A Relationship Between the Fibonacci Sequence and Cantor's Ternary Set

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by

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For my wife Janna for all of her support

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ABSTRACT

A Relationship Between the Fibonacci Sequence

and Cantor's Ternary Set

The Fibonacci sequence and Cantor's ternary set are two objects of study in mathematics. There is much theory published about these two objects, individually. This paper provides a fascinating relationship between the Fibonacci sequence and Cantor's ternary set. It is a fact that every natural number can be expressed as the sum of distinct Fibonacci numbers. This expression is unique if and only if no two consecutive Fibonacci numbers are used in the expression--this is known as Zekendorf's representation of natural numbers. By Zekendorf's representation, a function F from the natural numbers into [0,0.603] will be defined which has the property that the closure of F(N) is homeomorphic to Cantor's ternary set. To accomplish this, it is shown that the closure of F(N) is a perfect, compact, totally disconnected metric space. This then shows that the closure of F(N) is homeomorphic to Cantor's ternary set and thereby establishing a relationship between the Fibonacci sequence and Cantor's ternary set.

CHAPTER 1

INTRODUCTION

There are many interesting objects that are studied in mathematics. Two such objects are the Fibonacci sequence and Cantor's ternary set. The Fibonacci sequence is studied in such disciplines as elementary number theory and combinatorics while Cantor's ternary set is studied in topology and real analysis. Much theory exists concerning each object, individually. However, in this thesis, an interesting relationship between the Fibonacci sequence and Cantor's ternary set is established.

The Fibonacci numbers form a recursive sequence. It is a fact that every natural number can be expressed as the sum of distinct Fibonacci numbers. (See page 10.) However, Zekendorf found that this representation is unique if and only if no two consecutive Fibonacci numbers are used in the sum. This representation will prove to be an important key to establishing the desired relationship.

The basis for the relationship is a function F that maps the natural numbers x into [0,0.603], where x is expressed using Zekendorf's representation. Some properties of the Fibonacci numbers will be proved and used to establish important properties of F. It is known that Cantor's ternary set is a perfect, compact, totally disconnected metric space. It will be shown that for the set of natural numbers **N**, the closure of F(**N**), denoted by cl(F), satisfies these conditions as well.

It is a fact that any two perfect, compact, totally disconnected metric spaces are homeomorphic to each other [4]. This then implies that cl(F) is homeomorphic to Cantor's ternary set. Two sets are said to be homeomorphic if there exists a one-to-one function that maps one of the sets onto the other; whereby, this one-to-one function and its inverse are continuous. The homeomorphism between cl(F) and Cantor's ternary set provides the relationship between the Fibonacci sequence and Cantor's ternary set.

CHAPTER 2

THE FIBONACCI SEQUENCE

Background information on Fibonacci and his sequence will provide a better understanding of and appreciation for this person and his great mathematical achievements. Some properties of the Fibonacci sequence are presented in order to provide a better knowledge of this powerful sequence.

Leonardo of Pisa, better known as Leonardo Fibonacci, was one of the most talented mathematicians of the Middle Ages. He is responsible for many advances in the study of discrete mathematics. He was born in about 1180 probably in Pisa, Italy. His father, Guglielmo, was appointed chief magistrate over the community of Pisan merchants in the north African port of Bugia (now Bejaia, Algeria). It is Leonardo's father who helped to enhance his understanding of mathematics, for he sent Leonardo to study calculation with an Arab master.

Fibonacci's extended trips to Egypt, Sicily, Greece, and Syria brought him in contact with eastern and Arabic mathematical practices.

He was so impressed with the superiority of the Hindu-Arabic methods of calculation that in 1202 he wrote the Liber Abaci. This book, devoted to arithmetic and elementary algebra, illustrates and advocates Hindu-Arabic notation and at the time helped to introduce these numerals into Europe. Featured are the nine Indian figures--the digits 9, 8, 7, 6, 5, 4, 3, 2, and 1 and the notion that with these 9 digits and the sign 0, any number may be written. Also included in the book are methods of calculation with integers and fractions, computation of square roots and cube roots, and the solution of linear and quadratic equations by false position and by algebraic processes [2].

One of the most interesting problems posed and solved by Fibonacci, which is found in the Liber Abaci, is the following:

> A man put one pair of adult rabbits (of opposite sex) in a certain place entirely surrounded by a wall. Assume that each pair of adult rabbits produce one pair of young (of opposite sex) each month. It takes two months for each pair of young to become adults, at which time they produce their first pair. How many pairs of rabbits are present at the beginning of each month [8]?

Assuming that none of the rabbits die, then a pair is born during the first month, so at the beginning of the second month there are two pairs present. During the second month, the original pair has produced another pair. A month later, the original pair and the firstborn pair have produced new pairs, so now there is a total of five pairs, and so on. The following table illustrates the solution for the first ten months. (The numbers in the table indicate the count at the beginning of each month.)

MONTHS	ADULT PAIRS	1-MONTH-OLD PAIRS	NEWBORN PAIRS	TOTAL PAIRS
1	1	0	0	**1**
2	1	0	1	**2**
3	1	1	1	**3**
4	2	1	2	**5**
5	3	2	3	**8**
6	5	3	5	**13**
7	8	5	8	**21**
8	13	8	13	**34**
9	21	13	21	**55**
10	34	21	34	**89**

If the first term is defined to be 1, then when the above pattern is continued indefinitely, the sequence formed is

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, ...

The importance of the solution to this problem is that it is the Fibonacci sequence.

Definition 2.1. The Fibonacci sequence, defined recursively, is $u_1 = u_2 = 1$, $u_n = u_{n-2} + u_{n-1}$ for natural numbers $n \geq 3$. ∎

A table of the first fifteen Fibonacci numbers will serve as a useful reference.

$u_1 = 1$	$u_6 = 8$	$u_{11} = 89$
$u_2 = 1$	$u_7 = 13$	$u_{12} = 144$
$u_3 = 2$	$u_8 = 21$	$u_{13} = 233$
$u_4 = 3$	$u_9 = 34$	$u_{14} = 377$
$u_5 = 5$	$u_{10} = 55$	$u_{15} = 610$

The Fibonacci numbers form a recursive sequence. Fibonacci solved the rabbit problem; however, he did not write down this recursive rule for the sequence. This rule was not written until 1634 by Albert Girard.

The Fibonacci numbers possess a variety of remarkable properties. One such property follows.

Definition 2.2. The Fibonacci sequence grows at the variable rate $G_k = u_{k+1}/u_k$ for k = 1, 2, 3, ...

We have

$G_1 = u_2/u_1 = 1/1 =$ **1.000**	$G_2 = u_3/u_2 = 2/1 =$ **2.000**
$G_3 = u_4/u_3 = 3/2 =$ **1.500**	$G_4 = u_5/u_4 = 5/3 =$ **1.667**
$G_5 = u_6/u_5 = 8/5 =$ **1.600**	$G_6 = u_7/u_6 = 13/8 =$ **1.625**
$G_7 = u_8/u_7 = 21/13 =$ **1.615**	$G_8 = u_9/u_8 = 34/21 =$ **1.619** ... .

Theorem 2.3. The limit of G_k as $k \to \infty$ is τ, where $\tau = (1 + 5^{1/2})/2 = 1.618034$.... This limit is called the golden ratio [8].

Proof: The recursive formula for the Fibonacci numbers $u_k = u_{k-2} + u_{k-1}$ gives the characteristic equation $x^2 - x - 1 = 0$. By the quadratic formula, the characteristic roots are $\alpha = (1 + 5^{1/2})/2$ and $\beta = (1 - 5^{1/2})/2$. Next, by induction we will show that $u_k = (\alpha^k - \beta^k)/(\alpha - \beta)$ for all natural numbers k.

$$u_1 = \{[(1 + 5^{1/2})/2]^1 - [(1 - 5^{1/2})/2]^1\}/\{(1 + 5^{1/2})/2 - (1 - 5^{1/2})/2\} = 1.$$

$$u_2 = \{[(1 + 5^{1/2})/2]^2 - [(1 - 5^{1/2})/2]^2\}/\{(1 + 5^{1/2})/2 - (1 - 5^{1/2})/2\} = 1.$$

Suppose $u_k = (\alpha^k - \beta^k)/(\alpha - \beta)$.

Now, $(\alpha^{k+1} - \beta^{k+1})/(\alpha - \beta) = (\alpha^{k-1}\alpha^2 - \beta^{k-1}\beta^2)/(\alpha - \beta)$

$$= [\alpha^{k-1}(\alpha + 1) - \beta^{k-1}(\beta + 1)]/[\alpha - \beta]$$

since $\alpha^2 - \alpha - 1 = 0$ and $\beta^2 - \beta - 1 = 0$.

$$= (\alpha^k - \beta^k)/(\alpha - \beta) + (\alpha^{k-1} - \beta^{k-1})/(\alpha - \beta)$$

$$= u_k + u_{k-1}$$

$$= u_{k+1}.$$

Therefore, $\lim_{k\to\infty} G_k = \lim_{k\to\infty} [u_{k+1}/u_k]$

$$= \lim_{k\to\infty} [(\alpha^{k+1} - \beta^{k+1})/(\alpha - \beta)]/[(\alpha^k - \beta^k)/(\alpha - \beta)]$$

$$= \lim_{k\to\infty} [(\alpha^{k+1} - \beta^{k+1})/(\alpha^k - \beta^k)]$$

$$= \lim_{k\to\infty} \{[\alpha^{k+1}/\alpha^k - \beta^{k+1}/\alpha^k]/[\alpha^k/\alpha^k - \beta^k/\alpha^k]\}$$

$$= \alpha \quad \text{since } -1 < \beta < 0$$

$$= (1 + 5^{1/2})/2. \blacksquare$$

This ratio has the property that if one divides the line AB at C so that τ = AB/AC, then AB/AC = AC/CB. This number τ plays an important role in art, for rectangles with sides in the ratio τ:1 (called golden rectangles) are considered to be the most aesthetic. In fact, an entire book (De Divina Proportione by Piero della Francesca) was written about the applications of τ in the work of Leonardo de Vinci.

CHAPTER 3

CANTOR'S TERNARY SET

Georg Cantor is remembered chiefly for founding set theory, one of the greatest achievements of 19th-century mathematics. Cantor was born in St. Petersburg, Russia, but he spent most of his life in Germany. He took a strong interest in the arguments of medieval theologians concerning continuity and the infinite. He studied philosophy, physics, and mathematics in Zurich, Gottingen, and Berlin. Cantor attended the University of Berlin, where he learned higher mathematics from Karl Weierstrass, Ernst Kummer, and Leopold Kronecker. His doctoral thesis was titled "In Mathematics the Art of Asking Questions is More Valuable Than Solving Problems". He joined the faculty at the University of Halle, first as a lecturer, then as an assistant professor, then as a full professor in 1879.

One of his most intriguing discoveries is now known as Cantor's ternary set (or Cantor's discontinuum or Cantor's set).

Definition 3.1. Cantor's ternary set C is the set of points in [0,1] that remain after the "middle third" intervals have been successively removed.■

To construct Cantor's ternary set, begin by removing the middle third of [0,1]. Let C_1 be the points that remain, so

$$C_1 = [0,1/3] \cup [2/3,1].$$

Then remove the middle third of each of the intervals of C_1. Let C_2 be the set that remains, so

$$C_2 = [0,1/9] \cup [2/9,1/3] \cup [2/3,7/9] \cup [8/9,1].$$

Removing the middle thirds again yields

$$C_3 = [0,1/27] \cup [2/27,1/9] \cup [2/9,7/27] \cup [8/27,1/3] \cup$$
$$[2/3,19/27] \cup [20/27,7/9] \cup [8/9,25/27] \cup [26/27,1].$$

Continuing in this manner yields C_4, C_5, ..., C_n, ...

Cantor's ternary set C is the infinite intersection of the C_n's [7].

The following property will be useful in showing that Cantor's ternary set is totally disconnected.

Property 3.2. Cantor's ternary set has measure zero.

Proof: The length of the interval removed from [0,1] to construct C_1 is 1/3. The total length of the intervals removed from [0,1] to construct C_2 is 1/3 + 2/9, to construct C_3 is 1/3 + 2/9 + 4/27, ···, to construct C_n is 1/3 + 2/9 + 4/27 + 8/81 + ··· + $2^{n-1}/3^n$. So, the total length of the intervals removed from [0,1] to construct Cantor's ternary set is the infinite sum of $[2^{n-1}/3^n] = \Sigma_{n=1}^{\infty}(1/2)(2/3)^n$. This is a convergent geometric series with sum 1. Since the total length of the removed intervals is 1 and the length of [0,1] is 1, Cantor's ternary set has measure zero [1].■

A second definition of Cantor's ternary set is presented to provide a different view of the set. This definition offers a better understanding as to which points of [0,1] belong to the set.

Definition 3.3. (Alternative definition of Cantor's ternary set) Cantor's ternary set is the set C of real numbers in [0,1] which have a ternary (base 3) expansion using only the digits 0 and 2 [3].■

In other words, since every real number x in [0,1] can be expressed in base three as $x = \Sigma_{n=1}^{\infty} a_n/3^n$, where a_n = 0, 1, or 2, the set of all x in which $a_n \neq 1$ for all n is Cantor's ternary set [6]. So, it is easy to see that

1/3 is in Cantor's ternary set since

$$1/3 = (0.0222...)_3$$

$$= 0/3 + 2/9 + 2/27 + \cdots + 2/3^n + \cdots$$

$$= (2/9)\Sigma_{n=0}^{\infty}(1/3)^n = 1/3$$

We can find other points in the set. For example,

$$x = (0.002002...)_3$$

$$= \Sigma_{n=1}^{\infty}[2/3^{3n}]$$

$$= 2/3^3 + 2/3^6 + 2/3^9 + \cdots$$

$$= (1/3^3)(2 + 2/3^3 + 2/3^6 + \cdots) \text{ which implies}$$

$$x = (1/3^3)(2 + x) \text{ which implies}$$

$x = 1/13$ is a point in Cantor's ternary set.

Also, we know that 20/27 is in Cantor's ternary set since

$$20/27 = 2/3 + 0/3^2 + 2/3^3 + 0/3^4 + 0/3^5 + \cdots$$

$$= (0.202000...)_3.$$

There are an infinite number of points in Cantor's ternary set. This is seen by noting that $(0.2)_3$, $(0.22)_3$, $(0.222)_3$, $(0.2222)_3$, ... are all in the set. In fact, Cantor's ternary set can be put into a one-to-one correspondence with the points of [0,1]. These are just a few of the interesting facts about Cantor's ternary set.

CHAPTER 4

PRELIMINARIES

Recall Definition 2.1 (page 6) of the Fibonacci sequence. Zekendorf found that under certain conditions that a natural number can be written uniquely as the sum of distinct Fibonacci numbers. Before examining Zekendorf's representation of natural numbers, we need the following theorem.

Theorem 4.1. Every natural number N can be expressed as the sum of distinct Fibonacci numbers $u_{i_1} < u_{i_2} < u_{i_3} < \cdots < u_{i_k}$ where the u_{i_w} $(1 \leq w \leq k)$ are elements of the subset of Fibonacci numbers $\{u_2, u_3, u_4, \ldots\}$. So, $N = u_{i_1} + u_{i_2} + \cdots + u_{i_k}$.

Proof: Note that the Fibonacci numbers $u_{n_1}, u_{n_2}, u_{n_3}, \ldots, u_{n_k}$ used in this proof form a decreasing sequence. Since the Fibonacci sequence $\{u_n\} \to \infty$ as $n \to \infty$,

$$u_{n_1} \leq N < u_{n_1+1} \text{ for some } n_1.$$

Notice that $N - u_{n_1} < u_{n_1+1} - u_{n_1} = u_{n_1-1} < u_{n_1}$.

If $N = u_{n_1}$, then done--otherwise,

$$u_{n_2} \leq N - u_{n_1} < u_{n_2+1} \text{ for some } n_2 \text{ since } N - u_{n_1} \geq 1.$$

The above shows $u_{n_2} < u_{n_1}$, similar reasoning shows $u_{n_3} < u_{n_2}$, and so on.

If $N - u_{n_1} = u_{n_2}$ then done--otherwise,

$$u_{n_3} \leq N - u_{n_1} - u_{n_2} < u_{n_3+1} \text{ for some } n_3 \text{ since } N - u_{n_1} - u_{n_2} \geq 1.$$

Continuing in this manner results in either

1) $N - u_{n_1} - u_{n_2} - \cdots - u_{n_{k-1}} = u_{n_k}$ where $n_k > 2$, so

$N = u_{n_k} + u_{n_{k-1}} + \cdots + u_{n_2} + u_{n_1}$ for $n_k > 2$

$= u_{i_1} + u_{i_2} + \cdots + u_{i_k}$ or

2) $u_{n_k} = 1 \leq N - u_{n_1} - u_{n_2} - \cdots - u_{n_{k-1}} < 2$ which implies

$N - u_{n_1} - u_{n_2} - \cdots - u_{n_{k-1}} = 1$ which implies

$N = 1 + u_{n_{k-1}} + \cdots + u_{n_2} + u_{n_1}$

$= u_{i_1} + u_{i_2} + \cdots + u_{i_k}$. ∎

Zekendorf found that if no consecutive Fibonacci numbers u_n, u_{n+1} are used in the sum, then the representation is unique. For the proof that this representation is unique, see [9]. In summary, we have the following:

Property 4.2. (Zekendorf's representation of natural numbers)

For any given natural number N, there are Fibonacci numbers $u_{i_1}, u_{i_2}, \cdots, u_{i_k}$ taken from the set $\{u_2, u_3, u_4, \ldots\}$ such that the following conditions are satisfied:

Z1. $i_1 < i_2 < \cdots < i_k$;

Z2. $|i_m - i_n| \geq 2$ for $m \neq n$ and $1 \leq m,n \leq k$;

Z3. $N = u_{i_1} + u_{i_2} + \cdots + u_{i_k}$ and the representation is unique.

That is, if $N = u_{j_1} + u_{j_2} + \cdots + u_{j_h}$ where $j_1 < j_2 < \cdots < j_h$ and $|j_m - j_n| \geq 2$ for $m \neq n$, $1 \leq m,n \leq h$, then $k = h$ and $i_m = j_m$ for $1 \leq m \leq k$ [9].∎

Zekendorf's representation of 108, for example, is

$$\begin{aligned} 108 &= 1 + 5 + 13 + 89 \\ &= u_2 + u_5 + u_7 + u_{11} \end{aligned}$$

Notice that the above three conditions are satisfied; therefore, the above representation of 108 is unique.

Some preliminary information about Cantor's ternary set and basic set properties are important for understanding the relationship between the Fibonacci sequence and Cantor's ternary set. We will explore the concept of a metric space, the idea of compactness in the set of real numbers, the concept of a perfect set, and the idea of a totally disconnected set.

Recall that a metric space is a set S together with a distance function d which satisfies the following properties:

(1) $d(x,y) \geq 0$ for all points x and y in S, $d(x,y) = 0$ if and only if $x = y$,

(2) $d(x,y) = d(y,x)$ for all points x and y in S, and

(3) $d(x,y) + d(y,z) \geq d(x,z)$ for all points x, y, and z in S.

The distance function d used with Cantor's ternary set is defined by $d(x,y) = |x-y|$ for all x and y in Cantor's ternary set.

Property 4.3. Cantor's ternary set C with the distance function d is a metric space.

Proof: The set of real numbers **R** with the distance function $d(x,y) = |x-y|$ is a metric space. Since Cantor's ternary set with distance function d is a subset of **R**, it is a metric space.$\blacksquare$

A subset of the real numbers with the distance function d is compact if it is closed and bounded. This is the Heine-Borel theorem. This proof can be found in many topology books, specifically see [3].

Property 4.4. Cantor's ternary set C with distance function d is compact.

Proof: Since C is the intersection of the closed sets C_n, it is closed. Clearly, C is a subset of [0,1]; thus, it is bounded. Therefore, by the Heine-Borel theorem, Cantor's ternary set with distance function d is compact.■

Recall that a space S is perfect if every point in S is a limit point of S. A point p is defined to be a limit point of a set S if each open disc centered at p with radius ε contains a point of S other than p.

Property 4.5. Cantor's ternary set C with distance function d is perfect.

Proof: Let x be an element of C and ε a positive number. Let N be a positive integer for which $2/3^N < \varepsilon$. Since $x = (0.x_1x_2x_3...)_3$ in C has a ternary expansion where each x_n is 0 or 2, we let $y = (0.y_1y_2y_3...)_3$ be the real number having the indicated ternary expansion:

$y_n = x_n$ for $n \neq N$ and y_N differing from x_N as follows:

y_N is 0 if x_N is 2, and y_N is 2 if x_N is 0.

Then y is an element of C, and $|x - y| = 2/3^N < \varepsilon$. Thus, y is an element of C within distance ε of x, so x is a limit point of C. Since x is any point of C, every point of C is a limit point of C. Therefore, Cantor's ternary set is perfect [3]. $\blacksquare$

A set S is totally disconnected provided that every component of S consists of a single point. A component S_0 of a set S is a maximal connected subset. In other words, S_0 is a connected subset of S such that there is no connected set in S containing S_0, other than S_0 itself [5]. A connected set S is a set that cannot be expressed as the union of two disjoint, non-empty open sets. We need to explore the concept of connected sets on the real line **R** or on any subinterval of **R**. We must show that 1) intervals on **R** are connected sets and 2) the only connected sets on **R** are precisely the intervals.

Lemma 4.6. The real line **R** as well as any subinterval of **R** with the usual topology is connected.

Proof: It will suffice to show this for the real line **R**, then it will follow for any subinterval of **R**. Suppose **R** is disconnected. Then **R** = A U B for some disjoint, non-empty open sets A and B of **R**. Since A = **R**\B (all points in **R** excluding the points of B) and B = **R**\A, A and B are closed as well as open. Consider points a and b where a < b and a is in A and b is in B. Let $A^* = A \cap [a,b]$. Now A^* is a closed and bounded subset of **R** which implies that it is compact and contains its least upper bound g. Note that $g \neq b$ since A and B are disjoint. So, $g < b$. Since A contains no points of (g,b], (g,b] is a subset of B. This implies g is in the closure of B. However, B is closed, so g is in B. Thus, g is in both A and B. This contradicts the assumption that A and B are disjoint. Therefore, **R** is connected. Following similar reasoning for a subinterval of **R** represented as the union of two disjoint, non-empty open sets will result in the same outcome [3].∎

Theorem 4.7. The connected subsets of R are precisely the intervals.

Proof: By Lemma 4.6, every interval of **R** is connected. So, it remains only to be proved that a subset D of **R** which is not an interval must be disconnected. Let D be a subset of **R** that is not an interval. Then there are members s and t in D and a real number w with $s < w < t$ for which w is not in D. Then there exists open sets $U = (-\infty, w)$ and $V = (w, \infty)$ satisfying the following properties:

D1. s is in $U \cap D$, so $U \cap D \neq \{\ \}$;

D2. t is in $V \cap D$, so $V \cap D \neq \{\ \}$;

D3. $U \cap V = \{\ \}$, so $U \cap V \cap D = \{\ \}$; and

D4. D is a subset of $U \cup V$.

So, by the definition of disconnected sets, D is disconnected. Hence, every connected subset of **R** must be an interval [3].∎

Property 4.8. Cantor's ternary set C with distance function d is totally disconnected.

Proof: Since Cantor's ternary set C has measure zero, it cannot contain any proper intervals. If C contained intervals, it would have measure greater than zero. By Theorem 4.7, the connected subsets of **R** are precisely the intervals. Therefore, the components of C consist of single points.∎

So, Cantor's ternary set C with distance function d defined by $d(x,y) = |x-y|$ for all points x and y in C is a compact, perfect, totally disconnected metric space. These key facts about C were presented to provide the framework for the relationship between the Fibonacci sequence and Cantor's ternary set. It will be shown that when viewed properly, Cantor's ternary set and the closure of the range of a function that uses the Fibonacci numbers are homeomorphic.

Now, we give the conditions necessary for a set to be homeomorphic to Cantor's ternary set.

Characterization of Cantor's ternary set [4]. A topological space T is homeomorphic to Cantor's ternary set if and only if

T1. T is a metric space;

T2. T is compact;

T3. T is perfect; and

T4. T is totally disconnected.∎

We have explored the Fibonacci sequence and some of its properties. The construction of Cantor's ternary set C was presented, and it was shown that Cantor's ternary set with distance function d defined by $d(x,y) = |x-y|$ for all points x and y in C is a perfect, compact, totally disconnected metric space. It was shown that every natural number can be expressed as the sum of distinct Fibonacci numbers. Zekendorf found that this representation is unique if and only if two consecutive Fibonacci numbers are not used. This uniqueness will prove essential for defining the function that is the key to the relationship between the Fibonacci sequence and Cantor's ternary set.

A function F that maps the natural numbers into [0,0.603] using Zekendorf's representation will be defined. Several properties of F and the Fibonacci numbers will be proved and used to show that F is an injection from the set of natural numbers into [0,0.603]. Cl(F) with distance function d defined by $d(x,y) = |x-y|$ for all x and y in cl(F) will be shown to be a perfect, compact, totally disconnected metric space. It then follows by the characterization of Cantor's ternary set that cl(F) is homeomorphic to Cantor's ternary set.

CHAPTER 5

THE FUNCTION F

Now we will define the function F that maps the set **N** of natural numbers into the interval [0,0.603]. The choice of 0.603 will be explained later. This function F is based on Zekendorf's representation of natural numbers. In the next chapter, we will prove that cl(F) is homeomorphic to Cantor's ternary set.

Definition 5.1. Let **N** be the set of natural numbers, x be an element of **N**, and $x = u_{i_1} + u_{i_2} + \cdots + u_{i_k}$ be Zekendorf's representation of x. Define the function F by

$$F(x) = 1/(2u_{i_1}) + 1/(2^2u_{i_2}) + 1/(2^3u_{i_3}) + \cdots + 1/(2^ku_{i_k}) \text{ for all } x \text{ in } \mathbf{N}.$$

Note that F is well-defined because Zekendorf's representation of x is unique.∎

To illustrate how to find F(x) for a given natural number x, consider x = 120. Find Zekendorf's representation of 120 by taking the Fibonacci numbers (with indices that differ by at least two) that sum to 120.

$$120 = 2 + 8 + 21 + 89$$

$$= u_3 + u_6 + u_8 + u_{11}$$

$$F(120) = 1/(2\cdot 2) + 1/(4\cdot 8) + 1/(8\cdot 21) + 1/(16\cdot 89) = 0.2879046$$

The following table displays the computation of some values of F(x) and is provided for reference.

x	Zekendorf's Rep.	F(x)	
1	1	1/(2·1)	= .500
2	2	1/(2·2)	= .250
3	3	1/(2·3)	= .167
4	1 + 3	1/(2·1) + 1/(4·3)	= .583
5	5	1/(2·5)	= .100
6	1 + 5	1/(2·1) + 1/(4·5)	= .550
7	2 + 5	1/(2·2) + 1/(4·5)	= .300
8	8	1/(2·8)	= .063
9	1 + 8	1/(2·1) + 1/(4·8)	= .531
10	2 + 8	1/(2·2) + 1/(4·8)	= .281

Next, we need to examine the series $\Sigma_{m=1}^{\infty}[1/(2^m u_{2m})]$. We will show that this series is convergent by comparing it to the convergent geometric series $\Sigma_{n=1}^{\infty} (½)^n$. Let $a_n = (½)^n$ and $b_n = (½)^n(1/u_{2n})$ for n in **N**. It is obvious that $(½)^n(1/u_{2n}) < (½)^n$ for $n > 1$ since $0 < 1/u_{2n} < 1$. Therefore, by the comparison test, this series is convergent.

Theorem 5.2. The sum $c = \Sigma_{m=1}^{\infty}[1/(2^m u_{2m})] = 0.6026368274$ (to 10 decimal places) is an upper bound of F(**N**).

Proof: It was just shown that this series is convergent, so c is finite. To see that this is an upper bound of F(**N**), examine the terms $1/(2u_2)$, $1/(4u_4)$, $1/(8u_6)$, etc.. Notice that since u_1 is not used in Zekendorf's representation, u_2 is the smallest value that can occupy u_{i_1} in the first term of $F(x) = \Sigma_{m=1}^{\infty}[1/(2^m u_{i_m})]$ which implies $1/(2u_2) \geq 1/(2u_{i_1})$. Next, u_4 is the smallest value that can occupy u_{i_2} in the second term (since Zekendorf's representation requires that the indices of the Fibonacci numbers used differ by at least two), thus making $1/(4u_4) \geq 1/(4u_{i_2})$. Continuing in this manner shows that $c \geq F(x) \geq 0$ for all x in **N**. ■

The results of the next two lemmas are necessary for proving Lemma 5.5, which will be used to prove that F is a one-to-one function.

Lemma 5.3. Let u_i for i = 1, 2, 3,... be the Fibonacci sequence. If $i > j > 2$, then $1/u_i \leq 2/(3u_j)$.

Proof: Since $i > j$ implies $i-1 \geq j$, we have

$$\begin{aligned} 3u_i &= 3(u_{i-1} + u_{i-2}) \\ &\geq 3(u_j + u_{j-1}) \end{aligned}$$

$= 3u_j + 3u_{j-1}.$

Hence, $3u_i \geq 3u_j + 3u_{j-1}$. **(1)**

Also, $u_j = u_{j-1} + u_{j-2} \leq 2u_{j-1}$. **(2)**

Multiplying (2) by 3/2 yields $(3/2)u_j \leq 3u_{j-1}$ **(3)**

By (1) and (3), $3u_i \geq 3u_j + (3/2)u_j = (9/2)u_j$

This results in $1/u_i \leq 2/(3u_j)$. ∎

Lemma 5.4. Let x be an element of **N**. If Zekendorf's representation of x is $u_{i_1} + u_{i_2} + \cdots + u_{i_k}$, then

$$F(x) < (4/3)[1/(2u_{i_1})] = 2/(3u_{i_1}).$$

Proof: (using induction) It is easily seen that

$$u_{i_2} = u_{i_2-1} + u_{i_2-2} \geq 2u_{i_2-2} \geq 2u_{i_1}.$$

If $u_{i_m} \geq 2^{m-1}u_{i_1}$, then

$$u_{i_{m+1}} = u_{i_{m+1}-1} + u_{i_{m+1}-2} \geq 2u_{i_{m+1}-2} \geq 2u_{i_m} \geq 2^m u_{i_1}.$$

So, $F(x) = 1/(2u_{i_1}) + 1/(2^2u_{i_2}) + \cdots + 1/(2^ku_{i_k})$

$\leq 1/(2u_{i_1}) + 1/(2^3u_{i_1}) + \cdots + 1/(2^{2k-1}u_{i_1})$ by induction hypothesis

$< [1/(2u_{i_1})](1 + 1/2^2 + \cdots + 1/2^{2k-2} + \cdots)$

$= [1/(2u_{i_1})][1/(1-¼)]$

$= (4/3)[1/(2u_{i_1})]$. Therefore,

$F(x) < 2/(3u_{i_1})$. ∎

Lemma 5.5. Let x_1, x_2 be in **N** and $x_1 = u_{i_1} + \cdots + u_{i_k}$ and $x_2 = u_{j_1} + \cdots + u_{j_h}$ be Zekendorf's representations of x_1 and x_2, respectively. If $F(x_1) < F(x_2)$, then $F(x_2) - F(x_1) > (0.1)[1/(2^h u_{j_h})]$.

Proof: We discuss the following three cases.

Case 1: $u_{i_m} = u_{j_m}$ for $m = 1, 2, \ldots, k$. Then $h \geq k + 1$, otherwise $F(x_1) = F(x_2)$.

It is easily seen that
$$\begin{aligned} F(x_2) - F(x_1) &= 1/(2^{k+1}u_{j_{k+1}}) + \cdots + 1/(2^h u_{j_h}) \\ &\geq 1/(2^h u_{j_h}) \\ &> (0.1)[1/(2^h u_{j_h})]. \end{aligned}$$

Case 2: There is a natural number $2 \leq g \leq k$ such that $u_{i_m} = u_{j_m}$ for $m = 1, 2, \ldots, g-1$, but $u_{i_g} \neq u_{j_g}$. There are two subcases:

(i) $i_g > j_g$. Similar to Lemma 5.4, we can prove by induction that $u_{i_{g+s}} \geq 2^s u_{i_g}$. Then

$$\begin{aligned} F(x_1) - \sum_{m=1}^{g-1}[1/(2^m u_{i_m})] &= 1/(2^g u_{i_g}) + 1/(2^{g+1}u_{i_{g+1}}) + \cdots + 1/(2^k u_{i_k}) \\ &\leq 1/(2^g u_{i_g}) + 1/(2^{g+2}u_{i_g}) + \cdots + 1/(2^{2k-g}u_{i_g}) \\ &< [1/(2^g u_{i_g})](1 + 1/2^2 + \cdots + 1/2^{2k-2g} + \cdots) \\ &= [1/(2^g u_{i_g})][1/(1 - ¼)] \\ &= (4/3)[1/(2^g u_{i_g})] \end{aligned}$$

Since $i_g > j_g$, by Lemma 5.3, we have $(4/3)[1/(2^g u_{i_g})] \leq (8/9)[1/(2^g u_{j_g})]$.

So, $F(x_1) - \sum_{m=1}^{g-1}[1/(2^m u_{i_m})] < (8/9)[1/(2^g u_{j_g})]$. **(4)**

Therefore, $F(x_2) - F(x_1) \geq 1/(2u_{j_1}) + \cdots + 1/(2^g u_{j_g}) - F(x_1)$

$= 1/(2^g u_{j_g}) - \{F(x_1) - \Sigma_{m=1}^{g-1}[1/(2^m u_{i_m})]\}$

$> 1/(2^g u_{j_g}) - (8/9)[1/(2^g u_{j_g})]$ by (4)

$= (1/9)[1/(2^g u_{j_g})]$

$> (0.1)[1/(2^g u_{j_g})]$

$\geq (0.1)[1/(2^h u_{j_h})]$.

(ii) $i_g < j_g$. This case is impossible because consideration of $F(x_2) - \Sigma_{m=1}^{g-1}[1/(2^m u_{j_m})]$ in (i) instead of $F(x_1) - \Sigma_{m=1}^{g-1}[1/(2^m u_{i_m})]$ leads to $F(x_1) - F(x_2) > 0$, which is a contradiction of $F(x_2) > F(x_1)$.

Case 3: $u_{i_1} \neq u_{j_1}$. There are two subcases:

(i) $i_1 > j_1$.

$F(x_1) < (4/3)[1/(2u_{i_1})]$ by Lemma 5.4

$\leq (8/9)[1/(2u_{j_1})]$ by Lemma 5.3.

So, $F(x_1) < (8/9)[1/(2u_{j_1})]$. **(5)**

Then $F(x_2) - F(x_1) \geq 1/(2u_{j_1}) - F(x_1)$

$> 1/(2u_{j_1}) - (8/9)[1/(2u_{j_1})]$ by (5)

$= (1/9)[1/(2u_{j_1})]$

$> (0.1)[1/(2u_{j_1})]$

$\geq (0.1)[1/(2^h u_{j_h})]$.

(ii) $i_1 < j_1$.

$F(x_2) - F(x_1) \leq F(x_2) - 1/(2u_{i_1})$

$< (4/3)[1/(2u_{j_1})] - 1/(2u_{i_1})$ by Lemma 5.4

$\leq (4/3)[1/(3u_{i_1})] - 1/(2u_{i_1})$ by Lemma 5.3

$= 4/(9u_{i_1}) - 1/(2u_{i_1})$

$= -1/(18u_{i_1})$

< 0.

Hence, $F(x_2) - F(x_1) < 0$. This contradicts the hypothesis that $F(x_1) < F(x_2)$. Therefore, this case is impossible.■

Recall that $c = \Sigma_{m=1}^{\infty}[1/(2^m u_{2m})] = 0.6026368274$ to 10 decimal places is an upper bound for $F(\mathbf{N})$. Next, we give an important property of the function F.

Theorem 5.6. The function F is an injection from **N** into [0, c].

Proof: Suppose that there are x_1, x_2 in **N** such that $F(x_1) = F(x_2)$. We will prove that this implies $x_1 = x_2$. Assume that $x_1 \neq x_2$. Let $x_1 = u_{i_1} + \cdots + u_{i_k}$ and $x_2 = u_{j_1} + \cdots + u_{j_h}$ be Zekendorf's representations of x_1 and x_2, respectively. Since Zekendorf's representation of a natural number is unique, $x_1 \neq x_2$ implies there is a natural number g such that $u_{i_g} \neq u_{j_g}$ or without loss of generality $h > k$. Using the method in the proof of Lemma 5.5, we know that $i_g > j_g$ implies $F(x_2) > F(x_1)$, $i_g < j_g$ implies $F(x_1) > F(x_2)$, and $h > k$ implies $F(x_2) > F(x_1)$. These are contradictory to the condition $F(x_1) = F(x_2)$. ∎

CHAPTER 6

THE MAIN RESULT

Now we prove the main result which is that cl(F) is homeomorphic to Cantor's ternary set. It follows from the characterization of Cantor's ternary set that it suffices to show that cl(F) with distance function d defined by d(x,y) = |x-y| for all x and y in cl(F) is a perfect, compact, totally disconnected metric space.

Theorem 6.1. Cl(F) with distance function d is homeomorphic to Cantor's ternary set.

Proof: It has already been shown that cl(F) is a subset of the interval [0, c]. (See Theorem 5.2 for explanation of c). As previously discussed, it then follows that cl(F) with d(x,y) = |x-y| for x and y in cl(F) is a metric space.

Cl(F) is a bounded subset of **R**. Since cl(F) is the intersection of all closed sets that contain F(**N**), it is closed. Since cl(F) is closed and bounded on **R**, the Heine-Borel Theorem states that it is a compact set [3].

To show that cl(F) is a perfect set, we have to prove that every point in F(**N**) is a limit point of F(**N**). Suppose p is an element of F(**N**). Then there is an x in **N** such that p = F(x). If $x = u_{i_1} + u_{i_2} + \cdots + u_{i_k}$ is Zekendorf's representation of x, then consider the sequence x_r where (r = 1, 2, 3, ...) defined as follows:

$$\begin{aligned} x_r &= (u_{i_1} + u_{i_2} + \cdots + u_{i_k}) + u_{i_k+r+1} \\ &= x + u_{i_k+r+1}. \end{aligned}$$

It is easily seen that x_r is in **N** and the above expression is Zekendorf's representation of x_r since $i_k+r+1 \geq i_k+2$. A direct computation gives

$$\begin{aligned} F(x_r) &= F(x) + 1/(2^{k+1}u_{i_k+r+1}) \\ &= p + 1/(2^{k+1}u_{i_k+r+1}) \end{aligned}$$

Surely, $F(x_r)$ is an element of F(**N**) since x_r is an element of **N**. Therefore, $\lim_{r\to\infty} F(x_r) = p$ since $1/(2^{k+1}u_{i_k+r+1}) \to 0$. Thus, p is a limit point of F(**N**) and since p is any point of F(**N**), every point of F(**N**) is a limit point of F(**N**). So, cl(F) is a perfect set.

To show that cl(F) is totally disconnected, we have to prove that each connected component of cl(F) is a single point. Assume that there exists a connected component S of cl(F) which is not a single point. Since S is a connected set and connected sets in the space of real numbers are intervals by Theorem 4.7, we know that there are two real numbers u, v such that u < v and the interval (u, v) is a subset of S. Let

q be an element of (u, v). Then q is a limit point of $F(\mathbf{N})$, so there exists a sequence x_r in $\mathbf{N}$ such that $F(x_r)$ converges to q. Hence, there exists x_{r_0} in $\mathbf{N}$ such that $F(x_{r_0})$ is a point in (u, v). Using Zekendorf's representation, $x_{r_0} = u_{j_1} + u_{j_2} + \cdots + u_{j_h}$. Consider $I = (F(x_{r_0}) - (0.1)[1/(2^h u_{j_h})], F(x_{r_0}))$. We have $F(\mathbf{N}) \cap I = \{\ \}$ since x in $\mathbf{N}$ and $F(x) < F(x_{r_0})$ imply $F(x)$ is not in I by Lemma 5.5. Notice that I is an open set and that $cl(F) \cap I = \{\ \}$. Hence, $S \cap I = \{\ \}$ and $(u, v) \cap I = \{\ \}$. This is impossible because letting $c = \max\{u, F(x_{r_0})-(0.1)(1/2^h u_{j_h})\}$, we have $(c, F(x_{r_0}))$ is a subset of $(u, v) \cap I$. This contradiction proves that S must be a single point.■

CHAPTER 7

GAPS IN THE CLOSURE OF F(N)

In the construction of Cantor's ternary set C, the intervals of [0,1] that are removed leave behind gaps between the points of C. Likewise, there are gaps in [0,0.603] with respect to cl(F). As was previously shown, the gap between $F(x_1)$ and $F(x_2)$, where $F(x_1) < F(x_2)$ for natural numbers $x_1 = u_{i_1} + u_{i_2} + \cdots + u_{i_k}$ and $x_2 = u_{j_1} + u_{j_2} + \cdots + u_{j_h}$, is greater than $(0.1)[1/(2^h u_{j_h})]$. These gaps will be put into different classes, and some of the gap lengths will be calculated. All gap lengths are accurate to eight decimal places.

Definition 7.1. A **class 1 gap** is the interval G = (a, b), with b = F(x) where the Zekendorf's representation of x has one term ($x = u_i$) and a is the closest point of cl(F) to b with a < b. Note $G \cap cl(F) = \{\ \}$.■

Before we can compute the length of the gap to the left of any point b, where $b = F(x) = 1/(2u_{i_1})$, we need to know how to find the closest point to b, which we call a, where $a < b$. The method for computing this point a for a class 1 gap will be presented as the following theorem.

Theorem 7.2. The point a closest to $b = F(x)$ with $x = u_{i_1}$ and $a < b$ when figuring a class 1 gap is defined as follows:

$$a = 1/(2u_{i_1+1}) + 1/(2^2u_{i_1+3}) + 1/(2^3u_{i_1+5}) + \cdots + 1/(2^nu_{i_1+(2n-1)}) + \cdots$$

Proof: We must show $t \le a < b$ for all t in cl(F) where $t < b$. First, we will show that $a < b$. We need to show that

$$1/(2^nu_{i_1+(2n-1)}) < 1/(3^nu_{i_1}) \text{ for all natural numbers n.} \qquad \textbf{(1)}$$

Recall the results of Lemma 5.3, which is $1/u_i \le 2/(3u_j)$, where $i > j > 2$. The equality holds only when $u_i = 3$ and $u_j = 2$, so this implies that $1/(u_{i_1+1}) = 2/(3u_{i_1})$ only when $u_{i_1+1} = 3$ and $u_{i_1} = 2$. So, we will prove inequality (1) for all other possibilities of u_{i_1}. We will prove (1) by induction.

For $n = 1$, we have $1/(2u_{i_1+1}) < 1/(3u_{i_1})$.

This implies $1/(u_{i_1+1}) < 2/(3u_{i_1})$, which is true by Lemma 5.3.

Now, assume true for $n = k$: $1/(2^ku_{i_1+(2k-1)}) < 1/(3^ku_{i_1})$.

Now, $1/(2^{k+1}u_{i_1+(2(k+1)-1)}) = 1/(2^{k+1}u_{i_1+(2k+1)})$

$< 1/(2^{k+1}(3/2)u_{i_1+2k})$ by Lemma 5.3

$= 1/(3\cdot 2^ku_{i_1+2k})$

$< 1/(3\cdot 2^k(3/2)u_{i_1+(2k-1)})$ by Lemma 5.3

$= 2/(3^2 \cdot 2^k u_{i_1+(2k-1)})$

$< 2/(9 \cdot 3^k u_{i_1})$ by induction hypothesis

$= 2/(3 \cdot 3^{k+1} u_{i_1})$

$< 1/(3^{k+1} u_{i_1})$

Therefore, $1/(2^n u_{i_1+(2n-1)}) < 1/(3^n u_{i_1})$ for all natural numbers n.

So by (1), $a = \Sigma_{n=1}^{\infty}[1/(2^n u_{i_1+(2n-1)})] < \Sigma_{n=1}^{\infty}[1/(3^n u_{i_1})]$. **(2)**

Since the sum on the right of the inequality in (2) equals

$(1/u_{i_1})[1/3 + 1/3^2 + 1/3^3 + \cdots] = (1/u_{i_1})[(1/3)/[1-(1/3)]] = 1/(2u_{i_1})$,

$a < 1/(2u_{i_1})$, which implies $a < b$.

Now, we must show that for t in cl(F), $a \geq t$ for all $t < b$. Recall a is defined as $1/(2u_{i_1+1}) + 1/(2^2 u_{i_1+3}) + \cdots + 1/(2^n u_{i_1+(2n-1)}) + \cdots$. The Fibonacci number that is used in the first term of the expression for a must be greater than u_{i_1}; otherwise, $a > b$. So, to maximize the first term, the Fibonacci number used must be the smallest one allowed (this is u_{i_1+1}). To maximize the second term, the smallest Fibonacci number allowed by Zekendorf's representation is u_{i_1+3}. Following in this manner, term by term, will maximize this sum, thus making $a \geq t$ for all t in cl(F) where $t < b$. Therefore, the point a, as defined above is the closest point to b where $a < b$. ∎

The values of b in this class are F(1) = 1/2, F(2) = 1/(2·2) = 1/4, F(3) = 1/(2·3) = 1/6, F(5) = 1/(2·5) = 1/10, F(8), F(13), F(21), etc. since Zekendorf's representation of x requires only one term. We will investigate the nature of these gaps by computing the lengths of some gaps. First, the gap length to the left of 1/2, which is F(1), will be figured. Notice $a = 1/(2\cdot 2) + 1/(2^2\cdot 5) + 1/(2^3\cdot 13) + 1/(2^4\cdot 34) + 1/(2^5\cdot 89) + 1/(2^6\cdot 233) + 1/(2^7\cdot 610) + 1/(2^8\cdot 1597) + 1/(2^9\cdot 4181) + 1/(2^{10}\cdot 10946) + 1/(2^{11}\cdot 28657) + \cdots = 0.31188712$ (accurate to eight decimal places). The sum of the remaining terms (not shown) is less than $1/(2^{12}\cdot 2^{16}) + 1/(2^{13}\cdot 2^{17}) + 1/(2^{14}\cdot 2^{18}) + \cdots = (1/2^{28})/(1-¼) < 5\cdot 10^{-9}$. Thus, we have accuracy to eight decimal places. Similar calculations have been performed for the tails of the subsequent "gap" series. These are not included. So, the length of the gap to the left of F(1) is 0.50000000 - 0.31188712 = 0.18811288 (accurate to eight decimal places).

Now, we will compute the length of the gap to the left of 1/4, which is F(2). Notice F(3) = 1/(2·3) = 1/6. Notice that $a = 1/(2\cdot 3) + 1/(2^2\cdot 8) + 1/(2^3\cdot 21) + 1/(2^4\cdot 55) + 1/(2^5\cdot 144) + 1/(2^6\cdot 377) + 1/(2^7\cdot 987) + 1/(2^8\cdot 2584) + 1/(2^9\cdot 6765) + 1/(2^{10}\cdot 17711) + 1/(2^{11}\cdot 46368) + \cdots = 0.20527365$. Therefore, the length of the gap to the left of F(2) is 0.25000000 - 0.20527365 = 0.04472635.

Now, we will figure the length of the gap to the left of 1/6, which is F(3). Note that F(5) = 1/(2·5) = 1/10. So, a = $1/(2\cdot5) + 1/(2^2\cdot13) + 1/(2^3\cdot34) + 1/(2^4\cdot89) + 1/(2^5\cdot233) + 1/(2^6\cdot610) + 1/(2^7\cdot1597) + 1/(2^8\cdot4181) + 1/(2^9\cdot10946) + 1/(2^{10}\cdot28657) + \cdots = 0.12377526$. This makes the gap length to the left of 1/6 = 0.16666666 - 0.12377526 = 0.04289140.

Some additional gap lengths were figured, but the calculations are not shown. The gap length to the left of 1/10, which is F(5), was computed and found to be 0.02278603. Finally, the gap length to the left of 1/16, which is F(8), was found to be 0.01494970.

The method for calculating a, the point closest to b where a < b, as used in figuring a class 1 gap was proved. However, the method for finding the closest point to b in the classes of gaps to follow has not been proven. The method will be offered as a conjecture.

Conjecture 7.3. The point a closest to b = F(x) where a < b and $x = u_{i_1} + u_{i_2} + \cdots + u_{i_k}$ when figuring a class k gap where $k \geq 2$ is defined as follows: $a = 1/(2u_{i_1}) + 1/(2^2u_{i_2}) + 1/(2^3u_{i_3}) + \cdots + 1/(2^{k-1}u_{i_{k-1}})$

$$+\ 1/(2^k u_{i_{k+1}}) + 1/(2^{k+1}u_{i_{k+3}}) + 1/(2^{k+2}u_{i_{k+5}}) + \cdots.\blacksquare$$

Definition 7.4. A **class 2 gap** is the interval $G = (a, b)$, with $b = F(x)$ where the Zekendorf's representation of x has two terms ($x = u_{i_1} + u_{i_2}$) and a is the closest point of $cl(F)$ to b with $a < b$. Note $G \cap cl(F) = \{\ \}$.■

The values of b in this class are F(4), F(6), F(7), F(9), F(10), F(11), F(14), etc. since $4 = u_{i_1} + u_{i_2} = 1 + 3$, $6 = 1 + 5$, $7 = 2 + 5$, $9 = 1 + 8$, $10 = 2 + 8$, etc.. First, we will figure the length of the gap to the left of 7/12, which is F(4). Notice that $F(6) = 1/(2\cdot 1) + 1/(2^2\cdot 5) = 11/20$. Notice $a = 1/(2\cdot 1) + 1/(2^2\cdot 5) + 1/(2^3\cdot 13) + 1/(2^4\cdot 34) + 1/(2^5\cdot 89) + 1/(2^6\cdot 233) + 1/(2^7\cdot 610) + 1/(2^8\cdot 1597) + 1/(2^9\cdot 4181) + 1/(2^{10}\cdot 10946) + 1/(2^{11}\cdot 28657) + \cdots = 0.56188763$. So, the length of the gap to the left of 7/12 is $0.58333333 - 0.56188763 = 0.02144570$.

Next, we will compute the length of the gap to the left of 11/20, which is F(6). So, $a = 1/(2\cdot 1) + 1/(2^2\cdot 8) + 1/(2^3\cdot 21) + 1/(2^4\cdot 55) + 1/(2^5\cdot 144) + 1/(2^6\cdot 377) + 1/(2^7\cdot 987) + 1/(2^8\cdot 2584) + 1/(2^9\cdot 6765) + 1/(2^{10}\cdot 17711) + 1/(2^{11}\cdot 46368) + \cdots = 0.53860698$. Thus, the gap length to the left of F(6) is $0.55000000 - 0.53860698$, which equals 0.01139302.

We will calculate another gap length, this one to the left of 3/10, which is F(7). Here, $a = 1/(2\cdot 2) + 1/(2^2\cdot 8) + 1/(2^3\cdot 21) + 1/(2^4\cdot 55) + 1/(2^5\cdot 144) + 1/(2^6\cdot 377) + 1/(2^7\cdot 987) + 1/(2^8\cdot 2584) + 1/(2^9\cdot 6765) +$

$1/(2^{10} \cdot 17711) + 1/(2^{11} \cdot 46368) + \cdots = 0.28860698$. So, the gap to the left of F(7) is 0.30000000 - 0.28860698 = 0.01139302.

A few gap lengths were figured in class 2. Now, we will define a class 3 gap and figure the lengths of the gaps to the left of some of the gaps in this class.

Definition 7.5. A **class 3 gap** is the interval G = (a, b), with b = F(x) where the Zekendorf's representation of x has three terms ($x = u_{i_1} + u_{i_2} + u_{i_3}$) and a is the closest point of cl(F) to b with a < b. Note $G \cap cl(F) = \{\ \}$.∎

The values of b in this class are F(12), F(17), F(19), F(20), F(25), F(27), F(28), etc. since 12 = 1 + 3 + 8, 17 = 1 + 3 + 13, 19 = 1 + 5 + 13, etc.. Consider the gap to the left of 115/192, which is F(12). Notice that $a = 1/(2 \cdot 1) + 1/(2^2 \cdot 3) + 1/(2^3 \cdot 13) + 1/(2^4 \cdot 34) + 1/(2^5 \cdot 89) + 1/(2^6 \cdot 233) + 1/(2^7 \cdot 610) + 1/(2^8 \cdot 1597) + 1/(2^9 \cdot 4181) + 1/(2^{10} \cdot 10946) + \cdots = 0.59522096$. So, the length of the gap to the left of 115/192 is 0.59895833 - 0.59522096 = 0.00373737.

Now, $F(17) = 1/(2 \cdot 1) + 1/(4 \cdot 3) + 1/(8 \cdot 13) = 185/312$. To find the length of the gap to the left of 185/312, use $a = 1/(2 \cdot 1) + 1/(2^2 \cdot 3) + 1/(2^3 \cdot 21) + 1/(2^4 \cdot 55) + 1/(2^5 \cdot 144) + 1/(2^6 \cdot 377) + 1/(2^7 \cdot 987) + 1/(2^8 \cdot 2584)$

$+ 1/(2^9 \cdot 6765) + 1/(2^{10} \cdot 17711) + 1/(2^{11} \cdot 46368) + \cdots = 0.59069031.$

Thus, the gap length to the left of 185/312 is 0.59294871 - 0.59069031 = 0.00225840.

The first three classes of gaps were defined. Now, we define a class in general i.e. the n^{th} class.

Definition 7.6. A **class n gap** is the interval G = (a, b), with b = F(x) where the Zekendorf's representation of x has n terms ($x = u_{i_1} + u_{i_2} + \cdots + u_{i_n}$) and a is the closest point of cl(F) to b with a < b. Note $G \cap cl(F) = \{\ \}$. ∎

Clearly, there are an infinite number of classes each with an infinite number of gaps. Some of the gaps of the first three classes were computed. More computation might reveal a pattern for the gap lengths within a specific class.

CHAPTER 8

CONCLUSION AND SUGGESTIONS FOR FUTURE RESEARCH

The relationship between the Fibonacci sequence and Cantor's ternary set was found using the function F which utilizes Zekendorf's representation of natural numbers. Cl(F) with distance function d defined by $d(x,y) = |x-y|$ for all x and y in cl(F) was shown to be homeomorphic to Cantor's ternary set. This then means that there exists a one-to-one function H(x) from cl(F) onto Cantor's ternary set, where H(x) and $H^{-1}(x)$ are both continuous. Explicitly stating H seems to be a difficult task.

Several functions were examined before choosing the function F defined by $F(x) = 1/(2u_{i_1}) + 1/(2^2u_{i_2}) + \cdots + 1/(2^nu_{i_n})$. The function F was shown to be one-to-one, and cl(F) was shown to be homeomorphic to Cantor's ternary set. The function G defined by $G(x) = 1/u_{i_1} + 1/u_{i_2} + 1/u_{i_3} + \cdots + 1/u_{i_n}$ was tried first; however, there was some difficulty in showing that cl(G) was totally disconnected. It would be interesting to see if cl(G) could be proved to be a perfect, compact, totally disconnected metric space.

It was found that there are gaps to the left of the points of cl(F). These gaps were put into different classes. Further computing might uncover a pattern for the lengths of gaps within specific classes, and it would be interesting to see if Conjecture 7.3 could be proved.

The Fibonacci sequence and Cantor's ternary set are interesting objects studied in mathematics. Separately, much theory can be found about the properties of each object. However, this thesis has presented theory that establishes an intriguing relationship between two seemingly different areas of study. A relationship between the Fibonacci sequence and Cantor's ternary set was established. Continued research into this area is encouraged to determine if there is more to this relationship.

BIBLIOGRAPHY

1. Bartle, Robert G. The Elements of Real Analysis. 2nd ed. New York: John Wiley and Sons, 1964.

2. Boyer, Carl B. A History of Mathematics. Princeton: Princeton University Press, 1985.

3. Croom, Fred H. Principles of Topology. Philadelphia: Saunders College, 1989.

4. Hocking, John G., and Gail S. Young. Topology. Reading: Addison-Wesley, 1961.

5. Kasriel, Robert H. Undergraduate Topology. Philadelphia: W. B. Saunders, 1971.

6. Kuratowski, Kazimierz. Introduction to Set Theory and Topology. Oxford: Pergamon Press, 1972.

7. Marsden, Jerrold E. Elementary Classical Analysis. New York: W. H. Freeman, 1974.

8. Roberts, Fred S. Applied Combinatorics. Englewood Cliffs: Princeton-Hall, 1984.

9. Vajda, S. Fibonacci & Lucas Numbers, and the Golden Section: Theory and Applications. Chichester: Ellis Horwood Limited, 1989.

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