Embedding Graphs on Surfaces and Graph Minors

Tracy Leung, Mya Salas, Dylan Wilson
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PLANAR GRAPHS

A graph G is called **planar** if G can be drawn in a plane so that no two edges cross each other. **Embedding** is simply the "drawing" of G in the plane without edges crossing.

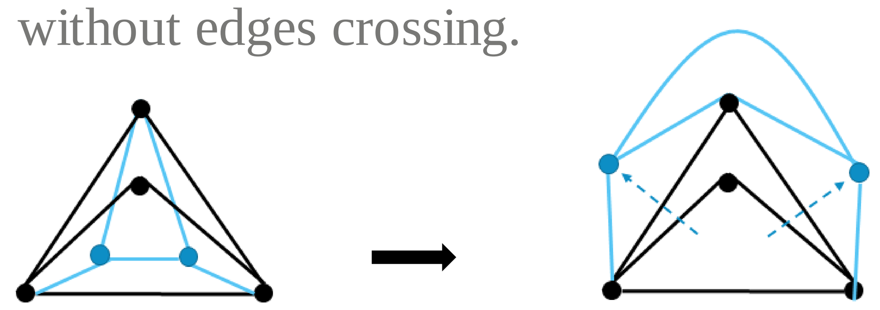

Figure 1. A planar graph.

CONSTRUCTING A TORUS

A **torus** is a donut shaped surface.

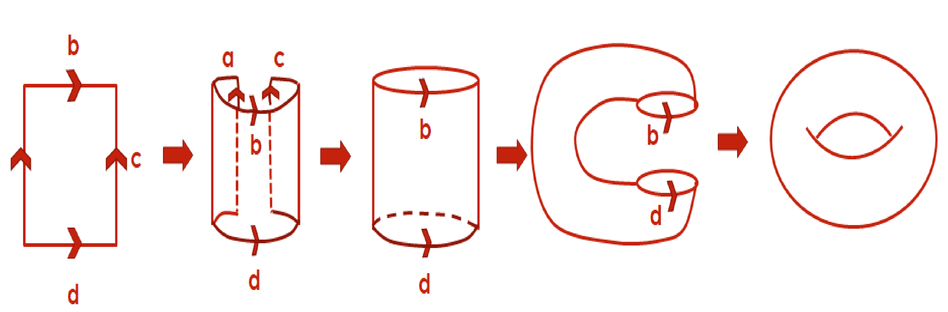

Figure 2. A Construction of a torus.

We begin with a sphere and drill two holes in its surface. Then we attach a "handle" to the sphere by placing the ends of the handle over the two holes.

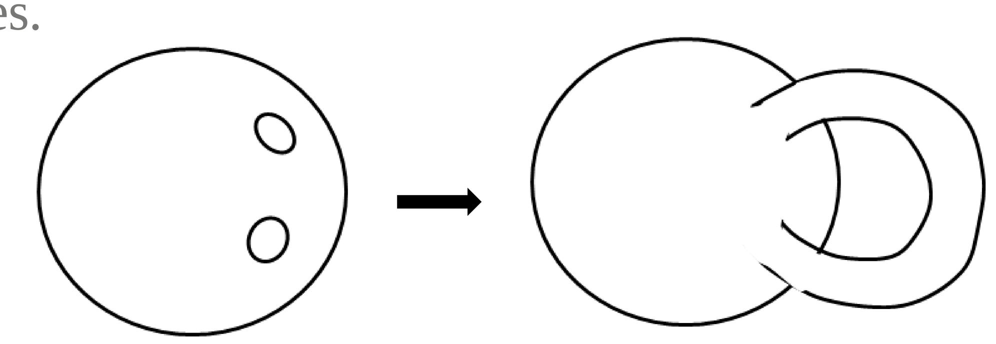

Figure 3. A torus is a sphere with one handle.

A sphere with k handles attached is denoted by S_k, which is also called a **surface of genus** $\boldsymbol{k}$. The smallest nonnegative integer k, such that G can be embedded on S_k, is called the genus of G, denoted by $\gamma(G)$.

2-CELL EMBEDDING

Every embedding of a connected graph in a plane is a **2-cell** embedding. A region **is 2-cell** if any closed curve that is drawn in that region can be continuously contracted (or shrunk) in that region to a single point.

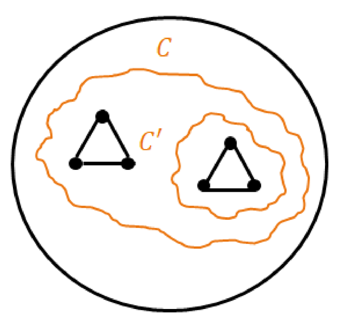

Figure 4. An embedding of $2K_3$ on the sphere that is not a 2-cell embedding.

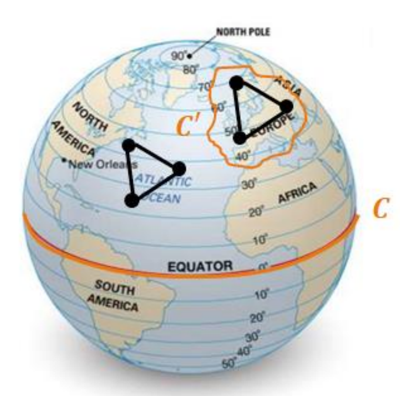

Figure 5. a 3D representation of the embedding.

VERTEX DELETIONS

The graph $G–W$ is obtained from G by removing all vertices in a set W and all edges incident to those vertices. If W is a singleton, e.g. $W=\{v\}$, then $G–W$ is denoted $G–v$. The graph $G–v$ is obtained through a **vertex deletion.**

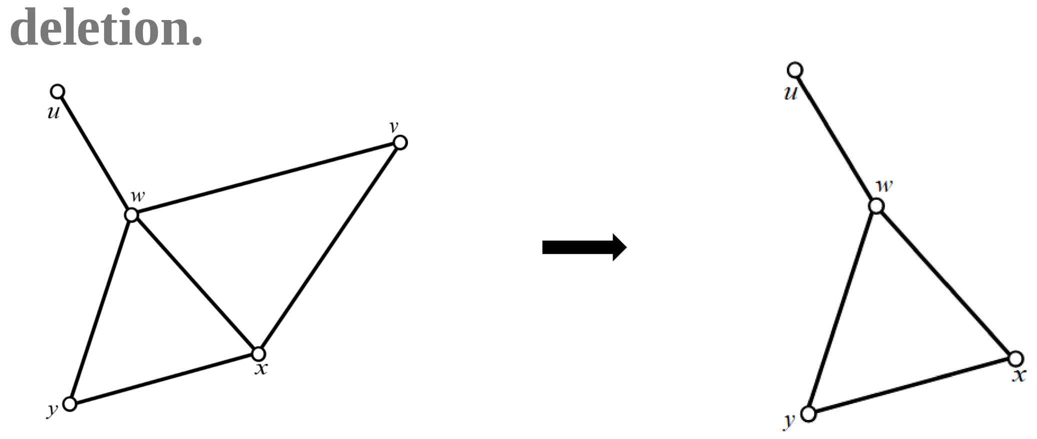

Figure 6. A graph obtained through a vertex deletion.

EDGE DELETIONS

For graphs G and G', identify an edge $e=uv \in E(G)$, If G' is isomorphic to the graph obtained by joining u in the graph $G-v$ to any neighbor of v not already adjacent to u, the graph G' is said to be obtained from G by **contracting the edge** e or by **edge contraction.**

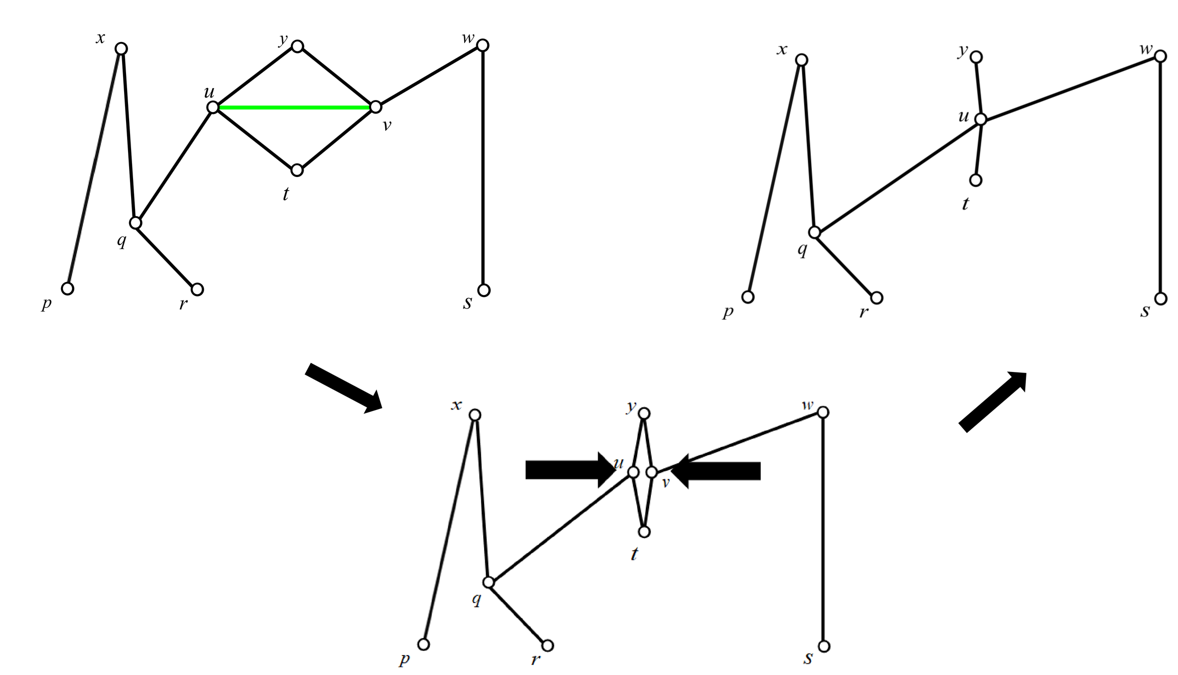

Figure 7. The process of edge contraction.

GRAPH MINORS

A graph H is called a **minor** of a graph G if a graph isomorphic to H can be obtained from G by a succession of edge contractions, edge deletions or vertex deletions in any order.

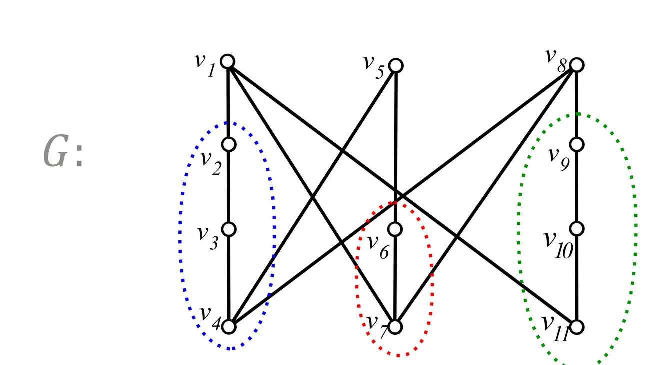

Figure 8. A graph G showing possible edge contractions.

MINIMALLY NONEMBEDABLE

Let F be a graph such that F cannot be embedded on S_k. Create successive **graph minors** of F until a graph F' is obtained that cannot be embedded on S_k. However, any **minor** of F' will result in a graph that is embeddable on S_k. Such a graph is called **minimally nonembeddable**.

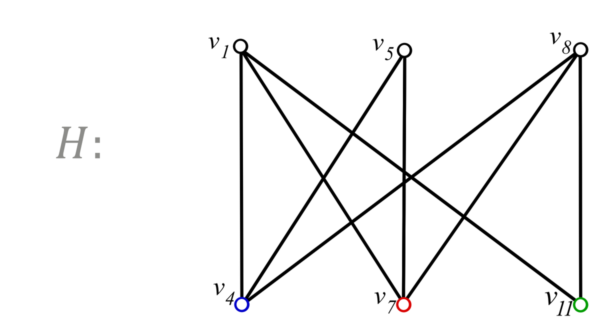

Figure 9. A graph H as a minor of G.

REFERENCES

[1] Chartrand, G., Zhang, P., A First Course in Graph Theory, Dover Publications, 2012.

[2] Chartrand, G., Zhang, P., Chromatic Graph Theory, CRC Press, 2009.

[3] Diestel R., Graph Theory, Springer Nature, 2017.

[4] Gross, J. L., Yellen, J., Zhang, P., Handbook of Graph Theory, CRC Press, 2014.

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