

# Embedding Graphs on Surfaces and Graph Minors

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## PLANAR GRAPHS

A graph  $G$  is called **planar** if  $G$  can be drawn in a plane so that no two edges cross each other. **Embedding** is simply the “drawing” of  $G$  in the plane without edges crossing.

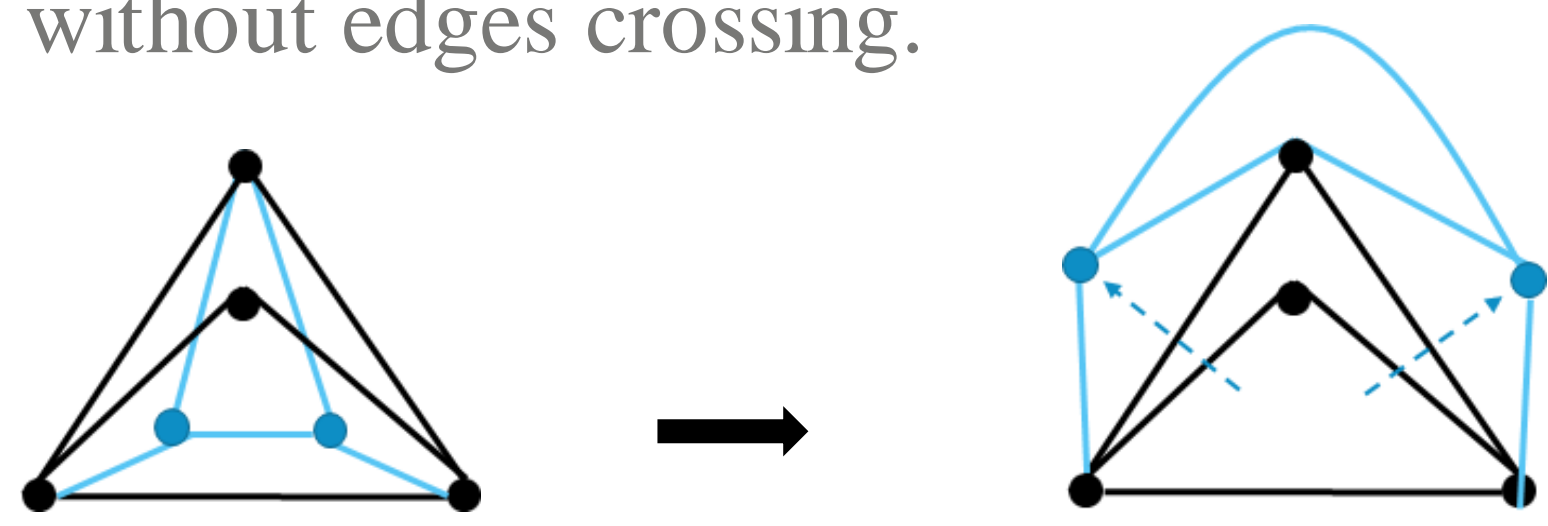


Figure 1. A planar graph.

## CONSTRUCTING A TORUS

A **torus** is a donut shaped surface.

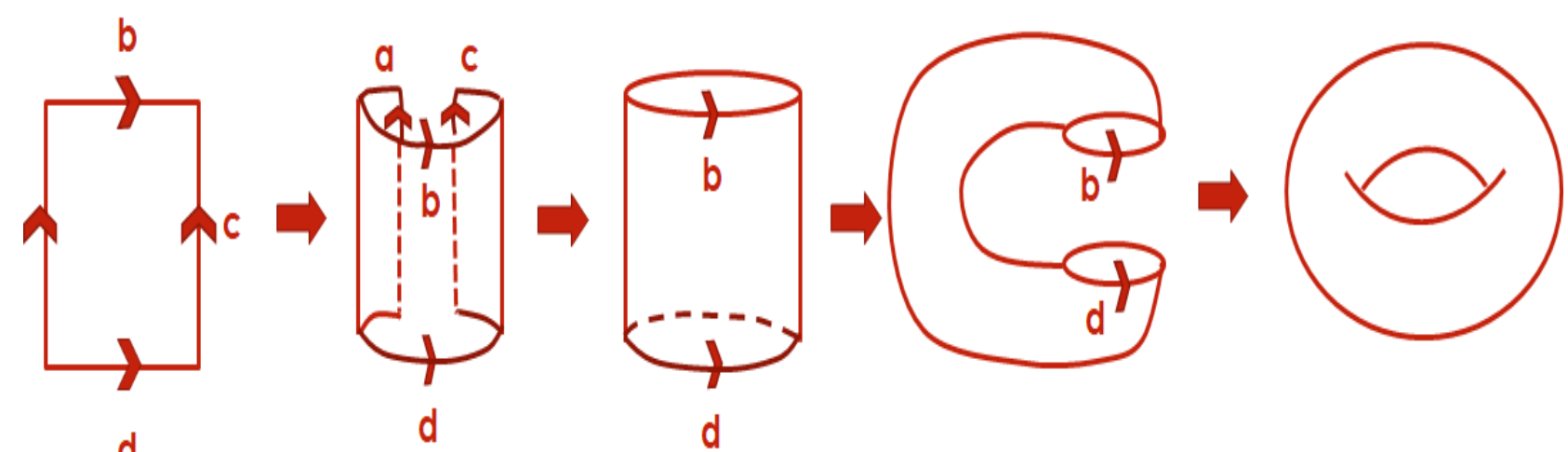


Figure 2. A Construction of a torus.

We begin with a sphere and drill two holes in its surface. Then we attach a “handle” to the sphere by placing the ends of the handle over the two holes.

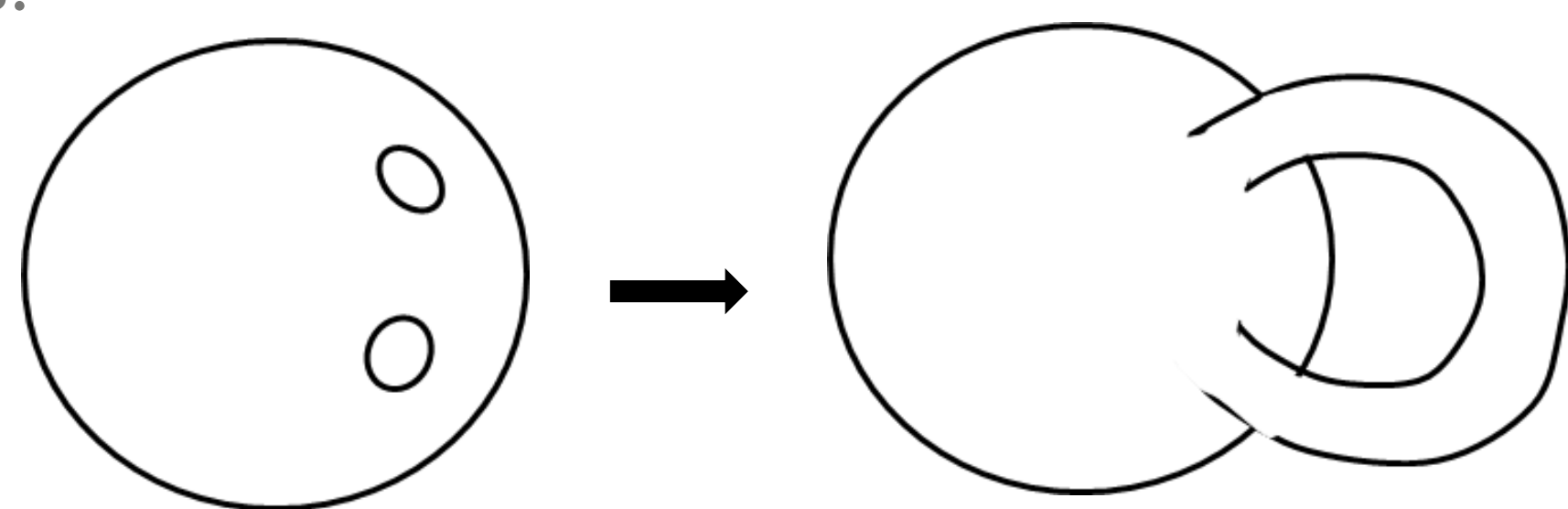


Figure 3. A torus is a sphere with one handle.

A sphere with  $k$  handles attached is denoted by  $S_k$ , which is also called a **surface of genus  $k$** . The smallest nonnegative integer  $k$ , such that  $G$  can be embedded on  $S_k$ , is called the **genus of  $G$** , denoted by  $\gamma(G)$ .

## 2-CELL EMBEDDING

Every embedding of a connected graph in a plane is a **2-cell** embedding. A region is **2-cell** if any closed curve that is drawn in that region can be continuously contracted (or shrunk) in that region to a single point.

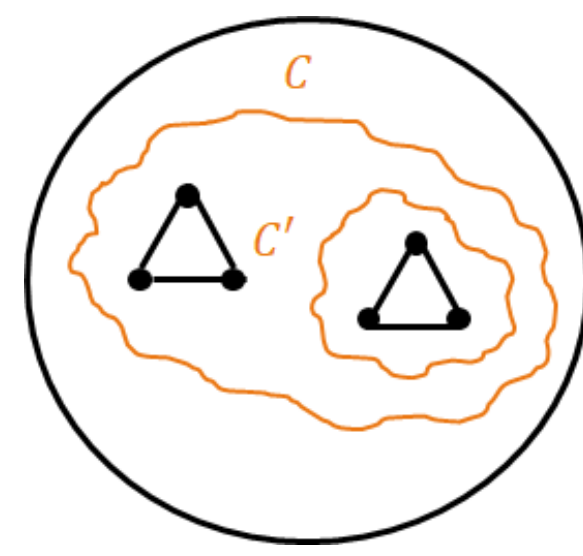


Figure 4. An embedding of  $2K_3$  on the sphere that is not a 2-cell embedding.

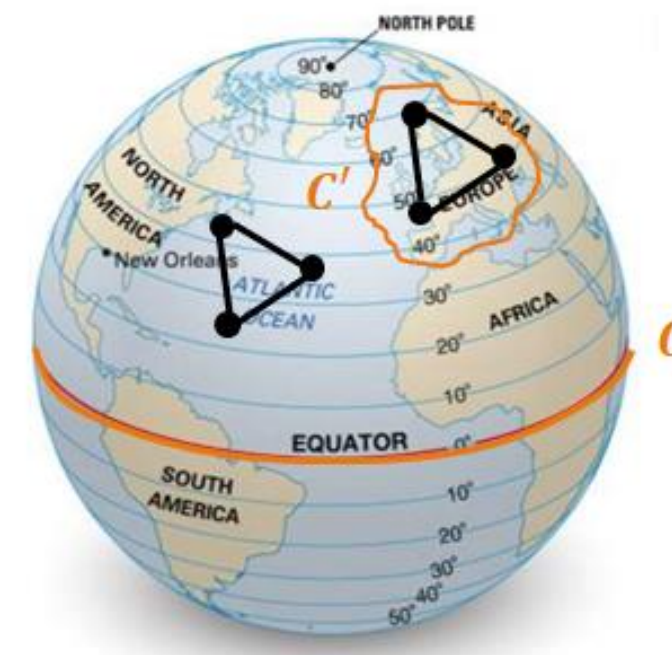


Figure 5. a 3D representation of the embedding.

## VERTEX DELETIONS

The graph  $G-W$  is obtained from  $G$  by removing all vertices in a set  $W$  and all edges incident to those vertices. If  $W$  is a singleton, e.g.  $W=\{v\}$ , then  $G-W$  is denoted  $G-v$ . The graph  $G-v$  is obtained through a **vertex deletion**.

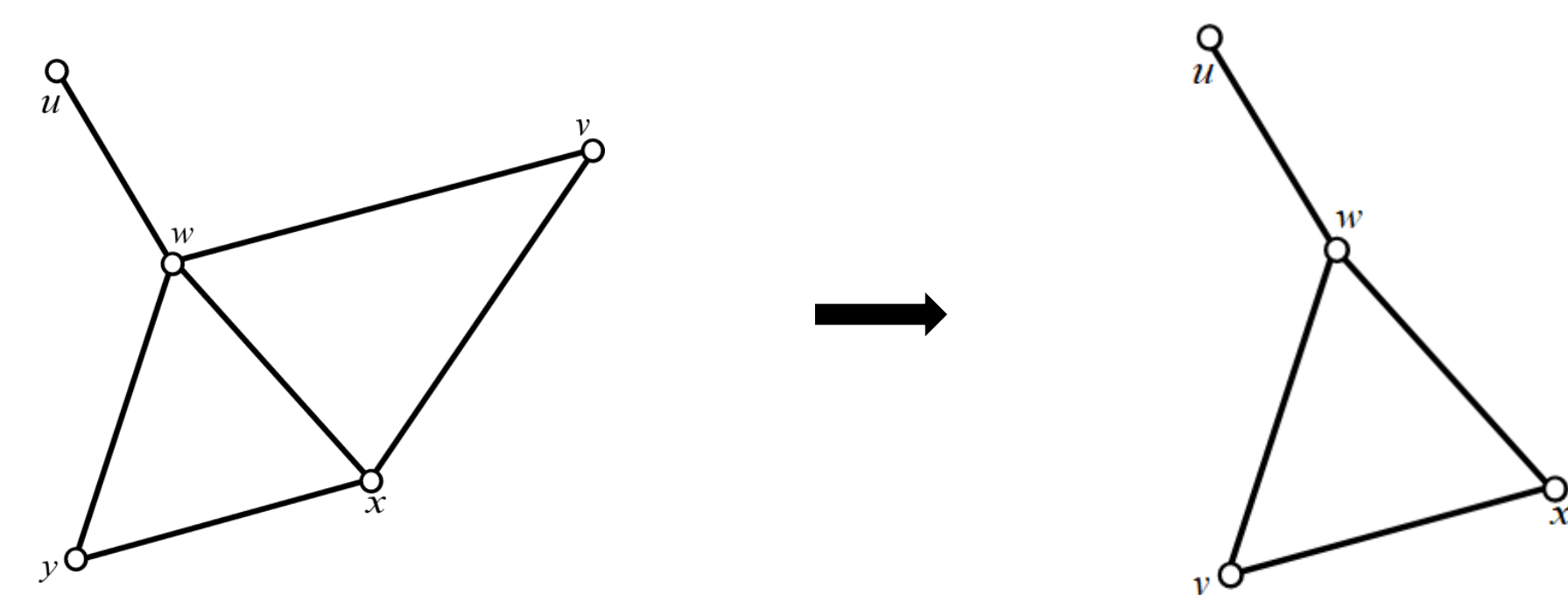


Figure 6. A graph obtained through a vertex deletion.

## EDGE DELETIONS

For graphs  $G$  and  $G'$ , identify an edge  $e=uv \in E(G)$ . If  $G'$  is isomorphic to the graph obtained by joining  $u$  in the graph  $G-v$  to any neighbor of  $v$  not already adjacent to  $u$ , the graph  $G'$  is said to be obtained from  $G$  by **contracting the edge  $e$**  or by **edge contraction**.

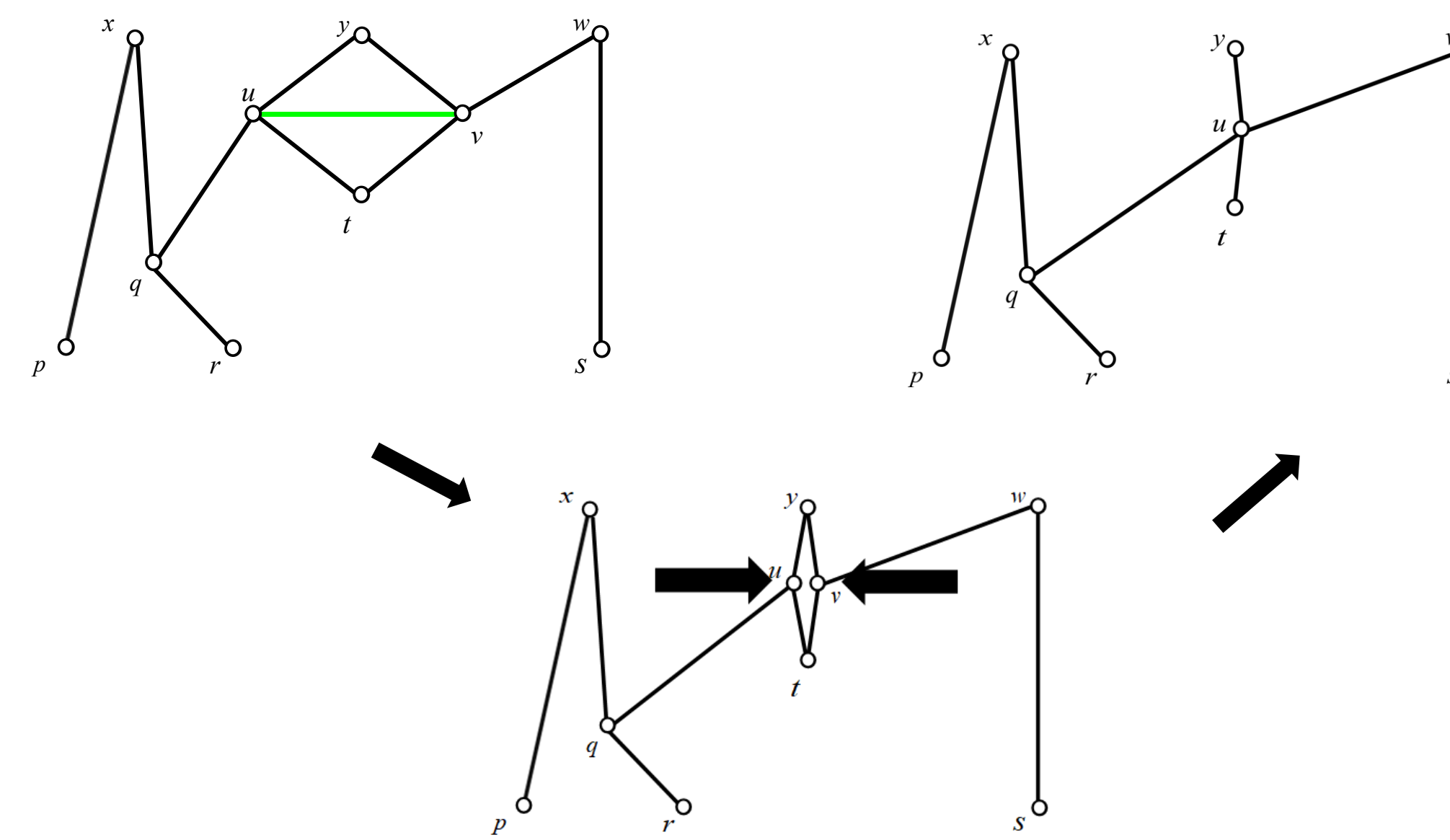


Figure 7. The process of edge contraction.

## GRAPH MINORS

A graph  $H$  is called a **minor** of a graph  $G$  if a graph isomorphic to  $H$  can be obtained from  $G$  by a succession of edge contractions, edge deletions or vertex deletions in any order.

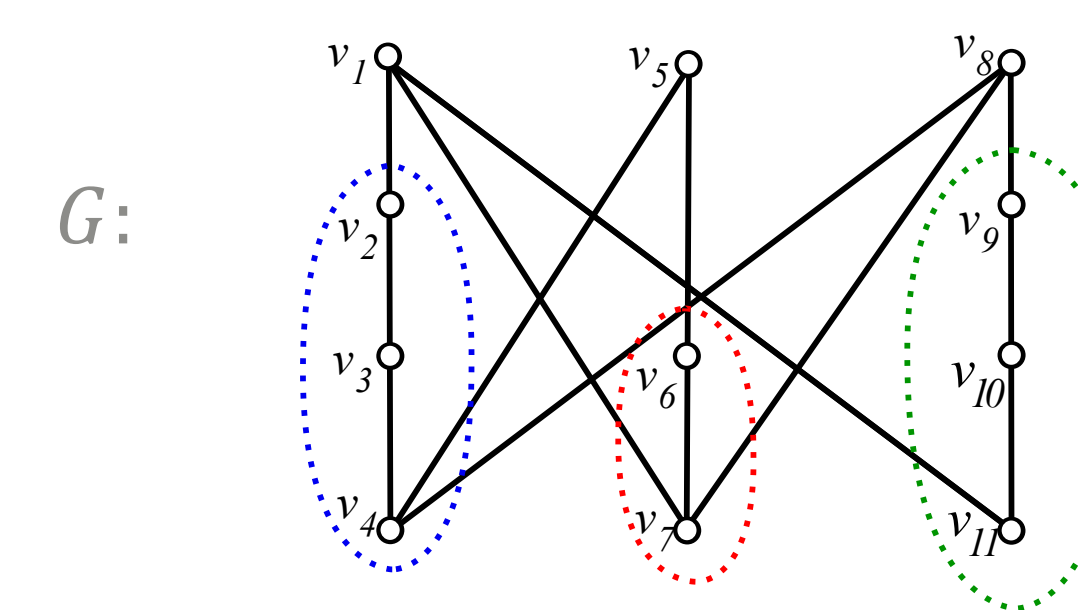


Figure 8. A graph  $G$  showing possible edge contractions.

## MINIMALLY NONEMBEDDABLE

Let  $F$  be a graph such that  $F$  cannot be embedded on  $S_k$ . Create successive **graph minors** of  $F$  until a graph  $F'$  is obtained that cannot be embedded on  $S_k$ . However, any **minor** of  $F'$  will result in a graph that is embeddable on  $S_k$ . Such a graph is called **minimally nonembeddable**.

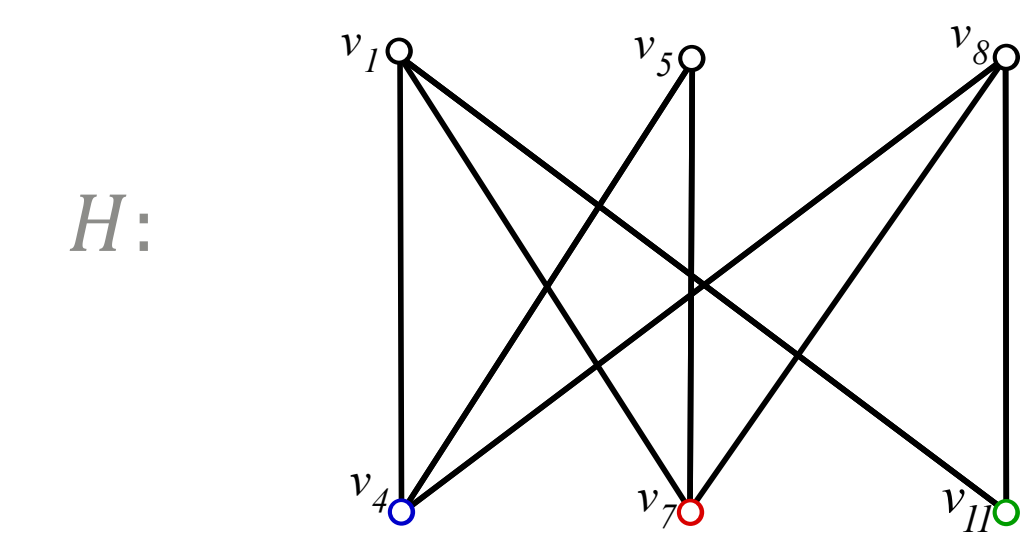


Figure 9. A graph  $H$  as a minor of  $G$ .

## REFERENCES

- [1] Chartrand, G., Zhang, P., A First Course in Graph Theory, Dover Publications, 2012.
- [2] Chartrand, G., Zhang, P., Chromatic Graph Theory, CRC Press, 2009.
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- [4] Gross, J. L., Yellen, J., Zhang, P., Handbook of Graph Theory, CRC Press, 2014.