \sigma^2, y, U_{obs}, \Omega) = \frac{p(y, \tilde{v}, \sigma^2 | y, U_{obs}, \Omega)}{\int_{\mathbb{R}^n} p(y, \tilde{v}, \sigma^2 | y, U_{obs}, \Omega) d\tilde{v}}.
\]

This becomes the following after integration and simplification:

\[
p(\tilde{v} | y, \sigma^2, y, U_{obs}, \Omega) \propto \exp\left\{-\frac{1}{2\sigma^2} \sum_{i=1}^{n_1} (\tilde{v}_i - v_i^*)^2\right\}.
\]

Hence, \( p(\tilde{v} | y, \sigma^2, y, U_{obs}, \Omega) \) is multivariate normal with mean vector \( v^* \) and covariance matrix \((\sigma^2_1)^2 I_{n_1}\), or \( (\tilde{v} | y, \sigma^2, y, U_{obs}, \Omega) \overset{i.i.d.}{\sim} N(v^*_i, \sigma^2_1)\), \( \forall i = 1, \ldots, n_1 \).

The conditional posterior density function of \( \tilde{w} \) is also multivariate normal;

\[
p(\tilde{w} | y, U_{obs}, \gamma, \sigma^2, \Omega) \propto \exp\left\{-\frac{1}{2\sigma^2} \sum_{j=1}^{n_2} (\tilde{w}_j - w_j^*)^2\right\}.
\]

Thus, \( (\tilde{w} | y, \sigma^2, y, U_{obs}, \Omega) \overset{i.i.d.}{\sim} N(w_j^*, \sigma^{*2}_2), \forall j = 1, \ldots, n_2 \). Now we will find the conditional density of \( \sigma^2 \).
Note that

\[ p(\sigma^2 | \tilde{v}, \tilde{w}, \gamma, y, U_{obs}, \Omega) = \frac{p(\gamma, \tilde{v}, \tilde{w}, \sigma^2 | y, U_{obs}, \Omega)}{\int_0^{\sigma^2} p(\gamma, \tilde{v}, \tilde{w}, \sigma^2 | y, U_{obs}, \Omega) d\sigma^2}. \]

After substituting (3.10) into the above and integrating, we get

\[ p(\sigma^2 | \tilde{v}, \tilde{w}, \gamma, y, U_{obs}, \Omega) \propto \frac{1}{(\sigma^2)^{\frac{z+1}{2}}} \exp\left\{ -\frac{1}{2\sigma^2} (y - U_\gamma)'(y - U_\gamma) \right\}, \]

or

\[ p(\sigma^2 | \tilde{v}, \tilde{w}, \gamma, y, U_{obs}, \Omega) \propto \frac{1}{(\sigma^2)^{\frac{z+1}{2}}} \exp\left\{ -\frac{1}{2\sigma^2} A \right\}, \]

where

\[ A = (y - U_\gamma)'(y - U_\gamma). \]

Hence, \( (\sigma^2 | \tilde{v}, \tilde{w}, \gamma, y, U_{obs}, \Omega) \) is distributed as an inverted gamma with the parameters \( \frac{n}{2} \) and \( \frac{A}{2} \); \( \frac{1}{\sigma^2} \) follows a gamma distribution. The conditional density for \( \gamma \) is given below:

\[ p(\gamma | \tilde{v}, \tilde{w}, \sigma^2, y, U_{obs}, \Omega) = \frac{p(\gamma, \tilde{v}, \tilde{w}, \sigma^2 | y, U_{obs}, \Omega)}{\int_{\gamma} p(\gamma, \tilde{v}, \tilde{w}, \sigma^2 | y, U_{obs}, \Omega) d\gamma}. \] (3.14)
Remember the following from (3.11):

\[ p(\tilde{v}, \tilde{w}, \gamma, \sigma^2 | y, U, \Omega) \propto \]

\[ \frac{1}{(\sigma^2)^n} \exp \left\{ -\frac{1}{2\sigma^2} \left[ (y - U\hat{\gamma})'(y - U\hat{\gamma}) \right] \right\} \]

\[ \times \exp \left\{ -\frac{1}{2\sigma_1^2} \sum_{i=1}^{n_1} (\tilde{v}_i - \tilde{v}_i^*)^2 \right\} \times \exp \left\{ -\frac{1}{2\sigma_2^2} \sum_{j=1}^{n_2} (\tilde{w}_j - \tilde{w}_j^*)^2 \right\}. \]

Substituting the above into equation (3.14) and integrating, we obtain

\[ p(y|\tilde{v}, \tilde{w}, \sigma^2, y, U_{obs}, \Omega) \propto \frac{1}{(\sigma^2)^n} \exp \left\{ -\frac{1}{2\sigma^2} \left[ (y - U\hat{\gamma})'(y - U\hat{\gamma}) \right] \right\}. \]

We can rewrite \((y - U\hat{\gamma})'(y - U\hat{\gamma})\) as \([y - U\hat{\gamma} - U(\gamma - \hat{\gamma})]'[y - U\hat{\gamma} - U(\gamma - \hat{\gamma})]\),

Which equals \((y - U\hat{\gamma})'(y - U\hat{\gamma}) + (\gamma - \hat{\gamma})'(U'U)(\gamma - \hat{\gamma})\), where \(\hat{\gamma} = (U'U)^{-1}U'y\) is the least-squares estimate [see, e.g. Ghamsary (1997, p.30)].

Now we can rearrange and substitute in the aforementioned conditional density and acquire the subsequent proportion:

\[ p(y|\tilde{v}, \tilde{w}, \sigma^2, y, U_{obs}, \Omega) \propto \exp \left\{ -\frac{1}{2\sigma^2} (y - \tilde{\gamma})'(U'U)(y - \tilde{\gamma}) \right\}. \]

Thus, \((y|\tilde{v}, \tilde{w}, \sigma^2, y, U_{obs}, \Omega)\) is multivariate normal with mean vector \(\tilde{\gamma} = (U'U)^{-1}U'y\) and covariance matrix \(\sigma^2(U'U)^{-1}\).
These full conditional distributions for each parameter are needed to implement the Gibbs sampler. The full conditional distributions are given below:

1. \( (\gamma | \tilde{v}, \tilde{w}, \sigma^2, y, U, \Omega) \sim N((U'U)^{-1}U'U, y, \sigma^2 (U'U)^{-1}) \) \hfill (3.15)

2. \( (\sigma^2 | \tilde{v}, \tilde{w}, \gamma, U, y, \Omega) \sim IG \left( \frac{n}{2}, \frac{1}{2} (y - U\gamma)'(y - U\gamma) \right) \) \hfill (3.16)

3. \( (\tilde{v}_i | y, \gamma, \sigma^2, U_{obs}, \Omega) \sim N \left( \nu_i^*, (\sigma_i^*)^2 \right), \forall i = 1, 2, ..., n_1 \) \hfill (3.17)

4. \( (\tilde{w}_j | y, \gamma, \sigma^2, U_{obs}, \Omega) \sim N \left( w_j^*, (\sigma_j^*)^2 \right), \forall j = 1, 2, ..., n_2 \). \hfill (3.18)
3.2 TWO SCIENTISTS WITH ONE COMMON BINARY VARIABLE AND ONE COMMON CONTINUOUS VARIABLE

In this section we will analyze the situation in which scientist 1 and scientist 2 both observe a continuous variable $Y$, a binary variable $X_1$, and a continuous variable $X_2$. In addition, scientist 1 observes continuous variable $W$, and scientist 2 observes continuous variable $V$. Scientist 1 observes $(Y, X_1, X_2, W)$ $n_1$ times and runs a conditional regression $Y$ on $(X_1, X_2, W)$. Scientist 2 observes $(Y, X_1, X_2, V)$ $n_2$ times and runs a conditional regression $Y$ on $(X_1, X_2, V)$. The data of size $n_1 + n_2$ $(Y, X_1, X_2, W, V)$ is obtained by combining the two scientists' data. Like section 3.1, scientist 1 did not observe $V$ and scientist 2 did not observe $W$. We will again call these missing observations $\tilde{v}$ and $\tilde{w}$, respectively. Therefore, scientist 1 has the data $\{(y_1)_{n_1}, (x_{11})_{n_1}, (x_{21})_{n_1}, (w)_{n_1}, (\tilde{v})_{n_1}\}_{n_1}^{n_2}$ and scientist 2 has the data $\{(y_2)_{n_2}, (x_{12})_{n_2}, (x_{22})_{n_2}, (\tilde{w})_{n_2}, (v)_{n_2}\}_{n_2}^{n_2}$. Consider, as before, running a regression on the above data.

Model

The model is the same as in the previous section. The assumption is made that the regressors $v$ and $w$ are jointly normal with mean vector $\mu_{2 \times 1}$ and variance-covariance matrix $\Sigma_{2 \times 2}$. Again, it is assumed that the vector of observations can be written as

\[
\begin{pmatrix}
y_{n \times 1} \\
\end{pmatrix} = \begin{pmatrix}
U_{n \times 5} \\
\end{pmatrix} + \begin{pmatrix}
u_{n \times 1} \\
\end{pmatrix}, \quad (3.19)
\]

where $U$, $\gamma$, and error terms $u_i$ are defined in (3.20). In this section $X_1$ is assumed to be normal, while $X_2$ is the binary variable.
Let $U_{n \times 5} = \begin{bmatrix} e_n & x_{11} & x_{21} & w & \tilde{v} \\ e_n & x_{12} & x_{22} & \tilde{w} & v \end{bmatrix}$, $y_{n \times 1} = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, $e_{n \times 1} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $i = 1, 2$, $y_{5 \times 1} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \\ y_5 \end{bmatrix}$.

$n = n_1 + n_2$, and $u \sim N(0, \sigma^2 I)$.

(3.20)

Likelihood

Since all $u_i \sim N(0, \sigma^2)$, we have that $y \sim N(U\gamma, \sigma^2 I_n)$ from equation (3.19). Thus, the joint probability density function for $\gamma$, given $U$, $y$, and $\sigma^2$, is

$$p(y|U, \gamma, \sigma^2) \propto \frac{1}{(\sigma^2)^{n/2}} \exp\left\{ -\frac{1}{2\sigma^2} (y - U\gamma)'(y - U\gamma) \right\},$$

or if $A = (y - U\gamma)'(y - U\gamma)$,

$$p(y|U, \gamma, \sigma^2) \propto \frac{1}{(\sigma^2)^{n/2}} \exp\left\{ -\frac{1}{2\sigma^2} A \right\}. \tag{3.21}$$

Now, $U$ can be rewritten as the following:

$$U = \begin{pmatrix} e_n & x_{11} & x_{21} & w & 0 \\ e_n & x_{12} & x_{22} & 0 & v \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 & \tilde{v} \\ 0 & 0 & 0 & \tilde{w} & 0 \end{pmatrix} = U_{\text{observed}} + U_{\text{missing}},$$

where $U_{\text{observed}} = \begin{pmatrix} e_n & x_{11} & x_{21} & w & 0 \\ e_n & x_{12} & x_{22} & 0 & v \end{pmatrix}$ is the observed part of the design matrix,
and \( U_{mis} = \begin{pmatrix} 0 & 0 & 0 & 0 & \tilde{v} \\ 0 & 0 & 0 & \tilde{w} & 0 \end{pmatrix} \) is the missing part of the design matrix.

Substituting \( U \) into \( A \), we obtain 
\[
A = \left[ y - (U_{obs} + U_{mis}) \right] \left[ y - (U_{obs} + U_{mis}) \right]'
\]

Rearranging the above product produces the following:
\[
A = (y - U_{obs} \gamma)'(y - U_{obs} \gamma) + \left[ y' U_{mis} \gamma - 2(y - U_{obs} \gamma)' U_{mis} \gamma \right]
\]

Now, let \( g(\tilde{v}, \tilde{w}) = \left[ y' U_{mis} \gamma - 2(y - U_{obs} \gamma)' U_{mis} \gamma \right] \)

and \( A_i = (y - U_{obs} \gamma)'(y - U_{obs} \gamma) \)

so \( A = (y - U_{obs} \gamma)'(y - U_{obs} \gamma) + g(\tilde{v}, \tilde{w}) = A_i + g(\tilde{v}, \tilde{w}) \).

Also note that \( g(\tilde{v}, \tilde{w}) \) is a quadratic equation in terms of \( \tilde{v} \) and \( \tilde{w} \). Note that
\[
U_{mis} \gamma = \begin{pmatrix} 0 & 0 & 0 & 0 & \tilde{v} \\ 0 & 0 & 0 & \tilde{w} & 0 \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \end{pmatrix} = \begin{pmatrix} \tilde{v} \gamma_5 \\ \tilde{w} \gamma_4 \end{pmatrix}
\]

and
\[
U_{mis} \gamma = \begin{pmatrix} e_1 & x_{11} & x_{21} & w & 0 \\ e_2 & x_{12} & x_{22} & 0 & v \end{pmatrix} \begin{pmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \\ \gamma_4 \\ \gamma_5 \end{pmatrix} = \begin{pmatrix} \gamma_1 e_1 + x_{11} \gamma_2 + x_{12} \gamma_3 + w \gamma_4 \\ \gamma_1 e_2 + x_{21} \gamma_2 + x_{22} \gamma_3 + v \gamma_5 \end{pmatrix}
\]

Thus,
\[
g(\tilde{v}, \tilde{w}) = \left[ (U_{mis} \gamma)' U_{mis} \gamma - 2(y - U_{obs} \gamma)' U_{mis} \gamma \right]
\]
where \((U_{mis\gamma})'(U_{mis\gamma}) = \begin{pmatrix} \bar{\nu}_5 \\ \bar{\nu}_4 \end{pmatrix}' \begin{pmatrix} \bar{\nu}_5 \\ \bar{\nu}_4 \end{pmatrix} = \gamma_4 \bar{\nu}_5 \bar{\nu}_4 + \gamma_4 \bar{\nu}_5 \bar{\nu}_4\),

or \((U_{mis\gamma})'(U_{mis\gamma}) = \sum_{i=1}^{n_1} \gamma_5^2 \bar{\nu}_i^2 + \sum_{j=1}^{n_2} \gamma_4^2 \bar{\nu}_j^2\). Let us define the following:

\[ K_1 = \gamma_1 - (\gamma_1 e_n + x_{11} \gamma_2 + x_{12} \gamma_3 + \nu \gamma_4) \text{ and} \]
\[ K_2 = \nu_2 - (\gamma_1 e_n + x_{21} \gamma_2 + x_{22} \gamma_3 + \nu \gamma_4) . \]

Thus,

\[ -2(y - U_{obs\gamma})'(U_{mis\gamma}) = -2(y - U_{obs\gamma})' \begin{pmatrix} \bar{\nu}_5 \\ \bar{\nu}_4 \end{pmatrix} = \]
\[ -2 \begin{pmatrix} \gamma_1 - (\gamma_1 e_n + x_{11} \gamma_2 + x_{12} \gamma_3 + \nu \gamma_4) \\ \nu_2 - (\gamma_1 e_n + x_{21} \gamma_2 + x_{22} \gamma_3 + \nu \gamma_4) \end{pmatrix}' \begin{pmatrix} \bar{\nu}_5 \\ \bar{\nu}_4 \end{pmatrix} = -2K_1' \bar{\nu}_5 - 2K_2' \bar{\nu}_4 = \]
\[ -2\gamma_5 \bar{\nu}_5 K_1 - 2\gamma_4 \bar{\nu}_4 K_2 = -2 \sum_{i=1}^{n_1} k_{i1} \gamma_5 \bar{\nu}_i - 2 \sum_{j=1}^{n_2} k_{2j} \gamma_4 \bar{\nu}_j, \]

where \(k_{li}\) is the \(i\)th row of \(K_1\), and \(k_{2j}\) is the \(j\)th row of \(K_2\). Therefore, after substituting the above into equation (3.22) we obtain the following:

\[ g(\bar{\nu}, \bar{\nu}) = \sum_{i=1}^{n_1} \gamma_5^2 \bar{\nu}_i^2 + \sum_{j=1}^{n_2} \gamma_4^2 \bar{\nu}_j^2 - 2 \sum_{i=1}^{n_1} k_{i1} \gamma_5 \bar{\nu}_i - 2 \sum_{j=1}^{n_2} k_{2j} \gamma_4 \bar{\nu}_j . \] (3.23)

**Priors**

Assuming that \(\gamma, \sigma^2, \bar{\nu}, \text{ and } \bar{\nu}\) are independent, we know that

\[ p(\gamma, \sigma^2, \bar{\nu}, \bar{\nu}) = p(\gamma) p(\sigma^2) p(\bar{\nu}) p(\bar{\nu}) . \] (3.24)
If we use non-informative, or vague, priors for $\gamma$ and $\sigma^2$, again we obtain

$$p(\gamma)p(\sigma^2) \propto \frac{1}{\sigma^2}. \quad (3.25)$$

Just like section 3.1, $\tilde{v}_i$ are the rows of $\tilde{v}$ and $\tilde{w}_j$ are the rows of $\tilde{w}$. Also, if we assume that $\tilde{v}_i \overset{i.i.d.}{\sim} N(\mu_1, \sigma_1^2)$ and $\tilde{w}_j \overset{i.i.d.}{\sim} N(\mu_2, \sigma_2^2)$ where $\mu_1, \mu_2, \sigma_1^2,$ and $\sigma_2^2$ are known, then let us define $\Omega = (\mu_1, \mu_2, \sigma_1^2, \sigma_2^2)$. Therefore,

$$p(\tilde{v}|\mu_1, \sigma_1) = \prod_{i=1}^{n_1} p(\tilde{v}_i|\mu_1, \sigma_1) \propto \frac{1}{\sigma_1^{n_1/2}} \exp\left\{\frac{-1}{2\sigma_1^2} \sum_{i=1}^{n_1} (\tilde{v}_i - \mu_1)^2\right\} \quad \text{and}$$

$$p(\tilde{w}|\mu_2, \sigma_2) = \prod_{j=1}^{n_2} p(\tilde{w}_j|\mu_2, \sigma_2) \propto \frac{1}{\sigma_2^{n_2/2}} \exp\left\{\frac{-1}{2\sigma_2^2} \sum_{j=1}^{n_2} (\tilde{w}_j - \mu_2)^2\right\}.$$

Replacing parts of equation (3.24) gives us the following:

$$p(\gamma, \sigma^2, \tilde{v}, \tilde{w}) \propto \frac{1}{\sigma^2}$$

$$\times \exp\left\{\frac{-1}{2\sigma_1^2} \sum_{i=1}^{n_1} (\tilde{v}_i - \mu_1)^2\right\} \times \exp\left\{\frac{-1}{2\sigma_2^2} \sum_{j=1}^{n_2} (\tilde{w}_j - \mu_2)^2\right\}. \quad (3.26)$$

**Joint Posterior**

We can use Bayes' theorem to say

$$p(\tilde{v}, \tilde{w}, \gamma, \sigma^2|y, U, \Omega) \propto p(y|\tilde{v}, \tilde{w}, \gamma, \sigma^2, U, \Omega)p(\tilde{v}, \tilde{w}, \gamma, \sigma^2|\Omega) \quad (3.27)$$

After substituting (3.23) and (3.26) into (3.27), we get the joint posterior.
Equation (3.28) then becomes the following:

\[ p(\bar{v}, \bar{w}, \gamma, \sigma^2 \mid y, U, \Omega) \propto \frac{1}{\sqrt{\pi}^n} \exp \left\{ -\frac{1}{2\sigma^2} (y - U_{\text{obs}} \gamma)'(y - U_{\text{obs}} \gamma) \right\} \]

\[ \times \frac{1}{\sigma^2} \times \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (\bar{v}_i - \mu_1)^2 \right\} \times \exp \left\{ -\frac{1}{2\sigma^2} \sum_{j=1}^{n} (\bar{w}_j - \mu_2)^2 \right\}. \]  

(3.29)

Now, let \( \tilde{A} = \left\{ -\frac{1}{2\sigma_1^2} \sum_{i=1}^{n} (\bar{v}_i - \mu_1)^2 \right\} + \left\{ -\frac{1}{2\sigma_2^2} \sum_{j=1}^{n} (\bar{w}_j - \mu_2)^2 \right\} + \frac{1}{2\sigma^2} g(\bar{v}, \bar{w}). \)

This equation has all of the missing data that was in the joint posterior. Substituting \( \tilde{A} \) into (3.29) obtains the following:

\[ p(\bar{v}, \bar{w}, \gamma, \sigma^2 \mid y, U, \Omega) \propto \]

\[ \frac{1}{\sqrt{\pi}^n} \exp \left\{ -\frac{1}{2\sigma^2} (y - U_{\text{obs}} \gamma)'(y - U_{\text{obs}} \gamma) \right\} \]

\[ \times \frac{1}{\sigma^2} \times \exp \left\{ -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (\bar{v}_i - \mu_1)^2 \right\} \times \exp \left\{ -\frac{1}{2\sigma^2} \sum_{j=1}^{n} (\bar{w}_j - \mu_2)^2 \right\} \times \exp \left\{ -\frac{1}{2\sigma^2} g(\bar{v}, \bar{w}) \right\}. \] 

Now, \( \tilde{A} \) can be rearranged by combining like terms. Thus,
\[
\tilde{A} = \sum_{i=1}^{n} \left[ \left( \frac{\gamma_i^2}{\sigma^2} + \frac{1}{\sigma_i^2} \right) \tilde{\nu}_i^2 - 2 \left( \frac{k_i \gamma_i}{\sigma^2} + \frac{\mu_i}{\sigma_i^2} \right) \tilde{\nu}_i \right] \\
+ \sum_{j=1}^{n_k} \left[ \left( \frac{\gamma_j^2}{\sigma^2} + \frac{1}{\sigma_j^2} \right) \tilde{\omega}_j^2 - 2 \left( \frac{k_{j,j} \gamma_j}{\sigma^2} + \frac{\mu_j}{\sigma_j^2} \right) \tilde{\omega}_j \right].
\]

(3.30)

Completing the square gives us

\[
\tilde{A} = \sum_{i=1}^{n} \frac{(\tilde{\nu}_i - \nu_i^*)^2}{\sigma_i^*} + \sum_{j=1}^{n_k} \frac{(\tilde{\omega}_j - \omega_j^*)^2}{\sigma_j^*} + \text{constant},
\]

where \( \nu_i^* = (\sigma_i^*) \left( \frac{k_i \gamma_i}{\sigma^2} + \frac{\mu_i}{\sigma_i^2} \right) \) and \( \sigma_i^* = \left( \frac{\gamma_i^2}{\sigma^2} + \frac{1}{\sigma_i^2} \right)^{-1} \), and

where \( \omega_j^* = (\sigma_j^*) \left( \frac{k_{j,j} \gamma_j}{\sigma^2} + \frac{\mu_j}{\sigma_j^2} \right) \) and \( \sigma_j^* = \left( \frac{\gamma_j^2}{\sigma^2} + \frac{1}{\sigma_j^2} \right)^{-1} \).

Consequently, the joint posterior density is now

\[
p(\tilde{\nu}, \tilde{\omega}, \gamma, \sigma^2 | \nu, U, \Omega) \propto \\
\frac{1}{(\sigma^2)^{n+1}} \exp \left\{ -\frac{1}{2\sigma^2} \left[ (\nu - \nu_{obs})^T (\nu - \nu_{obs}) \right] \right\} \\
\times \exp \left\{ -\frac{1}{2\sigma_1^*} \sum_{i=1}^{n} (\tilde{\nu}_i - \nu_i^*)^2 \right\} \times \exp \left\{ -\frac{1}{2\sigma_2^*} \sum_{j=1}^{n_k} (\tilde{\omega}_j - \omega_j^*)^2 \right\}.
\]

(3.31)
Conditional Posteriors

As in the conditional posteriors portion of section 3.1, the conditional density is equivalent to the following:

\[
p(y \mid \bar{V}, \bar{W}, \sigma^2, y, U, \Omega) = \frac{p(y, \bar{V}, \bar{W}, \sigma^2 \mid y, U, \Omega)}{\int_{y'} p(y', \bar{V}, \bar{W}, \sigma^2 \mid y', U, \Omega) \, dy'},
\]

which implies

\[
p(y \mid \bar{V}, \bar{W}, \sigma^2, y, U, \Omega) \propto p(y, \bar{V}, \bar{W}, \sigma^2 \mid y, U, \Omega).
\]

There are \((n_1 + n_2 + 5 + 1)\) unknowns (i.e. \(\bar{V}_{n_1}, \bar{W}_{n_2}, \gamma, \sigma^2\)). The conditional distributions are found just as before. Thus, the full conditional distributions are given below:

1. \((\gamma \mid \bar{V}, \bar{W}, \sigma^2, y, U, \Omega) \sim N((U'U)^{-1}U'y, \sigma^2(U'U)^{-1})\)

2. \((\sigma^2 \mid \bar{V}, \bar{W}, y, U, \Omega) \sim IG\left(\frac{n}{2}, \frac{1}{2}(y - U\gamma)'(y - U\gamma)\right)\)

3. \((\bar{V} \mid y, \gamma, \sigma^2, U_{obs}, \Omega) \sim N(v_i^*, (\sigma_1^*)^2) \quad \forall i = 1, 2, ..., n_1\)

4. \((\bar{W} \mid y, \gamma, \sigma^2, U_{obs}, \Omega) \sim N(w_j^*, (\sigma_2^*)^2) \quad \forall j = 1, 2, ..., n_2\).

The model in this section can be extended to more than one common continuous variable with relative ease.
Chapter 4

4.1 NUMERICAL RESULTS

In this section we present the findings of a numerical application involving data from the University of North Florida. The information is from 306 college algebra students during the fall 1997 semester. The data consist of the student’s work load during the week ($V$), credit hours enrolled during current semester ($W$), computer use ($X$), professor ($X$), and final grade received in the class ($Y$). There were a total of 6 college algebra instructors for the fall 1997 semester. Two were professors and the rest were visiting instructors. These two groups will be our two “scientists.” Some of the data will be suppressed and can be thought of as the missing data. The summary statistics are given below:

Table 4.1 – Summary Statistics for College Algebra Data

<table>
<thead>
<tr>
<th>Variable</th>
<th>Mean</th>
<th>Std. Dev.</th>
<th>Std. Error</th>
<th>Minimum</th>
<th>Maximum</th>
</tr>
</thead>
<tbody>
<tr>
<td>Professors</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>credits</td>
<td>12.10</td>
<td>2.75</td>
<td>0.33</td>
<td>3.00</td>
<td>16.00</td>
</tr>
<tr>
<td>work hours</td>
<td>14.14</td>
<td>14.87</td>
<td>1.77</td>
<td>0.00</td>
<td>50.00</td>
</tr>
<tr>
<td>computer</td>
<td>0.79</td>
<td>0.41</td>
<td>0.05</td>
<td>0.00</td>
<td>1.00</td>
</tr>
<tr>
<td>professor</td>
<td>2.07</td>
<td>2.07</td>
<td>0.25</td>
<td>1.00</td>
<td>6.00</td>
</tr>
<tr>
<td>final grade</td>
<td>2.17</td>
<td>1.16</td>
<td>0.14</td>
<td>0.00</td>
<td>4.00</td>
</tr>
<tr>
<td>n1 = 70</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Visiting</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>credits</td>
<td>12.27</td>
<td>2.53</td>
<td>0.16</td>
<td>3.00</td>
<td>21.00</td>
</tr>
<tr>
<td>work hours</td>
<td>15.66</td>
<td>14.27</td>
<td>0.93</td>
<td>0.00</td>
<td>60.00</td>
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<tr>
<td>computer</td>
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<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>professor</td>
<td>4.09</td>
<td>1.11</td>
<td>0.07</td>
<td>2.00</td>
<td>5.00</td>
</tr>
<tr>
<td>final grade</td>
<td>2.50</td>
<td>1.17</td>
<td>0.08</td>
<td>0.00</td>
<td>4.00</td>
</tr>
<tr>
<td>n2 = 236</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Two scientists with one common binary variable

Consider the following data: student's work load during the week ($V$), credit hours enrolled during current semester ($W$), computer use ($X$), and final grade received in the class ($Y$). The variable $X$ is the common binary variable; either they used the computer or they did not. Suppose that the professors witnessed $W$ and not $V$. Also, suppose that the visiting instructors witnessed $V$ and not $W$. We set the dependent variable to be $Y$, the final grade received in the class. The following model can be used:

\[
(y_{306:1} | \gamma, \sigma^2) = U_{306:1} \gamma + u_{306:1}, \text{ where } u \sim N(0, \sigma^2 I_{505}).
\]

There were $n_1 = 70$ students under the direction of professors and $n_2 = 236$ students under the direction of visiting instructors. The regression procedure in SAS was used with the full data (without missing data) to calculate the least square estimates of the parameters. It was concluded that credit hours ($V$), hours worked per week ($W$), and computer usage ($X$) were significant factors in predicting the final grade in the class. The Gibbs sampler was used to estimate the regression coefficients with the missing data situation. The true values are estimated from the full data set using SAS. The results are located in the following table:

<table>
<thead>
<tr>
<th>Variable</th>
<th>Parameter</th>
<th>&quot;True&quot; Value</th>
<th>Gibbs Estimate</th>
<th>Standard Error</th>
<th>Within SE's</th>
</tr>
</thead>
<tbody>
<tr>
<td>intercept</td>
<td>$\gamma_0$</td>
<td>1.74</td>
<td>2.15</td>
<td>0.14</td>
<td>2.9</td>
</tr>
<tr>
<td>X</td>
<td>$\gamma_1$</td>
<td>-0.34</td>
<td>-0.35</td>
<td>0.10</td>
<td>0.1</td>
</tr>
<tr>
<td>W</td>
<td>$\gamma_2$</td>
<td>-0.01</td>
<td>-0.03</td>
<td>0.01</td>
<td>2</td>
</tr>
<tr>
<td>V</td>
<td>$\gamma_3$</td>
<td>0.08</td>
<td>0.06</td>
<td>0.01</td>
<td>2</td>
</tr>
<tr>
<td>$\sigma^2$</td>
<td></td>
<td>1.17</td>
<td>1.12</td>
<td>0.04</td>
<td>1.25</td>
</tr>
</tbody>
</table>
The within standard errors column indicates about how many standard errors the Gibbs estimate is from the “true” estimate.

Two scientists with one common binary variable and one common continuous variable

Consider the following data: student’s work load during the week \( (V) \), credit hours enrolled during current semester \( (W) \), computer use \( (X_1) \), professor \( (X_2) \), and final grade received in the class \( (Y) \). The variable \( X_1 \) is the common binary variable and \( X_2 \) is the common continuous variable. As before, suppose that the professors witnessed \( W \) and not \( V \). Also, suppose that the visiting instructors witnessed \( V \) and not \( W \). We set the dependent variable to be \( Y \), the final grade received in the class. The following model can be used:

\[
(y_{306x1} | U, \gamma, \sigma^2) = u_{306x1} + u_{306x1}, \text{ where } u \sim N(0, \sigma^2 I_{306}).
\]

There were \( n_1 = 70 \) students under the direction of professors and \( n_2 = 236 \) students under the direction of visiting instructors. Once more, the regression procedure in SAS was used with the full data (without missing data) to calculate the least square estimates of the parameters. It was again concluded that credit hours \( (V) \) and hours worked per week \( (W) \) were significant factors in predicting the final grade in the class. Once again, the Gibbs sampler was used to estimate the regression coefficients with the missing data situation. The true values are estimates from the full data set using SAS. The results are located in the following table:
Table 4.3 - Results of the Gibbs Sampler With One Common Binary Variable and One Common Continuous Variable

<table>
<thead>
<tr>
<th>Variable</th>
<th>Parameter</th>
<th>&quot;True&quot; Value</th>
<th>Gibbs Estimate</th>
<th>Standard Error</th>
<th>Within SE's</th>
</tr>
</thead>
<tbody>
<tr>
<td>intercept</td>
<td>$\gamma_0$</td>
<td>1.80</td>
<td>2.07</td>
<td>0.13</td>
<td>2.1</td>
</tr>
<tr>
<td>$X_1$</td>
<td>$\gamma_1$</td>
<td>-0.38</td>
<td>-0.33</td>
<td>0.10</td>
<td>0.5</td>
</tr>
<tr>
<td>$X_2$</td>
<td>$\gamma_2$</td>
<td>-0.01</td>
<td>0.00</td>
<td>0.01</td>
<td>1</td>
</tr>
<tr>
<td>$W$</td>
<td>$\gamma_3$</td>
<td>-0.01</td>
<td>-0.01</td>
<td>0.02</td>
<td>0</td>
</tr>
<tr>
<td>$V$</td>
<td>$\gamma_4$</td>
<td>0.08</td>
<td>0.04</td>
<td>0.01</td>
<td>4</td>
</tr>
<tr>
<td>$\sigma^2$</td>
<td></td>
<td>1.17</td>
<td>1.13</td>
<td>0.04</td>
<td>1</td>
</tr>
</tbody>
</table>

An IBM compatible PC was used to telnet to NERDC (Northeast Regional Data Center) in order to compile and run the FORTRAN Gibb Sampler program calling subroutines from IMSL (International Mathematical & Statistical Library). One simulation took approximately 115 minutes in CPU time.
4.1 CONCLUSION

When combining several studies for a meta-analysis, the obstacle of missing data frequently arises. The Bayesian approach to meta-analysis regards these missing data as additional parameters that need to be estimated. For the College Algebra data, there were 312 (1+5+70+236) parameters that needed to be approximated. The Gibbs sampler estimated jointly the missing data and parameters with relative ease. After using the Gibbs sampler with the College Algebra data, all but one of the estimates obtained were at most three standard errors away from the true value of the parameter.

In this thesis, vague priors were used for $\gamma$ and $\sigma^2$. It would be interesting to research the use of informative priors for these two parameters. It would be of interest as well to apply this Bayesian approach of Meta-analysis to multivariate observations.
APPENDIX

FORTRAN PROGRAM FOR THE GIBBS SAMPLER

program gibb
integer nca,nra,ncc,ncr,zm,zm,m,
& ncy,ncy,i,kout,ncg,ncd,nrg,nrd,
& ncf,nrf,a,k,n2,nout,ns,ldp,ldpsig,np,irank,ldt,
& ldsig,nt,lim,z,l
& iseed,lrd,ldr,ld
parameter (nca=5,nra=306,ng=1,
& k=5,ncy=1,nry=306,ncg=1,ns=1,nu=1,
& ncd=1,nrg=5,nrd=306,ncf=1,nrf=306,nout=16,nt=70,ldt=70,
& ncc=1,ncr=5,ldsig=5,ldp=236,ldpsig=236,n1=70,
& n=306,l=236,n2=236,ldsig=70)
real a(nra,nca), b(nca,nca), binv(nca,nca),alpha,j,gm(nrg),
& y(nry,ncy),g(ng,nrg),d(nrd,ncd),f(nrf,ncf),beta,ldv(n1,n1),
& e(ncf,ncf),y1(n1),y2(n2),x(n),n1(n1),n2(n2),k1(n1),k2(n2),
& xxx1(n1),xxx2(n2),en1(n1),en2(n2),w(n1),tsig(n2,n2),sinv,g4(nrg),
& v(n2),gee1(n1),gee2(n2),xxx1g2(n1),xxx2g2(n2),wwg4(n1),vgg5(n2),
& vcov(n1,n1),wcov(n2,n2),p(ldp,ldp),psig(ldp,ldp),g3(nrg),g5(nrg),
& tlg(ng),tolv(n2),tolw(n1),s1,s2,m1,mu1,mu2,s2star1,s2star2,
& vstar(n2),wstar(n1),tol,vcov(nrg,nrg),sum,ave,u(u,u),bu,au,
& r(nr,ldr),rsig(ldr,ldr),oldg(nrg),olds,oldv(n2),
& oldw(n1),c(ncr,ncc),q(nrc,ncc),idw(n2,n2),x(n,ldt),tol,o,
& vtild(n1),vtild(n2),ww(n),vv(n),temp1,temp2,
& aa(n),pp(n),as(n),pp1(n1),ppp2(n2),p1g3(n1),p2g3(n2)
external wrrm,mrrrr,mxtxf,mxyf,linrg,chfac,mmvn,rnset,
& rngan,sscal,sswap,umach,mun,sadd
iseed=0
call rnset(iseed)
tol = 0.001
do 500 i=1,n1
   do 510 j=1,n1
      if (i.eq.j) then
         idv(i,j)=1.0
      else
         idv(i,j)=0.0
      end if
      en1(i)=1.0
   510 continue
   500 continue
do 520 i=1,n2
   do 530 j=1,n2
      if (i.eq.j) then
         idw(i,j)=1.0
      else
      end if
   530 continue
   520 continue
end
idw(i,j)=0.0
end if
en2(i)=1.0
530 continue
do 540 i=1,n
read (11,999) ww(i),vv(i),aa(i),ss(i),y(i,1),x(i),pp(i)
999 format (f8.0,f9.0,2f9.0,f9.0,2f9.0)
540 continue
do 550 i=1,n1
  vtilde(i)=0.0
  w(i)=vv(i)
y1(i)=y(i,1)
  xxx1(i)=x(i)
  ppp1(i)=pp(i)
550 continue
do 560 i=1,n2
  wtilde(i)=0.0
  temp1=i+n1
  v(i)=ww(temp1)
y2(i)=y(temp1,1)
  xxx2(i)=x(temp1)
  ppp2(i)=pp(temp1)
560 continue
s1=14.39*14.39
s2=2.58*2.58
mu1=15.3
mu2=12.232
o=2.7726
ave=0.0
sum=0.0
limit=100
au=-16
bu=16
****** initial guess for sigma squared
do 300 lim=1,100
ave=0.0
sum=0.0
zz=1
do 490 mm=1,n
  a(mm,1)=1.0
  a(mm,2)=x(mm)
  a(mm,3)=pp(mm)
  if (mm.gt.n1) then
    a(mm,5)=v(mm-n1)
    a(mm,4)=0.0
  else
    a(mm,4)=w(mm)
    a(mm,5)=0.0
  end if
34
400 continue
do 310 z=1,limit
   do 410 ml=1,n
      a(ml,1)=1.0
      a(ml,2)=x(ml)
      if (ml.le.ml1) then
         a(ml,3)=pp(ml)
         a(ml,4)=w(ml)
         a(ml,5)=vtilde(ml)
      else
         a(ml,4)=wtilde(ml-nl)
         a(ml,5)=v(ml-nl)
      end if
   410 continue
   if (z .eq.l) then
      call run(nu,ng)
      call sscal(nu,au-au,ng,1)
      call sadd(nu,au,ng,1)
      g(1,1)=gm(1)
      g(2,1)=gm(2)
      g(3,1)=gm(3)
      g(4,1)=gm(4)
      g(5,1)=gm(5)
      call run(nu,ng)
      do 20 j=1,16,.001
         ave=(1/j+1/(j+.001))/o
         sum=sum+ave
         if (sum.gt.u(nu)) then
            s=j+.0005
            zz=2
            goto 30
         else
            goto 20
      end if
   else
      continue
   end if
****** u'u
30 call mxtcf (nra,nca,a,nra,nca,b,nca)
****** (u'u)^-1
   call linrg (nca,b,nca,binv,nca)
****** u'y
   call mxtg (nra,nca,a,nra,ary,ncy,y,ary,ncy,ncc,c,ncc)
****** (u'u)^-1 * u'y
   call mrrrt (nca,binv,nca,nrc,ncc,c,nrc,nrc,ncc,q,nrc)
   do 150 i=1,nrg
      do 160 j=1,nrg
         cov(i,j)=binv(i,j)*s
      160 continue
   150 continue
   if (z .ne.1) then
call umach(2,nout)
call chfac(k,cov,nrg,.00001,irank,rsig,ldrsig)
call rmvn(nr,k,rsig,ldrsig,r,ldr)
do 170 i=1,nrg
   oldg(i)=g(i,1)
170 continue
do 175 i=1,nrg
   g(i,1)=r(1,i)+q(i,1)
175 continue
else
   end if

****** u*gamma
   call mrrrr(nra,nca,a,nra,nrg,ncg,g,nrg,ncd,ncd,ncd)
****** y-u*gamma
   do 100 i=1,n
      f(i,1)=y(i,1)-d(i,1)
100 continue
****** (y-u*gamma)/(y-u*gamma)
   call mxtxf(mnf,ncf,f,mnf,ncf,e,ncf)
   if(z.ne.1) then
      alpha=n/2
      beta=2/e(ncf,ncf)
      call mgam(ns,alpha,sinv)
      call sscal(ns,beta,sinv,l)
      olds=s
      s=1/sinv
   else
      end if
   do 110 i=1,n1
      gee1(i)=g(1,1)*en1(i)
      xxx1g2(i)=g(2,1)*xxx1(i)
      wgg4(i)=g(4,1)*w(i)
      p1g3(i)=g(3,1)*ppp1(i)
      m1(i)=gee1(i)+xxx1g2(i)+wgg4(i)+p1g3(i)
      kl(i)=yl(i)-m1(i)
110 continue
   do 115 i=1,n2
      gee2(i)=g(1,1)*en2(i)
      xxx2g2(i)=g(2,1)*xxx2(i)
      vgg5(i)=g(5,1)*v(i)
      p2g3(i)=g(3,1)*ppp2(i)
      m2(i)=gee2(i)+xxx2g2(i)+vgg5(i)+p2g3(i)
      k2(i)=yl(i)-m2(i)
115 continue
   ssstar1=1/((g(5,1)**2)/s+1/s1)
   ssstar2=1/((g(4,1)**2)/s+1/s2)
   do 140 i=1,n1
      vstar(i)=ssstar1*(k1(i)*g(5,1)/s+mu1/s1)
140 continue
   do 145 i=1,n2
\[ \text{wstar}(i) = \text{sstar2} \times (k2(i) \times g(4,1)/s + \mu2/s2) \]

```
145 continue
   do 180 i=1,n1
       do 190 j=1,n1
           vcov(i,j) = sstar1 * idv(i,j)
190  continue
180  continue
   do 185 i=1,n2
       do 195 j=1,n2
           wcov(i,j) = sstar2 * idw(i,j)
195  continue
185  continue
   call umach(2,nout)
   call chfac(1,vcov,n1,0001,irank,tsig,ldtsig)
   call mmvn(nt,l,tsig,ldtsig,t,ldt)
   do 200 i=1,n1
       oldv(i) = vtilda(i)
       vtilda(i) = t(nt,i) + vstar(i)
200  continue
   call umach(2,nout)
   call chfac(1,vcov,n2,0001,irank,psig,ldpsig)
   call mmvn(np,ll,psig,ldpsig,p,ldp)
   do 210 i=1,n2
       oldw(i) = wtilda(i)
       wtilda(i) = p(np,i) + wstar(i)
210  continue
   do 220 i=1,nrg
      tolg(i) = abs(oldg(i) - g(i,1))
      tols = abs(olds - s)
      if (((tolg(1) .lt. tol) .and. (tolg(2) .lt. tol)) .and. (tolg(3) .lt. tol) .and. (tolg(4) .lt. tol) .and. (tols .lt. tol)
          .and. (z .ne. 1)) .or. (z .eq. limit) goto 320
310  continue
320  sums = sums + s
    sumg1 = sumg1 + g(1,1)
    sumg2 = sumg2 + g(2,1)
    sumg3 = sumg3 + g(3,1)
    sumg4 = sumg4 + g(4,1)
    sumg5 = sumg5 + g(5,1)
300  continue
   print *, 'gamma1 = ', sumg1/100
   print *, 'gamma2 = ', sumg2/100
   print *, 'gamma3 = ', sumg3/100
   print *, 'gamma4 = ', sumg4/100
   print *, 'gamma5 = ', sumg5/100
   print *, 'sigma'**2 = ', sums/100
end```

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BIBLIOGRAPHY


VITA

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BIRTHDATE:

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EDUCATION: 1996-1998 University of North Florida
            Master of Science in Mathematical Sciences
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