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Transitions in Line Bitangency Submanifolds for a One-Parameter Family of Immersion Pairs

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TRANSITIONS IN LINE BITANGENCY SUBMANIFOLDS FOR A ONE-PARAMETER FAMILY OF IMMERSION PAIRS

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CONTENTS

Abstract

Consider two immersed surfaces M and N. A pair of points $(p, q) \in M \times N$ is called a *line bitangency* if there is a common tangent line between them. Furthermore, we define the *line bitangency submanifold* as the union of all such pairs of points in $M \times N$. In this thesis we investigate the dynamics of the line bitangency submanifold in a one-parameter family of immersion pairs. We do so by translating one of the surfaces and studying the wide range of transitions the submanifold may undertake. We then characterize these transitions by the local geometry of each surface and provide examples of each transition.

INTRODUCTION

The motivation for studying bitangencies between two surfaces stems from the curiosity of mathematicians and computer scientists alike, and the present work aims to deepen our curiosity with new results which tie previously known geometry with that of bitangencies. In our effort, we have drawn inspiration from the important contributions of Dreibelbis, Banchoff, McCrory, Tari, Giblin, Bruce, Montaldi, Ballesteros, and many others. For the interested reader, we recommend [1], [5], [6] as references for the theory of bitangencies.

The purpose of Chapter 1 is to collect, organize, and emphasize those aspects of local geometry which are required for the study of bitangencies. In particular, we define the notions of smooth manifolds, tangent space, local properties of smooth maps, and we conclude by discussing special curves on immersed surfaces. Also in these chapters, are many of the central examples referenced throughout the thesis.

Chapter 2 is rather technical. We introduce the concepts of transversal intersection between manifolds, jet space, the compact-open topology on C^∞, and the *Monge form* of an immersed surface. These concepts lie at the heart of subsequent chapters, and they provide a convenient setting for future work.

Chapters 3 and 4 introduce the main submanifolds of interest–the line bitangency, double point, plane bitangency, and asymptotic line bitangency submanifolds. We show that these sets are indeed submanifolds of the immersion pair and proceed to characterize their behavior as the base surfaces

move relative to one another. Additionally, we characterize the types of singularities which occur in the projections of these submanifolds. Lastly, we provide specific examples of the transitions discussed.

DIFFERENTIAL GEOMETRY

The study of bitangencies is expressed in the language of differential geometry. We'd like to use phrases such as *smooth manifold*, *tangent space*, and *Monge form* without hesitation or ambiguity. To accomplish this, our first chapter develops the relevant theory from the ground up. Although this material may be found in many books (Consider [2], [4], [8], or [9], for instance), we are impelled to contextualize the subject matter, so that the reader can more easily see the connections with bitangencies. The reader is warned, however, that our treatment is far from exhaustive and is encouraged to peruse the cited references.

1 Smooth Manifolds

We begin with a metric space M, a point p in M, and an open subset $U \subseteq M$ which is centered about p. Since M is a metric space, there is a homeomorphism $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ which maps U to an open subset of $\mathbb{R}^n$.

1.1 Definition. We define a *chart* to be the ordered pair (φ, U). In this setting, the open set U is called a *coordinate neighborhood* of p, and the homeomorphism φ is called a *coordinate system* on U.

Suppose now that we have two charts (φ, U) and (ψ, V) on M. Since $U \cap V$ is open in M, and since the image $\varphi(U \cap V)$ is open in $\mathbb{R}^n$, it follows that $\varphi|_{U \cap V} : U \cap V \to \mathbb{R}^n$ is a coordinate system

on $U \cap V$. Likewise, $\psi|_{U \cap V} : U \cap V \to \mathbb{R}^n$ is also a coordinate system on $U \cap V$.

1.2 Definition. Two coordinate systems $\varphi : U \to \mathbb{R}^n$ and $\psi : V \to \mathbb{R}^n$ are said to be C^∞*-compatible* if the compositions,

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V) \quad \text{and} \quad \varphi \circ \psi^{-1} : \psi(U \cap V) \to \varphi(U \cap V)$$

are C^∞ as maps from $\mathbb{R}^n$ to $\mathbb{R}^n$. See Figure 1.

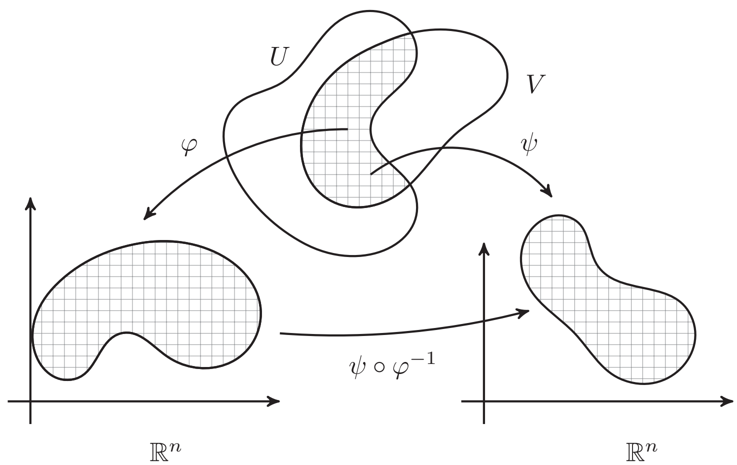


Figure 1: The map $\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V)$ seen as a smooth real map.

If $U \cap V$ is empty, then two charts are immediately C^∞-compatible. Also, with our future intentions in mind, we often omit 'C^∞' and say that two charts are compatible.

1.3 Definition. An *atlas* on a metric space M is a collection $\mathcal{A} = \{(\varphi_\alpha, U_\alpha)\}$ of pairwise compatible charts that cover M. A chart (φ, U) is said to be C^∞*-compatible with the atlas* $\mathcal{A}$ if it is compatible with each chart in $\mathcal{A}$.

1.4 Lemma. *Let $\mathcal{A} = \{(\varphi_\alpha, U_\alpha)\}$ be an atlas for a metric space M. If (φ, U) and (ψ, V) are both compatible with $\mathcal{A}$, then they are compatible with each other.*

Proof. If $U \cap V = \emptyset$, then the result holds vacuously. Otherwise, let $p \in U \cap V$. We aim to prove that $\psi \circ \varphi^{-1}$ is smooth at $\varphi(p)$. Since $\{(\varphi_\alpha, U_\alpha)\}$ is an atlas for M, there exists U_α (not necessarily unique) such that $p \in U_\alpha$. Hence, $p \in U \cap V \cap U_\alpha$. By rewriting $\psi \circ \varphi^{-1}$ as

$$\psi \circ \varphi^{-1} = (\psi \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \varphi^{-1}),$$

we see that $\psi \circ \varphi^{-1}$ is the composition of two smooth maps, and hence, it is smooth at $\varphi(p)$. Since p was arbitrarily chosen, it follows that $\psi \circ \varphi^{-1}$ is smooth on all of $\varphi(U \cap V)$. Likewise, $\varphi \circ \psi^{-1}$ is smooth on $\psi(U \cap V)$. $\square$

1.5 Definition. An atlas $\mathcal{M}$ on a metric space M is said to be *maximal* if it is not contained in any other atlas; in other words, if $\mathcal{A}$ is any other atlas containing $\mathcal{M}$, then $\mathcal{A} = \mathcal{M}$.

1.6 Definition. A *smooth* or *C^∞-manifold* is a metric space M together with a maximal atlas.

For convenience, we will often refer to a 1-dimensional manifold as a *curve*, a 2-dimensional manifold as a *surface*, and an n-dimensional manifold as an *n-manifold*. We write M^n to express that M is an n-dimensional smooth manifold.

As we will soon see, checking that a metric space M is a smooth manifold can be tiresome. Luckily, it follows from the proposition that we do not need to demonstrate a maximal atlas; any atlas on M will do.

1.7 Proposition. *Any atlas $\mathcal{A} = \{(\varphi_\alpha, U_\alpha)\}$ is contained in a unique maximal atlas for M.*

Proof. We aim to construct the maximal atlas from $\mathcal{A}$. Adjoin to $\mathcal{A}$ all charts (φ_i, U_i) that are compatible with $\mathcal{A}$. By Lemma 1.4, the charts (φ_i, U_i) are all compatible with one another. Hence, the enlarged collection of charts is also an atlas. Call the enlarged atlas $\mathcal{M}$. Any chart compatible with $\mathcal{M}$ must also be compatible with the original atlas $\mathcal{A}$, and so by construction belongs to $\mathcal{M}$. This proves that $\mathcal{M}$ is maximal.

Let $\mathcal{M}$ be the maximal atlas containing $\mathcal{A}$ that we have just constructed. If $\mathcal{M}'$ is another maximal atlas containing $\mathcal{A}$, then all the charts in $\mathcal{M}'$ are compatible with $\mathcal{A}$ and so by construction belong to $\mathcal{M}$. This proves that $\mathcal{M}' \subset \mathcal{M}$. Since both are maximal, $\mathcal{M}' = \mathcal{M}$. Therefore, the maximal atlas containing $\mathcal{A}$ is unique. $\square$

The following lemma gives us a typical use of the maximality condition, and it will be used implicitly throughout the remainder of the thesis.

1.8 Lemma. *Let M^n be a smooth manifold equipped with an atlas $\mathcal{A}$, and let $p \in M$. Then there exists a chart $(\varphi, U) \in \mathcal{A}$ such that $p \in U$ and $\varphi(p) = 0 \in \mathbb{R}^n$.*

Proof. Since $\mathcal{A}$ is an atlas, there exists $(\varphi, U) \in \mathcal{A}$ such that $p \in U$. Define a map $\tau : \mathbb{R}^n \to \mathbb{R}^n$ as the translation $\tau(y) = y - \varphi(p)$. Let $\psi = \tau \circ \varphi : U \to \mathbb{R}^n$. Then $\psi(p) = \tau(\varphi(p)) = \varphi(p) - \varphi(p) = 0$. Since τ is smooth (as a map from $\mathbb{R}^n \to \mathbb{R}^n$), and since φ is a coordinate system on U, it follows that $(\psi, U) \in \mathcal{A}$. $\square$

For the sake of brevity, we will simply speak of a "manifold" when we mean a C^∞ manifold. Furthermore, the terms "smooth" and "C^∞" will be used interchangeably. In the context of manifolds we denote the standard coordinates on $\mathbb{R}^n$ by $x^1, \ldots, x^n$. If (φ, U) is a chart on a manifold, we let $\varphi^i = x^i \circ \varphi$ be the ith component of φ and write $\varphi = (\varphi^1, \ldots, \varphi^n)$. Thus, for $p \in U$, $(\varphi^1(p), \ldots, \varphi^n(p))$ is a point in $\mathbb{R}^n$. The functions $\varphi^1, \ldots, \varphi^n$ are called *coordinates* or *local coordinates* in U. By a *chart* (φ, U) *centered about* p in a manifold M, we will mean a chart in the differentiable structure of M such that $p \in U$.

Smooth Maps Between Manifolds

Now that we have a feeling for smooth manifolds, we proceed by defining smooth maps between them.

1.9 Definition. Let M and N be manifolds. A map $f : M \to N$ is C^∞ *at a point* p *in* M if there are charts (ψ, V) about $f(p)$ in N and (φ, U) about p in M such that the composite function $\psi \circ f \circ \varphi^{-1}$ is C^∞ at $\varphi(p)$. The map $f : M \to N$ is said to be C^∞ if it is C^∞ at every point of M.

The next result is typical for smooth manifolds; it demonstrates that the definition above is independent of which charts are chosen on M and N.

1.10 Proposition. *Suppose $f : M \to N$ is C^∞ at $p \in M$. If (φ, U) is any chart about p in M, and (ψ, V) is any chart about $f(p)$ in N, then $\psi \circ f \circ \varphi^{-1}$ is C^∞ at $\varphi(p)$.*

Proof. Since f is C^∞ at $p \in M$, there are charts $(\varphi_\alpha, U_\alpha)$ about p in M and (ψ_β, V_β) about $f(p)$ in N such that $\psi_\beta \circ f \circ \varphi_\alpha^{-1}$ is C^∞. By the C^∞-compatibility of charts in a differentiable structure, both $\varphi_\alpha \circ \varphi^{-1}$ and $\psi \circ \psi_\beta$ are C^∞ on open subsets of Euclidean spaces. Hence, the composite

$$\psi \circ f \circ \varphi^{-1} = (\psi \circ \psi_\beta^{-1}) \circ (\psi_\beta \circ f \circ \varphi_\alpha^{-1}) \circ (\varphi_\alpha \circ \varphi^{-1})$$

is C^∞ at $\varphi(p)$. $\square$

Lastly, we are concerned with the notion of equivalence between two manifolds, and the following statements provide a natural setting for such inquiries.

1.11 Definition. A *diffeomorphism* $f : M \to N$ is a bijection such that f and its inverse f^{-1} are both smooth functions. Two manifolds M and N are said to be *diffeomorphic* if there exists a global diffeomorphism of M onto N.

1.12 Proposition. *If (φ, U) is a chart on a manifold M of dimension n, then the coordinate map $\varphi : U \to \varphi(U) \subseteq \mathbb{R}^n$ is a diffeomorphism.*

Proof. By assumption φ is a homeomorphism, and hence, it is both injective and surjective onto its image. To see that φ is smooth, consider the atlases $\{(\varphi, U)\}$ for U and $\{\mathrm{id}_{\varphi(U)}, \varphi(U)\}$ for $\varphi(U)$. Since $\mathrm{id}_{\varphi(U)} \circ \varphi \circ \varphi^{-1} : \varphi(U) \to \varphi(U)$ is the identity map, and is therefore smooth, it follows by definition that $\varphi : U \to \varphi(U)$ is also smooth.

To see that φ^{-1} is smooth, consider the map $\varphi \circ \varphi^{-1} \circ \mathrm{id}_{\varphi(U)} : \varphi(U) \to \varphi(U)$. This map is also the identity, so again we conclude that φ^{-1} is a smooth mapping. $\square$

Examples of Smooth Manifolds

i. The n-dimensional sphere $S^n = \{\mathbf{x} = (x_0, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum_{i=0}^{n} x_i^2 = 1\}$ is a smooth n-manifold. To see this, let $N = (1, 0, \ldots, 0)$ and $N^* = (-1, 0, \ldots, 0)$ denote the north and south poles of the sphere. Then the maps $\varphi_N : S^n \setminus N \to \mathbb{R}^{n+1}$ and $\psi_{N^*} : S^n \setminus N^* \to \mathbb{R}^{n+1}$ defined by

$$\varphi_N(\mathbf{x}) = \left(\frac{x_1}{1 - x_0}, \ldots, \frac{x_n}{1 - x_0} \right) \quad \text{and} \quad \psi_{N^*}(\mathbf{x}) = \left(\frac{x_1}{1 + x_0}, \ldots, \frac{x_n}{1 + x_0} \right).$$

provide us with an atlas for S^n.

ii. Any open subset V of a manifold M is also a manifold. If $\{(\varphi_\alpha, U_\alpha)\}$ is an atlas for M, then $\{(\varphi_\alpha\big|_{U_\alpha \cap V}, U_\alpha \cap V)\}$ is an atlas for V.

iii. Let V be a finite dimensional real vector space. We show that V is a smooth manifold. Consider an ordered basis $\mathbf{E} = \{E_1, \ldots, E_n\}$ for V. Then $\mathbf{E}$ defines an isomorphism $\phi : \mathbb{R}^n \to V$ by

$$\phi(x) = \sum_{i=1}^{n} x^i E_i.$$

This map is a homeomorphism, so (ϕ^{-1}, V) is a chart. If $\widetilde{\mathbf{E}} = \{\widetilde{E}_1, \ldots, \widetilde{E}_n\}$ is any other ordered basis and

$$\widetilde{\phi}(x) = \sum_{j=1}^{n} x^j \widetilde{E}_j$$

is the corresponding isomorphism, then there is some invertible matrix A_i^j such that

$$E_i = \sum_{j=1}^{n} A_i^j \widetilde{E}_j$$

for each $i = 1, \ldots, n$. The transition map between the two charts is given by $\widetilde{\phi}^{-1} \circ \phi(x) = \widetilde{x}$, where $\widetilde{x}$ is determined by

$$\sum_{j=1}^{n} \widetilde{x}^j \widetilde{E}^j = \sum_{i=1}^{n} x^i E_i = \sum_{i,j=1}^{n} x^i A_i^j \widetilde{E}_j.$$

It follows that $\widetilde{x}^j = \sum_{i,j=1}^{n} A_i^j x^i$. Thus the map sending x to $\widetilde{x}$ is an invertible linear map and hence a diffeomorphism, so any two charts are compatible. The collection of all such charts thus defines a smooth structure, called the *standard smooth structure on V*.

iv. Consider two C^∞ manifolds M^m and N^n. We show that the space $M \times N$ is also a smooth manifold. If (φ, U) and (ψ, V) are any two charts on M and N, the map

$$\varphi \times \psi : M \times N \to \mathbb{R}^m \times \mathbb{R}^n = \mathbb{R}^{m+n}$$

is an injection whose range $\varphi(U) \times \psi(V)$ is open in $\mathbb{R}^{m+n}$. It is therefore a chart for $M \times N$ with domain $U \times V$. If $\widetilde{\varphi} \times \widetilde{\psi}$ is another such chart whose domain $U' \times V'$ meets $U \times V$, the change of coordinates

$$\begin{aligned}(\widetilde{\varphi} \times \widetilde{\psi}) \circ (\varphi \times \psi)^{-1} &= (\widetilde{\varphi} \times \widetilde{\psi}) \circ (\varphi^{-1} \times \psi^{-1}) \\ &= (\widetilde{\varphi} \circ \varphi^{-1}) \times (\widetilde{\psi} \circ \psi^{-1})\end{aligned}$$

is a diffeomorphism. Consequently the set of all such charts is a C^∞ atlas and this defines a smooth structure of dimension $m + n$ on $M \times N$.

2 The Tangent Space

The concept of a differentiable map between manifolds was introduced in the previous section; however, we have yet to address the mechanics of taking the derivative of a smooth map. In other words, we have yet to describe what to do if one is given such a map and is asked to explicitly write its derivative. We discuss this, among other things, in the following section.

Partial Differentiation

Suppose that (φ, U) is a chart on M, and let $f : M \to \mathbb{R}$ be smooth. Then by its definition, $f \circ \varphi^{-1}$ is a smooth function from $\mathbb{R}^n$ to $\mathbb{R}$. Hence, we can consider its partial derivatives with respect to

the standard coordinates on $\mathbb{R}^n$:

$$\left.\frac{\partial(f\circ\varphi^{-1})}{\partial x^i}\right|_{\varphi(p)}.$$

We are now at liberty to define the partial derivatives of $f : M \to \mathbb{R}$ to be

$$\left.\frac{\partial f}{\partial \varphi^i}\right|_p = D_i(f\circ\varphi^{-1})\Big|_{\varphi(p)} = \left.\frac{\partial(f\circ\varphi^{-1})}{\partial x^i}\right|_{\varphi(p)}.$$

As one would expect, the functions $\dfrac{\partial f}{\partial \varphi^i}$ behave as normal partial derivatives do.

2.1 Proposition. *Let M be a smooth manifold with $p \in M$, and let $f, g : M \to \mathbb{R}$ be smooth maps. If (φ, U) and (ψ, V) are two charts about p, and if α, β are any two real numbers, then*

i.

$$\left.\frac{\partial(\alpha f + \beta g)}{\partial \varphi^i}\right|_p = \alpha \left.\frac{\partial f}{\partial \varphi^i}\right|_p + \beta \left.\frac{\partial g}{\partial \varphi^i}\right|_p$$

ii.

$$\left.\frac{\partial(fg)}{\partial \varphi^i}\right|_p = \left.\frac{\partial f}{\partial \varphi^i}\right|_p \cdot g(p) + f(p)\cdot \left.\frac{\partial g}{\partial \varphi^i}\right|_p$$

iii.

$$\left.\frac{\partial f}{\partial \psi^j}\right|_p = \sum_{i=1}^n \left.\frac{\partial f}{\partial \varphi^i}\right|_p \cdot \left.\frac{\partial \varphi^i}{\partial \psi^j}\right|_p$$

We leave the proofs of **i.** and **ii.** to the reader. Before we prove **iii.**, it would be helpful to recall the Chain-Rule for Euclidean spaces: if $F : \mathbb{R}^n \to \mathbb{R}$ and $G : \mathbb{R}^m \to \mathbb{R}^n$ are smooth, then

$$D_j(F\circ G)\Big|_a = \sum_{i=1}^n D_i F\Big|_{G(a)} \cdot D_j G^i\Big|_a.$$

For a proof, see [7].

Proof. We utilize the coordinate systems φ and ψ along with the Chain-Rule for Euclidean spaces:

$$\begin{aligned}
\left.\frac{\partial f}{\partial \psi^j}\right|_p &= D_j(f\circ\psi^{-1})\Big|_{\psi(p)} \\
&= D_j\left((f\circ\varphi^{-1})\circ(\varphi\circ\psi^{-1})\right)\Big|_{\psi(p)} \\
&= \sum_{i=1}^n D_i(f\circ\varphi^{-1})\Big|_{(\varphi\circ\psi^{-1})(\psi(p))} \cdot D_j(\varphi^i\circ\psi^{-1})\Big|_{\psi(p)} \\
&= \sum_{i=1}^n \left.\frac{\partial f}{\partial \varphi^i}\right|_p \left.\frac{\partial \varphi^i}{\partial \psi^j}\right|_p.
\end{aligned}$$

□

Tangent Vectors

Although we are primarily concerned with surfaces immersed in $\mathbb{R}^3$, where the notion of a tangent space is clear, we will have occasion in Chapter 2 to discuss more abstract manifolds. In particular, we'll be studying smooth submanifolds of function spaces. In this setting, the traditional sense of a tangent space is no longer applicable. However, we would like to salvage the idea, and this section demonstrates we can.

Let M^n be a smooth manifold with $p \in M$, and let f, g be smooth functions whose domains include p. The set of all such functions will be denoted by $\mathcal{F}(p)$. This space has the useful property that it is linear over $\mathbb{R}$ and is closed under multiplications. In other words, for any real numbers $\alpha, \beta \in \mathbb{R}$ the maps $\alpha f + \beta g$ and $f \cdot g$ are both elements of $\mathcal{F}(p)$. We would like to emphasize, however, that the domains of $\alpha f + \beta g$ and $f \cdot g$ is $\operatorname{dom} f \cap \operatorname{dom} g$. This definition is subtle but important.

2.2 Definition. Let α and β be any two real numbers, and let f and g be elements of $\mathcal{F}(p)$. A *linear operator* is a map $\Lambda : \mathcal{F}(p) \to \mathbb{R}$ which satisfies $\Lambda(\alpha f + \beta g) = \alpha\Lambda(f) + \beta\Lambda(g)$.

2.3 Proposition. *Let $\Lambda : \mathcal{F}(p) \to \mathbb{R}$ be a linear operator, and suppose that $f\big|_U = g\big|_U$, for some neighborhood U of p. Then $\Lambda(f) = \Lambda(g)$.*

Proof. This follows directly from the linearity of Λ. □

The next definition is the first step towards a tangent space for an abstract manifold.

2.4 Definition. A *derivation* on $\mathcal{F}(p)$ is a linear operator $\Lambda : \mathcal{F}(p) \to \mathbb{R}$ such that

$$\Lambda(fg) = f(p)\Lambda(g) + \Lambda(f)g(p).$$

It follows from Proposition 2.1 that if (φ, U) is a chart about p, then the function

$$\left.\frac{\partial}{\partial \varphi^i}\right|_p : \mathcal{F}(p) \to \mathbb{R}, \quad f \mapsto \partial f/\partial \varphi^i\big|_p$$

is a derivation on $\mathcal{F}(p)$.

2.5 Proposition. *Let Λ be a derivation on $\mathcal{F}(p)$, and let $f \in \mathcal{F}(p)$. If there exists a neighborhood U such that $f\big|_U = C$, for some constant value $C \in \mathbb{R}$, then $\Lambda(f) = 0$.*

Proof. Suppose we have the conditions as stated, and let $\mathbf{1}$ be the constant function with value 1. By Proposition 2.3, we have

$$\Lambda(f) = \Lambda(C\mathbf{1}) = C \cdot \Lambda(\mathbf{1}).$$

However, because

$$\Lambda(\mathbf{1}) = \Lambda(1 \cdot \mathbf{1}) = 1 \cdot \Lambda(\mathbf{1}) + \Lambda(\mathbf{1}) \cdot 1,$$

it follows that $\Lambda(\mathbf{1}) = 0$; and hence, $\Lambda(f) = 0$. $\square$

As previously declared, the set of all smooth maps defined at a point p of a smooth manifold M is closed under linear combinations over $\mathbb{R}$. Fortunately, the same can be said of the set of derivations defined on $\mathcal{F}(p)$. That is, if Λ and Ω are two derivations defined on $\mathcal{F}(p)$, and if α and β are two real numbers, then the function

$$\alpha\Lambda + \beta\Omega : \mathcal{F}(p) \to \mathbb{R}, \quad f \mapsto \alpha \cdot \Lambda(f) + \beta \cdot \Omega(f)$$

is also a derivation on $\mathcal{F}(p)$. In fact, more can be said of the set of all derivations.

2.6 Definition. The space of all derivations on $\mathcal{F}(p)$ is a real vector space which we call the *tangent space T_pM at p*. From now on, we will call any derivation on $\mathcal{F}(p)$ a *tangent vector at p*, and will it be denoted by $\mathbf{v}$ to emphasize the statements below.

Let's take a moment to discuss why we choose to identify tangent vectors with linear operators. Consider a vector $\mathbf{v} \in \mathbb{R}^3$ with base p and a map $f : \mathbb{R}^n \to \mathbb{R}$. Then $\mathbf{v}$ *induces* the derivation $f \mapsto \mathbf{v} \cdot \nabla f\big|_p$. In other words, we can associate a linear operator to any given vector in Euclidean space. Conversely, consider the directional derivative $D_{\mathbf{v}}f$. It is clearly a linear operator, and we wish to associate to it a vector in Euclidean space. To do so, we recover the vector $\mathbf{v}$ by observing its action on a basis. Therefore, we claim that it is natural to identify vectors with operators. The following statements rigorously demonstrate how to proceed, and we begin with a helpful lemma.

2.7 Lemma. *Let (φ, U) be a chart about $p \in M^m$ with $\varphi(p) = x_0$ and $f \in \mathcal{F}(p)$. Then there exist $f_i \in \mathcal{F}(p)$ such that*

$$f_i(p) = \left.\frac{\partial f}{\partial \varphi^i}\right|_p \qquad \text{for } i = 1, \ldots, m,$$

and

$$f(q) = f(p) + \sum_{i=1}^{m} \left(\varphi^i(q) - \varphi^i(p)\right) f_i(q). \tag{1}$$

Proof. Again, we'll take advantage of the local coordinate system. Let $x = \varphi(q)$, and let $x_0 = \varphi(p)$. Then (1) becomes

$$f(\varphi^{-1}(x)) = f(\varphi^{-1}(x_0)) + \sum_{j=1}^{m} \Big(\varphi^j(\varphi^{-1}(x)) - \varphi^j(\varphi^{-1}(x_0))\Big)\, f_j(\varphi^{-1}(x)).$$

Let g represent the function $f \circ \varphi^{-1} : \varphi^{-1}(U) \subseteq \mathbb{R}^m \to \mathbb{R}$, so that the functions $g_i : \varphi(U) \to \mathbb{R}$ are such that

$$g(x) = g(x_0) + \sum_{j=1}^{m} (x^j - x_0^j) g_j(x).$$

By shrinking the neighborhood U, we may assume that $\varphi(U) \subseteq \mathbb{R}^m$ is *star shaped* with respect to x_0; that is to say, for every $x \in \varphi(U)$ the line segment $\{x_0 + t(x - x_0) : t \in [0,1]\}$ is contained in $\varphi(U)$. Consider any *fixed* $x \in \varphi(U)$, and the associated curve $\gamma_x : [0,1] \to \varphi(U)$ defined by $\gamma_x(t) = x_0 + t(x - x_0)$. By implementing the Fundamental Theorem of Calculus and the Chain-Rule, we obtain

$$\begin{aligned} g(x) = g(\gamma_x(1)) &= g(\gamma_x(0)) + \int_0^1 \frac{d}{dt} g(\gamma_x(t))dt \\ &= g(\gamma_x(0)) + \int_0^1 \sum_{j=1}^{m} (D_j g)(\gamma_x(t)) \cdot \frac{d\gamma_x^j(t)}{dt} dt \\ &= g(\gamma_x(0)) + \sum_{j=1}^{m} (x^j - x_0^j) \int_0^1 (D_j g)(\gamma_x(t))dt \end{aligned}$$

We define $g_j(x) = (D_j g)(\gamma_x(t))$. Note that the constant derivative $\gamma_x' = x - x_0$ for the curve in $\mathbb{R}^m$- this makes no sense on a general manifold, but is coordinate dependent. One immediately verifies that with this definition $g_j(x_0) = (D_j g)(x_0)$. Consequently,

$$f_j(p) = g_j(\varphi(p)) = (D_j g)(\varphi(p)) = D_j(f \circ \varphi^{-1})\Big|_{\varphi(p)} = \frac{\partial f}{\partial \varphi^j}\Big|_p .$$

Since $g \in C^\infty$, also $g_j, f_j \in C^\infty$. $\square$

The fruit of our labor is that, as one would expect, the set of partial derivatives at a point is a basis for the tangent tangent space.

2.8 Proposition. *If φ is any chart of M whose domain includes a given point p, the vectors $\partial/\partial\varphi^i\big|_p$ $(i = 1, \dots, n)$ form a basis for T_pM.*

Proof. Suppose (φ, U) is a chart about $p \in M^m$, $\Lambda \in T_pM$, and $f \in \mathcal{F}(p)$. Using Lemma 2.7, Proposition 2.3, and Proposition 2.5, there exists, on some neighborhood of p, suitable functions f_i

such that we may write

$$\begin{aligned}
\Lambda(f) &= \Lambda\left[f(p) + \sum_{j=1}^{m}(\varphi^j - \varphi^j(p))f_j\right] \\
&= 0 + \sum_{j=1}^{m}\Big((\Lambda(\varphi^j) - 0)f_j(p) + (\varphi^j(p) - \varphi^j(p))\Lambda(f)\Big), \\
&= \sum_{j=1}^{m}(\Lambda\varphi^j)\cdot f_j(p).
\end{aligned}$$

Again, by Lemma 2.7, this is equal to

$$\Lambda(f) = \sum_{j=1}^{m}(\Lambda\varphi^j)\left.\frac{\partial f}{\partial\varphi^j}\right|_p.$$

Since this holds for all $f \in \mathcal{F}(p)$, we conclude

$$\Lambda = \sum_{j=1}^{m}(\Lambda\varphi^j)\left.\frac{\partial}{\partial\varphi^j}\right|_p. \tag{2}$$

□

The set of vectors $\partial/\partial\varphi^i\big|_p$ $(i = 1, \ldots, m)$ is called the *canonical basis* for T_pM associated with the coordinate system φ.

Derived Linear Functions

Let $p \in M$, $\phi : M \to N$ smooth with $\phi(p) = q$, and let $f \in \mathcal{F}(q)$. Then the map $f \circ \phi$ is also an element of $\mathcal{F}(p)$. In this way, a vector $\mathbf{v} \in T_pM$ determines a function $f \mapsto \mathbf{v}(f \circ \phi)$. This is a derivation on $\mathcal{F}(q)$ since

$$\begin{gathered}
(\alpha f + \beta g)\circ\phi = \alpha(f\circ\phi) + \beta(g\circ\phi) \quad \alpha, \beta \in \mathbb{R} \\
(f \cdot g)\circ\phi = (f\circ\phi)\cdot(g\circ\phi).
\end{gathered}$$

Hence, we conclude that $\mathbf{v}(f \circ \phi)$ is a tangent vector of T_qN, and we denote it by $\phi_{*p}(\mathbf{v})$. In this way, we define the linear map

$$\phi_{*p} : T_pM \to T_qN, \quad \mathbf{v} \mapsto \mathbf{v}(f\circ\phi),$$

called the *derived linear function* on T_pM. The map ϕ_{*p} is also referred to as the *tangent map* or *differential* of ϕ at p.

Consider the charts (φ, U) and (ψ, V) of M, N at p and $\phi(p) = q$ respectively. Then it follows from Equation (2) that the vector $\phi_{*p}(\mathbf{v})$ is equal to

$$\phi_{*p}(\mathbf{v}) = \sum_{j=1}^{n} \mathbf{v}(\psi^j \circ \phi) \left.\frac{\partial}{\partial \psi^j}\right|_q .$$

From which we conclude that ϕ_{*p} is determined by its action:

$$\left.\frac{\partial}{\partial \varphi^i}\right|_p \mapsto \sum_{j=1}^{n} \left.\frac{\partial(\psi^j \circ \phi)}{\partial \varphi^i}\right|_p \cdot \left.\frac{\partial}{\partial \psi^j}\right|_q \tag{3}$$

on the bases for T_pM associated with the chart φ. We notice that the matrix

$$\left[\left.\frac{\partial(\psi^j \circ \phi)}{\partial \varphi^i}\right|_p \right] = J_\Phi(\varphi(p))$$

where J_Φ is the Jacobian matrix of the coordinate representative $\Phi = \psi \circ \phi \circ \varphi^{-1}$ of the function ϕ. With this last statement in mind, we conclude that ϕ_{*p} is a linear map represented by the Jacobian matrix J_Φ, so that one should think of ϕ_{*p} as the *derivative* of the map ϕ at the point p. Indeed, we will often use the notation $D(\phi)_p$ if no confusion is possible.

A fundamental property of derived linear functions is given in the following proposition.

2.9 Proposition. *Let $\phi : M \to N$ and $\psi : N \to P$ be differentiable functions, and suppose p lies in the domain of $\psi \circ \phi$. Then*

$$(\psi \circ \phi)_{*p} = \psi_{*\phi(p)} \circ \phi_{*p}.$$

Proof. Suppose $\mathbf{v} \in T_pM$. If $f \in \mathcal{F}(\psi(\phi(p)))$,

$$\begin{aligned}
\big[(\psi \circ \phi)_{*p}(\mathbf{v})\big] f &= \mathbf{v}(f \circ (\psi \circ \phi)) = \mathbf{v}((f \circ \psi) \circ \phi) \\
&= \big[\phi_{*p}(\mathbf{v})\big] (f \circ \psi) \\
&= \Big[\psi_{*(\phi(p))}(\phi_{*p}\mathbf{v})\Big] f
\end{aligned}$$

From which it follows that $(\psi \circ \phi)_{*p}\mathbf{v} = \psi_{*\phi(p)}(\phi_{*p}\mathbf{v})$. □

2.10 Definition. The *rank* of a differentiable function $\phi : M \to N$ at p is the rank of the derived

linear function ϕ_{*p}. Equivalently, the rank of a differentiable function is the rank of the Jacobian matrix J_Φ, as defined previously.

Remark. It is a simple exercise to show that the rank of J_Φ is invariant under changes in coordinate charts, so that the rank of a differentiable function ϕ is well-defined. The importance of this definition becomes clear in the next section. We now have one of our most fundamental definitions.

2.11 Definition. Let $\phi : M^m \to N^n$ be a smooth map, and let $p \in M$. Then p is called a *critical point*, or a *singularity*, of ϕ if the rank of ϕ at p is less than n. If p is not a singularity of ϕ, it is called a *regular point* of ϕ. If p is a singularity of ϕ, the value $\phi(p)$ is called a *critical value* of ϕ. Other points of N are *regular values*; thus $q \in N$ is a regular value if and only if p is a regular point of ϕ for every $p \in \phi^{-1}(q)$.

3 Local Properties of Smooth Maps

In this section we discuss the local form of smooth maps, introduce two important classes of maps, and demonstrate how to construct smooth manifolds without having to refer to coordinate systems or atlases. We begin with a theorem.

3.1 Theorem (The Inverse Function Theorem)**.** *Let $\phi : M \to N$ be a smooth map between smooth manifolds, and let $p \in M$. Then the differential ϕ_{*p} is an isomorphism if and only if there exists a neighborhood U of p such that $\phi\big|_U$ is a diffeomorphism.*

For a proof, see [3]. We will be using this theorem implicitly and explicitly throughout the remainder of thesis, so we consider a simple example to see how it works.

3.2 Example. Given a smooth map $f : \mathbb{R}^n \to \mathbb{R}^m$, we define the *graph of f* to be the set

$$\Gamma(f) = \{(x, f(x)) \in \mathbb{R}^{n+m} : x \in \mathbb{R}^n\}.$$

We show that the graph of f is a smooth manifold by creating a chart on $\Gamma(f)$. Define the map $\varphi : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^m \times \mathbb{R}^n$ by $\varphi(x, y) = (x, y - f(x))$. Then $\varphi(x, y) = (x, 0)$ if and only if $(x, y) \in \Gamma(f)$; that is, $\varphi(x, y) = 0$ if and only if $f(x) = y$. This shows that φ maps surjectively onto $\Gamma(f)$. To see that φ is injective, suppose that $\varphi(x', y') = \varphi(x, y)$. Then $x = x'$, and hence $y - f(x) = y' - f(x)$;

from which it follows that $y = y'$. Thus, we conclude that φ is injective. Lastly, the derivative

$$D(\varphi) = \begin{pmatrix} I & 0 \\ -Df & I \end{pmatrix}$$

is nonsingular, so we conclude by Theorem 3.1 that there exists U such that $\varphi\big|_U$ is a diffeomorphism. Thus (φ, U) is a chart for the graph of f, making $\Gamma(f)$ a smooth manifold.

Immersions and Submersions

We now begin to study two important classes of smooth maps; so important, in fact, that they will occupy us for the remainder of this thesis.

3.3 Definition. Let $f : M^m \to N^n$ be a smooth map with $m \leq n$. Then f is called an *immersion* if $\operatorname{rank} f = m$ at every point of M. If f is an immersion, then the derived function on each tangent space is an injection.

To understand the importance of this class, consider a simple example where $\mathbf{s} : M^2 \to \mathbb{R}^3$ is an immersion. If $\mathbf{s}$ is parametrized by

$$\mathbf{s}(u, v) = (X(u, v), Y(u, v), Z(u, v)),$$

then by its definition, the Jacobian matrix

$$J = \begin{pmatrix} X_u & X_v \\ Y_u & Y_v \\ Z_u & Z_v \end{pmatrix}$$

has full rank. Therefore the two column vectors are linearly independent, and the surface $\mathbf{s}(M)$ has a well-defined tangent plane at every point. This property explains why we are interested in immersions. In fact, from now on, we will only be considering surfaces which are immersed in $\mathbb{R}^3$.

Now we introduce the second important class of maps.

3.4 Definition. Let $f : M^m \to N^n$ be a smooth map with $m \geq n$. Then f is called an *submersion* if $\operatorname{rank} f = n$ at every point of M. If f is a submersion, then the derived function on each tangent space is a surjection.

The following statements demonstrate the importance of submersions.

3.5 Lemma. *Let $f : M^{m+n} \to N^n$ be a submersion with $p \in M$. If $m \geq n$, then for any chart (ψ, V) about $f(p)$, there exists a coordinate system (φ, U) about p such that*

$$\psi \circ f \circ \varphi^{-1}(x^1, \ldots, x^{m+n}) = (x^1, \ldots, x^n).$$

Proof. Suppose that (φ, U) and (ψ, V) are charts of M and N about p and $f(p) = q$, respectively, and let $\alpha = \psi \circ f \circ \varphi^{-1}$. By hypothesis, the derivative $D(\alpha)_{\varphi(p)} : \mathbb{R}^{m+n} \to \mathbb{R}^n$ is surjective. Let $i : \mathbb{R}^n \to \mathbb{R}^{m+n}$ be the standard inclusion. Then it follows that $D(\alpha \circ i)\big|_{\varphi(p)} = D(\alpha)\big|_{\varphi(p)} \circ i : \mathbb{R}^n \to \mathbb{R}^n$ is an isomorphism. Now let $\tilde{\alpha}(x, y) = (x, \alpha(x, y))$. Then since $D(\alpha) \circ i$ is an isomorphism, it follows that $D(\tilde{\alpha})\big|_{\varphi(p)} : \mathbb{R}^m \oplus \mathbb{R}^n \to \mathbb{R}^m \oplus \mathbb{R}^n$ is also an isomorphism. We can now conclude by the Inverse Function Theorem that there exists a neighborhood U_0 of $\varphi(p)$ contained in $\varphi(U)$ such that $\tilde{\alpha}$ is a diffeomorphism from U_0 onto its image V_0; let $\tilde{\beta} : V_0 \to U_0$ be its smooth inverse, and let $\tilde{\varphi} = \tilde{\alpha} \circ \varphi$. Then $(\tilde{\varphi}, \varphi^{-1}(U_0)$ is a local chart of M about p, and

$$\psi \circ f \circ \tilde{\varphi}^{-1}(x, y) = \psi \circ f \circ \varphi^{-1} \circ \tilde{\beta}(x, y) = \alpha \circ \tilde{\beta}(x, y) = y.$$

□

The lemma above imposes importance on the concept of submersions since it allows for us to construct new manifolds without referring to coordinate systems or charts. To see this, consider the following definition.

3.6 Definition. Let M^m be a smooth manifold and let N be a subset of dimension n. Then N is a *submanifold* of M if for every point p in N, there exists a chart (φ, U) of M so that $\varphi^{-1}(V) = N \cap U$, where

$$V = \{(x^1, \ldots, x^m) \in \mathbb{R}^m : x^{n+1} = \cdots = x^m = 0\}.$$

Also, we define the *codimension* of N to be $\operatorname{codim} N = m - n$.

3.7 Proposition. *Let $f : M^{m+n} \to N^n$ be a smooth map, $p \in M$, and let $q \in N$ be such that $f^{-1}(q)$ is nonempty. If f is a submersion at all points of $P = f^{-1}(q)$, then P is a submanifold of M of dimension* $\dim M - \dim N$. *Moreover, for $p \in P$, we have $T_pP = \ker D(f)_p$.*

Proof. Our plan is to restrict the charts of M to cover P in such a way that P becomes a smooth manifold. Consider the charts $(\tilde{\varphi}, W = \varphi^{-1}(U_0))$ and (ψ, V) as given in the proof of Lemma 3.5, where we assume that $p \in U$ and $q \in V$. By Lemma 1.8 in the first section, we may assume that $\psi(q) = 0$. It is clear that $\tilde{\varphi}(W \cap P) = \tilde{\varphi}(W) \cap \mathbb{R}^m$, so $\tilde{\varphi}$ is a restricted chart of P about p. Lastly, by

Lemma 3.5, the map f at p can be expressed as the projection $\mathbb{R}^{m+n} \to \mathbb{R}^n$. This is a linear map with kernel $\mathbb{R}^m$, from which it follows that $\ker D(f)_p = D\tilde{\varphi}^{-1}_{\varphi(p)}(\mathbb{R}^n) = T_pP$. □

4 Special Curves on Surfaces

In this section, we digress from abstract machinery and return to a firm footing. Our goal is to describe the extrinsic geometry of a surface immersed in $\mathbb{R}^3$. We begin by considering a metric on the surface, and then we study the curvature of the surface. These ideas bring forth many new constructs on the surface itself, all of which will be useful in the future. Our first goal is to establish the notion of distance on a manifold.

4.1 Definition. Let M be a smooth manifold. The *first fundamental form* $\mathrm{I}_p(\cdot,\cdot)$ is the restriction of the usual dot product in $\mathbb{R}^3$ to the tangent space T_pM. Namely, for $\mathbf{v}, \mathbf{w}$ in T_pM, $\mathrm{I}_p(\mathbf{v},\mathbf{w}) = \mathbf{v}\cdot\mathbf{w}$.

This definition capitalizes on the ambient space $\mathbb{R}^3$ for a metric. Although this cannot be done in general, this notion of distance is sufficient for our purposes.

4.2 Proposition. *Let $\mathrm{I}_p(\cdot,\cdot) : T_pM \times T_pM \to \mathbb{R}$ be the first fundamental form at a point p on an immersed surface M. Given a coordinate system $\varphi(x,y) : U^2 \to \mathbb{R}^3$ about p, the matrix associated with the first fundamental form $\mathrm{I}_p(\cdot,\cdot)$ with respect to the basis $\{\varphi_x, \varphi_y\}$ is*

$$\mathrm{I}_p = \begin{pmatrix} E & F \\ F & G \end{pmatrix},$$

where $E = \varphi_x \cdot \varphi_x$, $F = \varphi_x \cdot \varphi_y$, and $G = \varphi_y \cdot \varphi_y$, and the symbols φ_x and φ_y denote the partial derivatives as defined previously.

Proof. We wish to express the inner product $\mathrm{I}_p(\cdot,\cdot)$ in terms of the standard basis on T_pM. Let

$$\mathbf{v} = v_1\varphi_x(p) + v_2\varphi_y(p)$$

and

$$\mathbf{w} = w_1\varphi_x(p) + w_2\varphi_y(p)$$

be two vectors in the tangent space. Then, if we calculate $\mathbf{v}\cdot\mathbf{w}$, we get

$$v_1w_1\varphi_x \cdot \varphi_x + v_1w_2\varphi_x \cdot \varphi_y + v_2w_1\varphi_x\varphi_y + v_2w_2\varphi_y\varphi_y.$$

Hence, from the calculation above, we see that

$$\mathrm{I}_p(\mathbf{v}, \mathbf{w}) = \mathbf{v}^\mathsf{T}\, \mathrm{I}_p\, \mathbf{w}.$$

$\square$

Our second definition characterizes the local curvature of the surface. A natural way to do this is to consider a curve $\gamma : [0,1] \to M$ on the surface along with a unit normal vector $\mathbf{n}$ whose base is at a point $\gamma(t)$. Then the curvature of the surface can be described by watching how $\mathbf{n}$ moves as its base goes along the curve $\gamma(t)$.

4.3 Definition. The *matrix of the second fundamental form* at $\varphi(\gamma(t)) = \varphi(x(t), y(t))$ is defined by

$$\mathrm{II} = \begin{bmatrix} L & M \\ M & N \end{bmatrix} = -\begin{bmatrix} \varphi_x \cdot \mathbf{n}_x & \varphi_y \cdot \mathbf{n}_x \\ \varphi_x \cdot \mathbf{n}_y & \varphi_y \cdot \mathbf{n}_y \end{bmatrix} = \begin{bmatrix} \varphi_{xx} \cdot \mathbf{n} & \varphi_{xy} \cdot \mathbf{n} \\ \varphi_{yx} \cdot \mathbf{n} & \varphi_{yy} \cdot \mathbf{n} \end{bmatrix} \tag{4}$$

Remark. There should not be any confusion between the matrix element M and the smooth manifold M.

The last inequality holds because $\varphi_x \cdot \mathbf{n} \equiv 0$ gives $\varphi_x \cdot \mathbf{n}_x + \varphi_{xx} \cdot \mathbf{n} = 0$; and hence, $\varphi_{xx} \cdot \mathbf{n} = -\varphi_x \cdot \mathbf{n}_y$. Furthermore, this calculation shows the symmetry of the matrix II, since $\varphi_{xy} = \varphi_{yx}$.

For future investigations, we would like to change our perspective on the matrix II. At the moment, it is simply a matrix which is dependent on the local parametrization of M. We would like to extend this matrix into a bilinear form on M. To do this, we define $\mathrm{II}_p(\mathbf{a}, \mathbf{b}) = (a_1, a_2)\, \mathrm{II} (b_1, b_2)^\mathsf{T}$, and call the map $\mathrm{II}_p(\cdot, \cdot) : T_pM \times T_pM \to \mathbb{R}$ the *second fundamental form* at the point p.

Via the matrix of the second fundamental form, we may characterize the points on a surface.

4.4 Definition. A point p on M is called

- *elliptic* if $\det \mathrm{II}_p > 0$;
- *hyperbolic* if $\det \mathrm{II}_p < 0$;
- *parabolic* if $\det \mathrm{II}_p = 0$.

Using this characterization, we see that, generally speaking, a surface is split into an elliptic region and a hyperbolic region, with a parabolic curve in between them. In fact, for most surfaces, the set of parabolic points forms a curve on the surface.

4.5 Definition. The parabolic points on a surface M form a curve or set of curves called collectively the *parabolic curve* on M.

There are other curves on the surface which are of interest to us that can be defined via the second fundamental form.

4.6 Definition. A tangent vector $\mathbf{v}$ at $p \in M$ is an *asymptotic vector* if $\mathrm{II}(\mathbf{v}, \mathbf{v}) = 0$. An *asymptotic curve* is a curve on the surface which is tangent to the asymptotic vectors at each point.

The next proposition should be clear from the definitions.

4.7 Proposition. *At an elliptic point there are no asymptotic directions; at a parabolic point there is one asymptotic direction; at a hyperbolic point there are two asymptotic directions.*

Monge Form of a Surface

Let $\mathbf{s} : M^2 \to \mathbb{R}^3$ be an immersion. Through an affine transformation, we may reparametrize M so that the surface passes through the origin and has a horizontal tangent plane. In symbols, we may reparametrize M so that $M = (x, y, f(x, y))$ and $f(0, 0) = 0$ and $f_x(0, 0) = f_y(0, 0) = 0$. If M is given in this form, then we may express f locally as

$$f(x, y) = ax^2 + 2bxy + cy^2 + \text{higher order terms.}$$

Clearly then, we have $2a = f_{xx}(0, 0)$, $b = f_{xy}(0, 0)$, $2c = f_{yy}(0, 0)$. By recalling the classification of points we had in Chapter 1, we see

4.8 Proposition. *Suppose M is given by $z = f(x, y)$ for some function f (that is, the surface is a graph). Then the surface is, at the origin,*

i. *elliptic if and only if $b^2 < ac$;*

ii. *hyperbolic if and only if $b^2 > ac$;*

iii. *parabolic if and only if $b^2 = ac$.*

The first fundamental form at $(0, 0)$ is not the identity, and the matrix of the second fundamental form II at the origin is

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}.$$

GENERIC GEOMETRY

1 Transversality

The concept of transversal intersection between two immersed manifolds M and N is essential to the theorems and proofs found in Chapters 3 and 4. To introduce it, consider a smooth map $f : M \to N$, a point p in M, and a submanifold $W \subset N$.

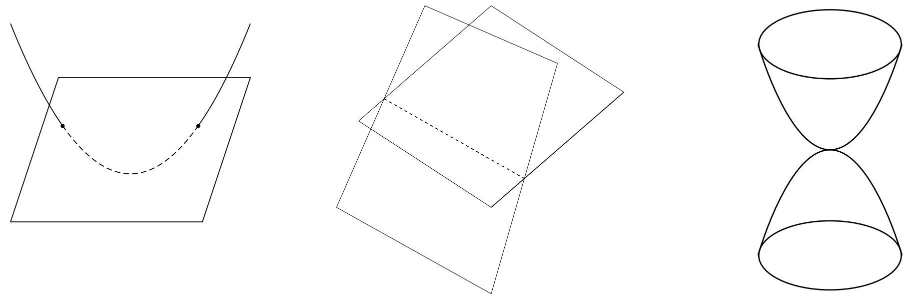
1.1 Definition. The map f is *transverse* to W at p, denoted by $f \pitchfork W$ at p, if one of the following conditions holds:

i. $f(p) \notin W$, or

ii. $f(p) \in W$ and $T_{f(p)}N = T_{f(p)}W + \text{Image}(Df_p)$.

Furthermore, if U is a subset of M then f is *transverse to* W *on* U, denoted by $f \pitchfork W$ on U, if $f \pitchfork W$ at p for all $p \in U$. Finally, f is simply *transverse to* W, denoted by $f \pitchfork W$, if f is transverse to W on all of M.

Geometrically, this definition says that if $f : M \to N$ is transverse to $W \subset N$, then the graph of f intersects W in a structurally stable manner. For examples, consider Figure 2 below.



(a) (b) (c)

Figure 2: Example (a) demonstrates a curve and a plane intersecting transversally. Example (b) demonstrates two planes intersecting transversally. Example (c) demonstrates a non-transversal intersection between paraboloids.

The next proposition will be used frequently.

1.2 Proposition. *Suppose that M^m and N^n are smooth manifolds such that $W \subset N$ is a submanifold of dimension k. Let $f : M \to N$ be a smooth map which is transverse to W. Then if inequality $m + k < n$ holds, it follows that $f(M) \cap W = \emptyset$. That is, if the codimension of W in N is larger than the dimension of M, then the image $f(M)$ misses W entirely.*

Proof. Suppose that $f(p) \in W$. Then by Proposition 2.8 in Chapte 1,

$$\begin{aligned}\dim(T_{f(p)}W + D(f)_p(T_pM)) &\leq \dim T_{f(p)}W + \dim T_pM \\ &= \dim W + \dim M < \dim N = \dim T_{f(p)}N.\end{aligned}$$

It is therefore impossible for f to be transverse to W. $\square$

The next lemma is useful in a theorem to come.

1.3 Lemma. *Let M^m and N^n be smooth manifolds with $W \subset N$ a submanifold and $f : M \to N$ a smooth map. If $p \in M$, $f(p) \in W$, and if there exists a neighborhood U about $f(p)$ such that $\varphi : U \to \mathbb{R}^n$ is a submersion with $W \cap U = \varphi^{-1}(0)$, then $f \pitchfork W$ at p if and only if $\varphi \circ f$ is a submersion at p.*

Remark: We can guarantee the existence of such a U. Consider the chart (ψ, U) about $f(p)$ and the decomposition $\mathbb{R}^m = \mathbb{R}^n \times \mathbb{R}^{m-n}$ so that $W \cap U = \psi^{-1}(0 \times \mathbb{R}^{m-n})$. By using the projection $\pi : \mathbb{R}^m \to \mathbb{R}^n$ we can then let $\varphi = \pi \circ \psi$.

Proof. By Proposition 3.7 in Chapter 1, we can immediately conclude that $\ker D(\varphi)_{f(p)} = T_{f(p)}W$. Furthermore, we have that $f \pitchfork W$ at p

$$\Leftrightarrow T_{f(p)}N = T_{f(p)}W + D(f)_p(T_pM)$$
$$\Leftrightarrow T_{f(p)}N = \ker D(\varphi)_{f(p)} + D(f)_p(T_pM).$$

Because $D(\varphi)_{f(p)}$ is onto, we conclude that $D(\varphi \circ f)_p$ is onto if and only if this last equality holds. Hence, $\varphi \circ f$ is a submersion at p if and only if $f \pitchfork W$ at p. □

Our hard work is now rewarded with one of the most useful consequences of transversality.

1.4 Theorem. *Consider two smooth manifolds M and N along with submanifold $W \subset N$ and smooth map $f : M \to N$ which is transverse to W. Then $f^{-1}(W)$ is a submanifold of M with* $\operatorname{codim} f^{-1}(W) = \operatorname{codim} W$.

Proof. For each $p \in f^{-1}(W)$ we construct a chart (ψ, V) in M such that $V \cap f^{-1}(W)$ is a submanifold. Consider the set U and map φ as defined in Lemma 1.3, and let V be a neighborhood about p such that $f(V) \subset U$. By Lemma 1.3 we conclude that $\varphi \circ f$ is a submersion on V. Thus $f^{-1}(W) \cap V = (\varphi \circ (f\big|_V))^{-1}(0)$ is a submanifold, by Proposition 3.7 in Chapter 1. □

Theorem 1.4 is quite useful. For example, consider the application below.

1.5 Example. Consider two smooth submanifolds M^m and N^n of a smooth manifold V^v which intersect transversally in V. Then by observing that the standard inclusions $i_M : M \to V$ and $i_N : N \to V$ are transverse to M and N respectively, we can conclude that the intersection $M \cap N$ will be a smooth submanifold of dimension $m + n - v$, or it will be empty.

The Basic Transversality Lemma

Let $M \subset \mathbb{R}^m$, $T \subset \mathbb{R}^k$, and $N \subset \mathbb{R}^n$ be smooth manifolds, and consider the smooth family of maps $F : M \times T \to N$. For our purposes, we think of this family as being parametrized by the manifold T; that is, for each t in T we consider the mapping $F(x,t) = f_t(x)$, where $x \in M$. If $F \pitchfork W$ for some submanifold $W \subset N$, then a fruitful question to ask is if f_t is transverse to W for all values of t. Proving that the answer is 'no' is easy; simply consider any of the examples in Figure 2, and translate one of the manifolds. It is easily seen that for a particular value of t, the intersection will no longer be transverse. However, it is just as clear that the intersection between two manifolds

is transverse *most* of the time. We would now like to make this notion rigorous and expose its consequences. We begin with a fundamental result of differential topology.

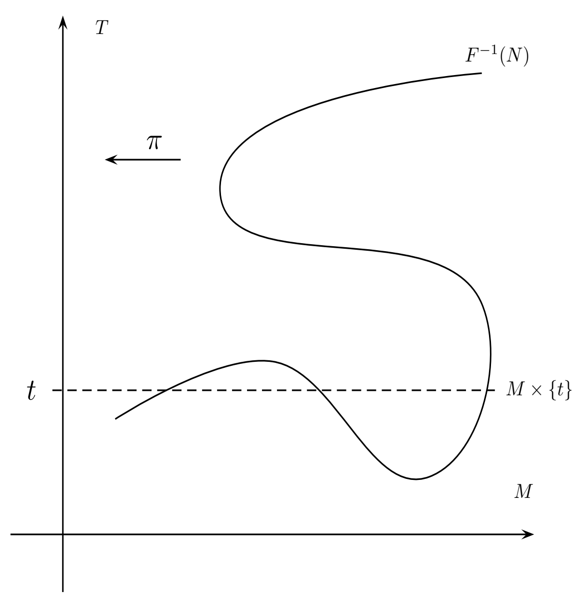
1.6 Theorem. *Suppose that $f_i : M_i \to N$ is a countable family of smooth mappings. Then the set of common regular values of the f_i is dense in N.*

Remark: Recall that a point $q \in f(M)$ is a regular value if f has maximal rank. This theorem is known as Sard's theorem, and the proof is quite lengthy and somewhat unrelated to our material. We therefore ask the reader to consider [8] for its proof.

The basic lemma we wish to prove in this section is this:

1.7 Lemma (Basic Transversality Lemma)**.** *Consider smooth manifolds $M \subseteq \mathbb{R}^m, T \subseteq \mathbb{R}^k$, and $N \subseteq \mathbb{R}^n$. If $F : M \times T \to \mathbb{R}^n$ is a smooth family of maps which is transverse to N, then for almost all $t \in T$ (all t outside a set of measure zero) the maps $f_t : M \to N$ given by $f_t(x) = F(x, t)$ are transverse to N.*

Proof. It follows from Theorem 1.4 that the set $F^{-1}(N)$ is a smooth submanifold of $M \times T$. We aim to show that $t \in T$ is a regular value of the projection $\pi : F^{-1}(N) \to T$ if and only if $f_t \pitchfork N$, so that the validity of the Lemma follows as a consequence of Sard's theorem. Throughout the proof, the reader is encouraged to keep the following picture in mind.



In fact, we need only prove our claim for an arbitrary point y in N since in a neighborhood of y, N can be parametrized as the preimage of of a regular value (say 0, for instance) of some smooth map $f : \mathbb{R}^n \to \mathbb{R}^p$. It is easy to see that F is transverse to N at this point if and only if $f \circ F$ has 0 as a regular value. Thus, we may consider N as consisting of a single point y.

Finally, the condition for F to be transverse to $\{y\}$ is that, for each $(x,t) \in M \times T$ with $F(x,t) = y$, we have

$$TF(x,t)(M_x \times T_t) = \mathbb{R}^n_y, \tag{5}$$

where $TF(x,t)(M_x \times T_t)$ is the image of the derivative. Consider now the condition for f_t to be transverse to N; this is the condition that for the same x's as in (5)

$$TF(x,t)(M_x \times \{0\}) = \mathbb{R}^n_y. \tag{6}$$

Furthermore, t is a regular value of the projection $\pi : F^{-1}(y) \to T$ if and only if for the same x as in (5) we have

$$T\pi(x,t)(F^{-1}(y))_{(x,t)} = T_t, \tag{7}$$

which is equivalent to

$$T\pi(x,t)(\ker TF(x,t)) = T_t. \tag{8}$$

Using the fact that $T\pi(x,t)$ is the natural projection to T_t it is now an easy exercise in linear algebra to show that (5) an (6) hold if and only if (5) and (8) hold. □

2 The Compact-Open Topology

In the previous section, we saw that two manifolds *usually* intersect transversally. The idea of studying properties which hold for most surfaces brings us to our main technical tool.

2.1 Definition. A *k-jet from M to N* is an equivalence class $[p,f,U]_k$ of triples (p,f,U), where $U \subset M$ is an open set, $p \in U$, and $f : U \to N$ is a smooth map. Two triples $[p,f,U]_k$ and $[p',f',U']_k$ are equivalent if $p = p'$ and for some pair of charts about p and p', the maps f and f' have the same derivatives up to order k.

As a shorthand, we denote an equivalence class by

$$[p,f,U]_k = j^k f\Big|_p ,$$

and call it the *k-jet of f at p*.

The set of all k-jets at a point p is denoted by $J_p^k(M,N)$, and union of these spaces

$$J^k(M,N) = \bigcup_{p\in M} J_p^k(M,N)$$

is called the *jet space*. Additionally, if $M = \mathbb{R}^m$ and $N = \mathbb{R}^n$, then we will implement the notation $J^k(m,n)$ for the jet space.

Let $U \subset \mathbb{R}^n$ be open, and let $f \in C^\infty(U,\mathbb{R}^n)$. Then the k-jet of f at $p \in U$ is canonically represented by the Taylor polynomial of f of order k at p. Recall that these polynomials are uniquely determined by the list of derivatives up to order k of f at p. This list belongs to the vector space

$$P^k(m,n) = \mathbb{R}^n \times \prod_{i=1}^{k} L^i_{\text{sym}}(\mathbb{R}^m,\mathbb{R}^n)$$

where $L^i_{\text{sym}}(\mathbb{R}^m,\mathbb{R}^n)$ denotes the vector space of symmetric i-linear maps from $\mathbb{R}^m$ to $\mathbb{R}^n$. Conversely any element of $P^k(m,n)$ comes from a unique jet in $J_p^k(M,N)$. In this way we have identifications

$$J_p^k(m,n) = P^k(m,n)$$

and

$$J^k(m,n) = \mathbb{R}^m \times P^k(m,n).$$

In particular $J^k(m,n)$ is a finite dimensional vector space (for k finite), and hence, it is also a smooth manifold. Lastly, it is clear that if $U \subset \mathbb{R}^m$ and $V \subset \mathbb{R}^n$ are open sets then $j^k(U,V)$ is an open subset of $J^k(m,n)$.

Our last goal for this section is to prove that for a smooth mapping $f : \mathbb{R}^m \to \mathbb{R}^n$, and a smooth manifold $W \subset \mathbb{R}^n$, we can find a smooth mapping $g : \mathbb{R}^m \to \mathbb{R}^n$ which is transverse to W, and as 'close' as we please to f. To make the notion of closeness rigorous, we induce a topology on $C^\infty(\mathbb{R}^m,\mathbb{R}^n)$ via jets.

Assume that $f : \mathbb{R}^m \to \mathbb{R}^n$ is smooth, $\varepsilon > 0$ is a small real number, $R > 0$ a large real number, and $k \geq 0$ is a positive integer. Then we associate to f a *fundamental neighborhood* in $C^\infty(\mathbb{R}^m,\mathbb{R}^n)$ comprising all those smooth mappings $g : \mathbb{R}^m \to \mathbb{R}^n$ for which for all $p \in \mathbb{R}^m$ with $|p| \leq R$ one has

$$\left\| j^k f\Big|_p - j^k g\Big|_p \right\| < \varepsilon,$$

where $||\cdot||$ a fixed norm on the jet-space $J^k(m,n)$. Furthermore, a subset $U \subset C^\infty(\mathbb{R}^m,\mathbb{R}^n)$ is

dense when given any smooth mapping $f : \mathbb{R}^m \to \mathbb{R}^n$ and any fundamental neighborhood V of f one can find a smooth mapping $g : \mathbb{R}^m \to \mathbb{R}^n$ in U with $g \in V$. Intuitively, any mapping can be approximated as closely as we please by mappings in U.

We are now in a position to state and prove an elementary transversality theorem.

2.2 Proposition (Thom's Transversality Theorem)**.** *The set of smooth mappings $f : \mathbb{R}^m \to \mathbb{R}^n$ which are transverse to given smooth submanifolds $W_1, \ldots, W_t$ of $\mathbb{R}^n$ is dense in $C^\infty(\mathbb{R}^m, \mathbb{R}^n)$.*

Proof. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ be smooth. We'll prove that f can be approximated as close as possible by mappings transverse to $W_1, \ldots, W_t$. Our strategy will be to create a smooth family $F : \mathbb{R}^m \times S \to \mathbb{R}^n$ which contains f, and with F transverse to $W_1, \ldots, W_t$. If this can be done, then we may apply the Basic Transversality lemma to conclude our result. To ensure that F is transverse to $W_1, \ldots, W_t$, we make it a submersion. Therefore we take $S = \mathbb{R}^n$ and define $F : \mathbb{R}^m \times \mathbb{R}^n \to \mathbb{R}^n$ by $(x, s) \mapsto f(x) + s$. It is clear that this is a submersion, and hence, it is transverse to $W_1, \ldots, W_t$. We now conclude from the Basic Transversality Lemma that there is a dense set of s for which f_s is transverse to $W_1, \ldots, W_t$. It now follows that if V is a fundamental neighborhood of f, then there exists an $s \neq 0$ for which f_s lies in V. $\square$

THE BITANGENT SUBMANIFOLDS

We are now ready to begin studying the line bitangencies between two immersed surfaces. The first section of this chapter formalizes the intuitive notions of line bitangency, double point, plane bitangency, and asymptotic line bitangency. The main result is that for a generic pair of immersed surfaces M and N, these notions lead to sets which are all smooth submanifolds of $M \times N$.

The remaining sections then study each of these submanifolds individually. In particular, they are devoted to the characterization of the singularities found in a one parameter family of immersion pairs. The singularities are classified via their severity and geometric consequence. Throughout this chapter, it is assumed that $\mathbf{s} : M \to \mathbb{R}^3$ and $\mathbf{r} : N \to \mathbb{R}^3$ are two immersed surfaces and that $\mathbf{n} : M \to S^2$ and $\mathbf{m} : N \to S^2$ are their unit normal vector fields.

1 The Line Bitangency Submanifolds

We begin by defining our basic objects of study.

1.1 Definition.

- A *line bitangency* is a pair of points $(p, q) \in M \times N$ such that the vector $\mathbf{s}(p) - \mathbf{r}(q)$ is an element of both $T_{\mathbf{s}(p)}\mathbf{s}(M)$ and $T_{\mathbf{r}(q)}\mathbf{r}(N)$.

- A *double point* is a pair of points $(p, q) \in M \times N$ such that $\mathbf{s}(p) = \mathbf{r}(q)$.

- A *plane bitangency* is a pair of points $(p, q) \in M \times N$ such that $T_{\mathbf{s}(p)}\mathbf{s}(M) = T_{\mathbf{r}(q)}\mathbf{r}(N)$, where we intend for the equality to mean *affine* equality, not just a vector space isomorphism.
- An *asymptotic line bitangency of* M is a line bitangency where the segment $\mathbf{s}(p) - \mathbf{r}(q)$ is an asymptotic vector at $p \in M$. We define asymptotic line bitangencies of N in a similar way.

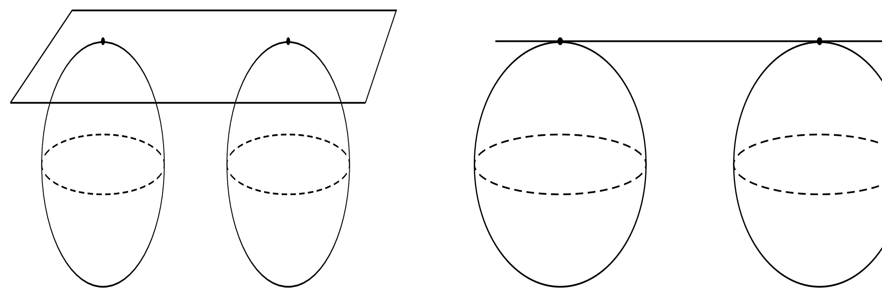


Figure 3: A plane bitangency and a line bitangency.

These definitions are intuitively satisfying but are useless when performing calculations. To aid ourselves in this pursuit, we define the *bitangency map* $\sigma : M \times N \to \mathbb{R}^2$ by

$$\sigma(p, q) = (\mathbf{n}(p) \cdot (\mathbf{s}(p) - \mathbf{r}(q)), \mathbf{m}(q) \cdot (\mathbf{s}(p) - \mathbf{r}(q))).$$

It is clear that the pair $(p, q) \in M \times N$ is either a line bitangency or a double point if and only if $\sigma(p, q) = (0, 0)$. Therefore, we define the *line bitangency* and *double point submanifolds* by

$$\Sigma_{LB} = \{(p, q) \in M \times N : \sigma(p, q) = (0, 0)\}$$
$$\Sigma_{DP} = \{(p, q) \in M \times N : \mathbf{s}(p) = \mathbf{r}(q)\}.$$

Similarly, we define the *plane bitangency* and *asymptotic line bitangency submanifolds* by

$$\Sigma_{PB} = \{(p, q) \in M \times N : \sigma(p, q) = \mathbf{0} \text{ and } \mathbf{n}(p) \times \mathbf{m}(q) = \mathbf{0}\}$$
$$\Sigma_{ALBM} = \{(p, q) \in M \times N : \sigma(p, q) = \mathbf{0} \text{ and } \mathrm{II}_M(\mathbf{s}(p) - \mathbf{r}(q), \mathbf{s}(p) - \mathbf{r}(q)) = 0\}$$
$$\Sigma_{ALBN} = \{(p, q) \in M \times N : \sigma(p, q) = \mathbf{0} \text{ and } \mathrm{II}_N(\mathbf{s}(p) - \mathbf{r}(q), \mathbf{s}(p) - \mathbf{r}(q)) = 0\}$$

The remainder of this section is dedicated to the study of these two submanifolds, which begins

with an important result.

1.2 Theorem. *For a generic choice of immersed surfaces* $\mathbf{s} : M \to \mathbb{R}^3$ *and* $\mathbf{r} : N \to \mathbb{R}^3$*, the sets* Σ_{LB}*,* Σ_{DP}*,* Σ_{PB}*,* Σ_{ALBM}*, and* Σ_{ALBN} *are all smooth submanifolds of* $M \times N$*.*

Proof. The proof of this result is an application of Thom's Transversality Theorem. To begin, consider the map $\phi_{\mathbf{s},\mathbf{r}} : M \times N \to S^2 \times S^2 \times \mathbb{R}^3$ defined by

$$\phi_{\mathbf{s},\mathbf{r}}(p, q) = (\mathbf{n}(p), \mathbf{m}(q), \mathbf{s}(p) - \mathbf{r}(q)).$$

Then $\phi_{\mathbf{s},\mathbf{r}}$ is a member of the family

$$\Phi : M \times N \times C^\infty(M, \mathbb{R}^3) \times C^\infty(N, \mathbb{R}^3) \to S^2 \times S^2 \times \mathbb{R}^3$$

which is given by

$$\Phi(p, q, \mathbf{s}, \mathbf{r}) = (\mathbf{n}_{\mathbf{s}}(p), \mathbf{m}_{\mathbf{r}}(q), \mathbf{s}(p) - \mathbf{r}(q)).$$

The map Φ is a submersion, and so by Thom's transversality theorem, for a generic choice of $\mathbf{s}$ and $\mathbf{r}$, the map ϕ will be transverse to any smooth submanifold of $S^2 \times S^2 \times \mathbb{R}^3$.

Define the subsets LB, DP, PB, $ALBM$, and $ALBN$ as

$$\begin{aligned}
LB &= \{(\mathbf{n}, \mathbf{m}, \mathbf{v}) : \mathbf{n} \cdot \mathbf{v} = \mathbf{m} \cdot \mathbf{v} = 0\} \\
DP &= \{(\mathbf{n}, \mathbf{m}, \mathbf{v}) : \mathbf{v} = 0\} \\
PB &= \{(\mathbf{n}, \mathbf{m}, \mathbf{v}) : \mathbf{n} \cdot \mathbf{v} = \mathbf{m} \cdot \mathbf{v} = 0, \mathbf{n} \times \mathbf{m} = \mathbf{0}\} \\
ALBM &= \{(\mathbf{n}, \mathbf{m}, \mathbf{v}) : \mathbf{n} \cdot \mathbf{v} = \mathbf{m} \cdot \mathbf{v} = 0, \mathrm{II}_M(\mathbf{v}, \mathbf{v}) = 0\} \\
ALBN &= \{(\mathbf{n}, \mathbf{m}, \mathbf{v}) : \mathbf{n} \cdot \mathbf{v} = \mathbf{m} \cdot \mathbf{v} = 0, \mathrm{II}_N(\mathbf{v}, \mathbf{v}) = 0\}
\end{aligned}$$

Each of these sets are submanifolds of $S^2 \times S^2 \times \mathbb{R}^3$, where $\operatorname{codim} LB = 2$ and $\operatorname{codim} DP = \operatorname{codim} PB = \operatorname{codim} ALBM = \operatorname{codim} ALBN = 3$. It now follows from Theorem 1.4 in Chapter 2 that the sets Σ_{LB}, Σ_{DP}, Σ_{PB}, Σ_{ALBM}, and Σ_{ALBN} are all smooth submanifolds of $M \times N$. Furthermore, $\dim \Sigma_{LB} = 2$ and $\dim \Sigma_{DP} = \dim \Sigma_{PB} = \dim \Sigma_{ALBM} = \dim \Sigma_{ALBN} = 1$. $\square$

2 Singularities of Σ_{LB} in a Family of Immersion Pairs

Now that we have established that Σ_{LB} is a smooth submanifold, we would like to study its behavior in a one-parameter family of immersion pairs. We are interested in characterizing the behavior of these submanifolds in a one-parameter family. To do this, we first assume that the immersions $\mathbf{s} : M \to \mathbb{R}^3$ and $\mathbf{r} : N \to \mathbb{R}^3$ are given in Monge form. In particular, assume that $\mathbf{s}(x, y) = (x, y, f_{\mathbf{s}}(x, y))$ and $\mathbf{r}(u, v) = (k + u, v\cos\theta + g_{\mathbf{r}}(u, v)\sin\theta, -v\sin\theta + g_{\mathbf{r}}(u, v)\cos\theta)$ where $f_{\mathbf{s}}(x, y)$ and $g_{\mathbf{r}}(u, v)$ are given below:

$$f_{\mathbf{s}}(x, y) = ax^2 + 2bxy + cy^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3 + \text{ higher order terms}$$
$$g_{\mathbf{r}}(u, v) = eu^2 + 2fuv + gv^2 + b_{30}u^3 + b_{21}u^2v + b_{12}uv^2 + b_{03}v^3 + \text{ higher order terms}$$

It is worth pointing out that, by Example 1.5 in Chapter 2, we are only interested in transitions which have codimension less than or equal to five. To see this, consider the family

$$F : M \times N \times \mathbb{R} \to C^\infty(M, \mathbb{R}^3) \times C^\infty(N, \mathbb{R}^3).$$

Since the dimension of $M \times N \times \mathbb{R}$ is 5, it follows that the image under F will not intersect any submanifolds of codimension higher than 5; moreover, the image under F will intersect submanifolds of codimension 5 will occur only at isolated points in the family.

2.1 Theorem. *Given a one-parameter family of immersion pairs, Σ_{LB} will have a singularity at the following points:*

- *Plane bitangent double points: A double point (p, q) such that $T_{\mathbf{s}(p)}\mathbf{s}(M) = T_{\mathbf{r}(q)}\mathbf{r}(M)$.*
- *Double asymptotic line bitangencies: A line bitangency (p, q) where the common tangent direction is asymptotic to both surfaces and where the curvature of each surface is related to the distance between them. In Monge form, the relation is $\sin^2\theta = 4bfk^2$.*
- *An asymptotic parabolic plane bitangency: A plane bitangency (p, q) such that either p or q is parabolic on its respective surface, and such that $\mathbf{s}(p) - \mathbf{r}(q)$ is an asymptotic vector at the parabolic point.*

Proof. We return to the bitangency map $\sigma : M \times N \to \mathbb{R}^2$ given by

$$\sigma(p, q) = (\mathbf{n}(p) \cdot (\mathbf{s}(p) - \mathbf{r}(q)), \mathbf{m}(q) \cdot (\mathbf{s}(p) - \mathbf{r}(q))).$$

Through an affine transformation and a reparametrization of the surfaces into Monge form, we may assume that the point of interest is at $\mathbf{0} = (0,0,0,0)$. Then the derivative of σ is given by the following matrix:

$$D(\sigma)_{\mathbf{0}} = \begin{pmatrix} 2ak & 2bk & 0 & \sin\theta \\ 0 & \sin\theta & 2ek & 2fk \end{pmatrix}.$$

By definition, a point $(p,q) \in M \times N$ will be a singularity of Σ_{LB} if and only if $\operatorname{rank} D(\sigma)_{\mathbf{0}}$ falls below 2. We proceed by investigating the possible ways this can happen.

If we let $k = \sin\theta = 0$, then the matrix of $D(\sigma)_{\mathbf{0}}$ becomes the zero matrix, which clearly has rank less than two. The reason why this singularity is called a plane bitangent double point is because if $\sin\theta = 0$, then by construction of the Monge form, the two surfaces have a plane bitangency. Additionally, if $k = 0$, then the two surfaces intersect at the origin, and hence, we have a plane bitangency double point. This singularity is extremely severe and a full investigation is lengthy. For these reasons, we have delegated its study to the next chapter.

Next, suppose that $a = e = 0$. Then the matrix of $D(\sigma)_{\mathbf{0}}$ becomes

$$D(\sigma)_{\mathbf{0}} = \begin{pmatrix} 0 & 2bk & 0 & \sin\theta \\ 0 & \sin\theta & 0 & 2fk \end{pmatrix},$$

so that the rank drops below two if and only if $\sin^2\theta = 4bfk^2$. Recall that if $a = e = 0$, then $(1,0,0)$ is an asymptotic direction for both $\mathbf{s}$ and $\mathbf{r}$ in Monge form, which explains why we call this a double asymptotic singularity. Furthermore, we observe that this singularity has codimension 5, and hence, it occurs only at isolated points in the family of immersions.

The equation $\sin^2\theta = 4bfk^2$ implies a geometric relationship between curvature of each surface, the distance between the two surfaces, and the angle between the surfaces respective tangent planes. Unfortunately, geometric content of this relation is not well understood.

The final way for the rank of $D(\sigma)_{\mathbf{0}}$ to fall below two is to let $\sin\theta = b = 0$. Then the derivative of σ becomes

$$D(\sigma)_{\mathbf{0}} = \begin{pmatrix} 2ak & 0 & 0 & 0 \\ 0 & 0 & 2ek & 2fk \end{pmatrix}.$$

Hence, if $a = 0$, then $\operatorname{rank} D(\sigma)_{\mathbf{0}} < 2$. The geometric implication of the conditions $\sin\theta = b = 0$ is that we have a plane bitangency and that the origin corresponds to a parabolic point on $\mathbf{s}$. Furthermore, if $a = 0$ then the $(1,0,0)$ as an asymptotic vector at the origin for $\mathbf{s}$. Again, this transition occurs only at isolated points. $\square$

Singularities in the Projection of Σ_{LB}

Now that we know where the transitions in Σ_{LB} occur, we'd like to be able to visualize them. Unfortunately, because Σ_{LB} is a surface in 4-space, we cannot do this perfectly. In this section, we characterize when the projection of Σ_{LB} onto M has singularities, and then we'll provide examples of each type of transition.

2.2 Theorem. *If $\pi : M \times N \to M$ is a projection onto the first component, then a point (p, q) is a singularity of the restriction $\pi : \Sigma_{LB} \to M$ if and only if (p, q) satisfies one of the following conditions:*

i. $(p, q) \in \Sigma_{PB}$

ii. $(p, q) \in \Sigma_{DP}$

iii. $(p, q) \in \Sigma_{ALBN}$.

A similar result holds when projecting into the second component, with the third condition changed to $\mathbf{s}(p) - \mathbf{r}(q)$ being an element of Σ_{ALBM}.

Proof. For a singularity in the projection to occur, the tangent space of Σ_{LB} should be nontransversal to the projection. If we are projecting into M, we get a fold if $D(\sigma)_{\mathbf{0}}$, $\mathbf{e}_1$, and $\mathbf{e}_2$ are linearly dependent. To calculate when this happens, consider the determinant below:

$$\det \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 2ak & 2bk & 0 & \sin\theta \\ 0 & \sin\theta & 2ek & 2fk \end{bmatrix} = -2ek \sin\theta.$$

Thus, these quantities are dependent when $-2ek \sin\theta = 0$. In our prescribed Monge form, if $k = 0$ then we have double point; if $e = 0$, then we have an asymptotic line bitangency on N; if $\sin\theta = 0$, then we have a plane bitangency. $\square$

Examples of Transitions

Up until this point, we have only demonstrated how we calculated singularities, but we would now like to visualize examples of these transitions. Consider first the double asymptotic line bitangency transition. In this situation, we can reparametrize the surfaces into an appropriate Monge form. A

simple example is:

$$\mathbf{s}(x, y) = (x, y, 2xy) \quad \text{and} \quad \mathbf{r}(u, v) = \left(\frac{1}{2} + u, 2uv, k - v\right).$$

Then as the surfaces vary in a one-parameter family, we obtain a transition which is homeomorphic to the following:

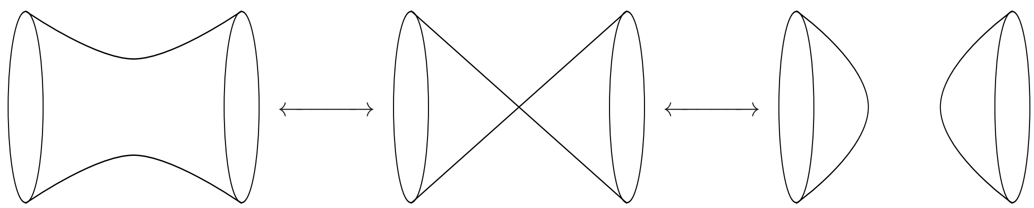


Figure 4: The double asymptotic and asymptotic parabolic plane bitangency transitions in Σ_{LB}.

If we now consider the asymptotic parabolic plane bitangency transition, we may again reparametrize the surfaces into an appropriate Monge form. For example, consider

$$\mathbf{s}(x, y) = (x, y, x^3 + y^2) \quad \text{and} \quad \mathbf{r}(u, v) = \left(1 + u, v, u^2 + 2v^2 + k\right).$$

Then the transition is again homeomorphic to Figure 4.

3 Singularities in Σ_{DP} in a Family of Immersion Pairs

The next submanifold we study is the double point submanifold. The characterization of its singularities is relatively simple compared to the other submanifolds. We begin with a theorem.

3.1 Theorem. *Given a one-parameter family of immersion pairs, the pair (p, q) is a singularity of Σ_{DP} if and only if (p, q) is a plane bitangent double point.*

Proof. Through an affine transformation and a reparametrization of the surfaces into Monge form, we may define the *double point map* $\sigma_{DP} : M \times N \to \mathbb{R}^3$ by $\sigma_{DP}(p, q) = \mathbf{s}(p) - \mathbf{r}(q)$. By inspection, we see that a point (p, q) is a double point if and only if $\sigma_{DP}(p, q) = \mathbf{0}$. Just as in the study of Σ_{LB}, our parametrization affords us the luxury of studying what happens at the origin. It follows that the origin is a singularity of Σ_{DP} if and only if the rank of $D(\sigma_{DP})_{\mathbf{0}}$ falls below three. We calculate

the derivative to be

$$D(\sigma_{DP})_{\mathbf{0}} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\cos\theta \\ 0 & 0 & 0 & \sin\theta \end{pmatrix}.$$

Hence, the rank of $D(\sigma_{DP}) < 3$ if and only if $\sin\theta = 0$. Again, the condition $\sin\theta = 0$ has the geometric implication of being a plane bitangency. $\square$

Just as we were interested in visualizing the transitions of the line bitangency submanifold, we are also interested in visualizing the transitions in the double point submanifold. This leads us to the following theorem.

3.2 Theorem. *If $\pi : M \times N \to M$ is a projection onto the first component, then a double point (p, q) is a singularity of the restriction $\pi : \Sigma_{DP} \to M$ if and if (p, q) is a plane bitangent double point.*

Proof. Consider again the derivative of the double point map evaluated at the origin,

$$D(\sigma_{DP})_{\mathbf{0}} = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -\cos\theta \\ 0 & 0 & 0 & \sin\theta \end{pmatrix}.$$

Then a double point (p, q) is a singularity of the projection $\pi : \Sigma_{DP} \to M$ if and only if the tangent space of (p, q) is nontransversal to the projection. A necessary and sufficient condition for this is for $D(\sigma_{DP})_{\mathbf{0}}$ to be spanned by $\mathbf{e}_3$ and $\mathbf{e}_4$. Clearly, however, this is only possible when $\sin\theta = 0$. That is, at a plane bitangent double point. $\square$

Due to the severity of the plane bitangency double point singularity, all further study has been delegated to the next chapter. There the reader can see sketches of the possible transitions in Σ_{DP}.

4 Singularities of Σ_{PB} in a Family of Immersion Pairs

In this section, we characterize the possible transitions in the plane bitangency submanifold. Recall that we have defined

$$\Sigma_{PB} = \{(p, q) \in M \times N : \sigma(p, q) = 0 \text{ and } \mathbf{n}(p) \times \mathbf{m}(q) = \mathbf{0}\}.$$

4.1 Theorem. *Given a one-parameter family of immersion pairs, Σ_{PB} will have singularities at the following points:*

- *Plane bitangent double points: A double point (p, q) such that $T_{\mathbf{s}(p)}\mathbf{s}(M) = T_{\mathbf{r}(q)}\mathbf{r}(N)$.*
- *Double parabolic plane bitangencies: A plane bitangency (p, q) such that both p and q are parabolic points on their respective surfaces.*
- *Asymptotic parabolic plane bitangencies: A plane bitangency (p, q) such that either p or q is parabolic with $\mathbf{s}(p) - \mathbf{r}(q)$ as an asymptotic vector.*

Proof. We employ a strategy similar to the one used for line bitangencies. Consider the *plane bitangency map* $\sigma_{PB} : M \times N \to \mathbb{R}^3$, which is given by

$$\sigma_{PB}(p, q) = (\mathbf{n}(p) \cdot (\mathbf{s}(p) - \mathbf{r}(q)), \mathbf{m}(q) \cdot (\mathbf{s}(p) - \mathbf{r}(q)), \mathbf{n}^2(p) - \mathbf{m}^2(q)).$$

Remark: Recall that $\mathbf{n}^2(p)$ denotes the second component of the normal vector $\mathbf{n}(p)$ and similarly for $\mathbf{m}^2(q)$. Additionally, the condition above is insufficient for general immersions but is validated if the surfaces are reparametrized into Monge form. To see this, consider the following: we are assuming that the plane bitangency occurs at the origin, and more so, we assume that through a rotation of M that the bitangent plane is the xy-plane. Under these conditions, if $\mathbf{n}^2(p) - \mathbf{m}^2(q)$ is made to be zero, then the condition of a plane bitangency will follow.

As in the case for line bitangencies, the origin (and hence any point after an affine transformation) is a singularity of Σ_{PB} if and only if the derivative drops below maximal rank. The derivative of σ_{PB} evaluated at $\mathbf{0}$ is

$$D(\sigma_{PB})_0 = \begin{pmatrix} 2ak & 2bk & 0 & 0 \\ 0 & 0 & 2ek & 2fk \\ -2b & -2c & 2f & 2g \end{pmatrix}$$

To determine when this matrix loses maximal rank, we need to determine when the row vectors are linearly dependent. After a quick calculation, we obtain the conditions to be:

$$-b(f^2 - eg)k^2 = 0, \quad a(f^2 - eg)k^2 = 0, \quad -(b^2 - ac)fk^2 = 0, \quad (b^2 - ac)ek^2 = 0$$

It is clear that if $k = 0$, which implies a plane bitangency double point in the prescribed Monge Form, then each condition will be satisfied. Otherwise, suppose that $f^2 - eg = 0$ and $b^2 - ac = 0$. Then we'd have both p and q as parabolic points on their respective surfaces, and again, each condition

is satisfied. This singularity has codimension 5. Lastly, we consider $a = b = 0$ or $e = f = 0$. This situation corresponds to an asymptotic parabolic point, which also has codimension 5. $\square$

Singularities in the Projection of Σ_{PB}

Just as we did with the line bitangencies, we would like to be able to visualize the transitions occuring in the plane bitangency submanifold. This pursuit leads us to the following theorem.

4.2 Theorem. *If $\pi : M \times N \to M$ is a projection onto the first component, then a plane bitangency (p, q) is a singularity of the restriction $\pi : \Sigma_{PB} \to M$ if and only if (p, q) satisfies one of the following conditions:*

- *The point q is a parabolic point in N.*
- *The pair (p, q) is an asymptotic parabolic plane bitangency.*
- *The pair (p, q) is a plane bitangent double point.*

Proof. For there to be a singularity in the restriction $\pi : \Sigma_{PB} \to M$, we need Σ_{PB} to be non-transversal to the projection. In terms of calculations, this boils down to requiring the conditions found in the proof of Theorem 4.1 to be spanned by $\mathbf{e}_3$ and $\mathbf{e}_4$. That is, we need the vector

$$\Big(-b(f^2 - eg)k^2, a(f^2 - eg)k^2, -(b^2 - ac)fk^2, (b^2 - ac)ek^2\Big)$$

to be spanned by $\mathbf{e}_3$ and $\mathbf{e}_4$. This happens if and only if $f^2 - eg = 0$, which implies that q is parabolic in N; or if $a = b = 0$, which implies that the pair (p, q) is an asymptotic parabolic plane bitangency; or if $k = 0$, which implies that (p, q) is a plane bitangent double point. $\square$

Examples of Transitions

We now demonstrate examples of the double parabolic plane bitangency and asymptotic parabolic plane bitangency transitions.

5 Singularities of Σ_{ALBM} and Σ_{ALBN} in a Family of Immersion Pairs

In this section, we characterize the possible transitions in the Σ_{ALBM} and Σ_{ALBN} submanifolds. Unfortunately, their structure is considerably more complicated than any of the other submanifolds

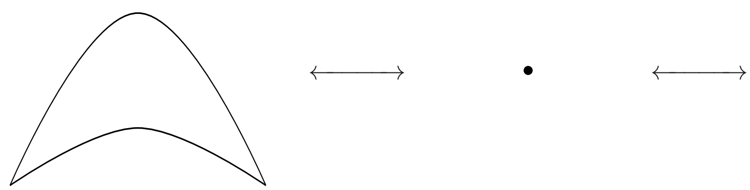


Figure 5: The double parabolic plane bitangency transition.

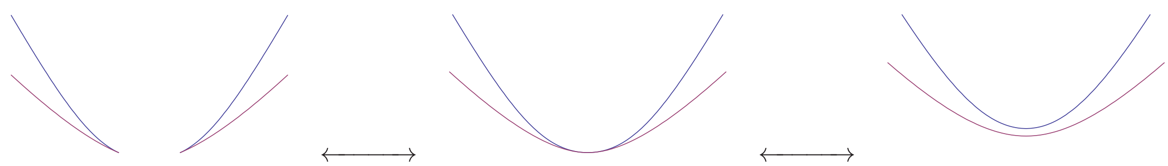


Figure 6: The double asymptotic plane bitangency transition.

we've investigated so far. This claim is demonstrated in the theorem below.

5.1 Theorem. *Given a one-parameter family of immersion pairs, Σ_{ALBM} will have singularities at the following points:*

- *Double points.*
- *Points whose third order terms satisfy the following equations:*

$$a_{30} = 0 \quad a_{21} = \frac{2(b + 2b^2 k \cot\theta)}{3k}$$

- *Double asymptotic line bitangencies where $a_{30} = 0$.*
- *Double asymptotic line bitangencies where the coefficients satisfy the relation: $\sin^2\theta = 4bfk^2$.*
- *Points on M that are parabolic plane bitangencies.*

A similar theorem can be constructed for Σ_{ALBN}.

Proof. Through an affine transformation and reparametrization, we may assume that the surfaces

are given in Monge form. We define a new map

$$\sigma_{ALBM} = (\mathbf{n}\cdot(\mathbf{s}-\mathbf{r}), \mathbf{m}\cdot(\mathbf{s}-\mathbf{r}), \mathrm{II}_M(\mathbf{s}-\mathbf{r}, \mathbf{s}-\mathbf{r})),$$

so that a point $(p, q) \in M \times N$ is an element of Σ_{ALBM} if and only if $\sigma_{ALBM}(p, q) = \mathbf{0}$. A similar map can be constructed to classify points in Σ_{ALBN}. Furthermore, a point (p, q) is a singularity of this map (and hence of Σ_{ALBM}) if and only if the rank of the derivative

$$D(\sigma_{ALBM})_{\mathbf{0}} = \begin{pmatrix} 0 & 2bk & 0 & \sin\theta \\ 0 & \sin\theta & 2ek & 2fk \\ 6a_{30}k^2 & -4bk + 6a_{21}k^2 & 0 & 4bk\cos\theta \end{pmatrix}.$$

falls below three. This is equivalent to having each of the following four equations to be satisfied:

$$4ek^2\left(4b^2k\cos\theta + 2b\sin\theta - 3a_{21}k\sin\theta\right) = 0, \quad 12a_{30}ek^3\sin\theta = 0,$$
$$6a_{30}k^2(4bfk^2 - \sin^2\theta) = 0, \quad -24a_{30}bek^4 = 0$$

Thus, a point (p, q) is a singularity of Σ_{ALBM} if and only if each of these components is zero. The reader may now determine that each of these conditions correspond to those found in the statement of the theorem. □

Singularities in the Projection of Σ_{ALBM}

As we have done previously, we would like to be able to visualize the transitions occuring in the Σ_{ALBM} and Σ_{ALBN} submanifolds. We are lead to the following theorem.

5.2 Theorem. *If $\pi : M \times N \to N$ is a projection onto the second component, then a point (p, q) is a singularity of the restriction $\pi : \Sigma_{ALBM} \to N$ if and only if (p, q) satisfies one of the following conditions:*

- *The segment $\mathbf{s}(p) - \mathbf{r}(q)$ is in an asymptotic direction of N at q, and the surfaces (expressed in Monge form) are such that their coefficients satisfy the condition $\sin^2\theta = 4bfk^2$.*
- *The surface M at p is such that $a_{30} = 0$.*
- *The point p is parabolic on M, and the pair (p, q) is a plane bitangency.*

A similar statement can also be made for the Σ_{ALBN} submanifold.

Unfortunately, due to the complexity of the Σ_{ALBM} and Σ_{ALBN} submanifolds, we have yet to visualize these transitions. Indeed, we would very much like to produce such pictures, and our current work is focused on this task.

THE PLANE BITANGENT DOUBLE POINT SINGULARITY

1 The Plane Bitangency Double Point Singularity

We are now ready to tackle the plane bitangency double point singularity. Due to the complexity of this singularity, we split the analysis into two cases. In the first case, we will assume that the immersion $\mathbf{s} : M \to \mathbb{R}^3$ is locally an elliptic paraboloid of a specific form. In particular, we assume that a local patch of $\mathbf{s} : M \to \mathbb{R}^3$ is given by

$$\mathbf{s}(x, y) = (x, y, x^2 + y^2).$$

This can be done if the immersion $\mathbf{s}$ is already of elliptic type. Otherwise, in the second case, we will assume that the surface M is given locally by a hyperbolic paraboloid of a specific form. In particular, we assume that a local patch of $\mathbf{s} : M \to \mathbb{R}^3$ is given by

$$\mathbf{s}(x, y) = (x, y, x^2 - y^2).$$

Of course, after these transformations are made, we have no control over the shape of the immersion $\mathbf{r} : N \to \mathbb{R}^3$. We now proceed with the analysis of the two cases.

2 The Immersion s is Elliptic

We assume first that $\mathbf{s} : M \to \mathbb{R}^3$ is of the elliptic type. This first case has the most diversity, and is therefore the more interesting one. Keep in mind that we are now studying the plane bitangency double point singularity, which led to the matrix of the derivative $D(\sigma)_{\mathbf{0}}$ to be the zero matrix, when the surfaces were in Monge form. Since this is a corank 2 singularity, we are encouraged to consider the hessians of σ_1 and σ_2, where $\sigma = (\sigma_1, \sigma_2)$ is the line bitangency map defined in Chapter 3. After a quick calculation, we see that

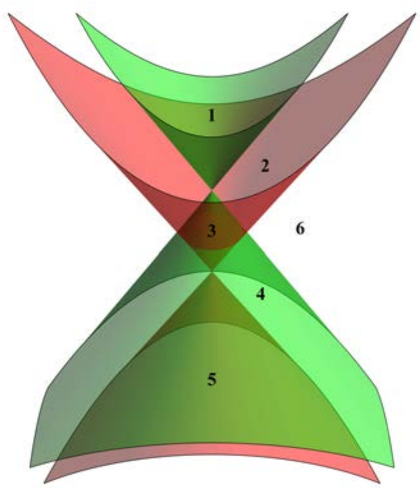
$$\text{Hess}(\sigma_1)_{\mathbf{0}} = \begin{pmatrix} -2 & 0 & 2 & 0 \\ 0 & -2 & 0 & 2 \\ 2 & 0 & -2e & -2f \\ 0 & 2 & -2f & -2g \end{pmatrix} \qquad \text{Hess}(\sigma_2)_{\mathbf{0}} = \begin{pmatrix} 2 & 0 & -2e & -2f \\ 0 & 2 & -2f & -2g \\ -2e & -2f & 2e & 2f \\ -2f & -2g & 2f & 2g \end{pmatrix}$$

Figure 7: Hessians evaluated at $\mathbf{0}$ for when $\mathbf{s}$ is of elliptic type.

We conclude that these Hessians will be singular if and only if one of the following two conditions are satisfied:

$$f^2 - eg = 0 \qquad f^2 - eg + e + g - 1 = 0.$$

Since we are only interested in non-degenerate singularities, we may safely avoid these conditions. We are therefore encouraged to consider the 6 regions shown below:



Notice that these two cones impose conditions on the coefficients e, f, and g of the immersion $\mathbf{r}$. By avoiding coefficients on these cones, we ensure that singularities remain non-degenerate.

Our aim now is to characterize the possible types of transitions within each region according to the signatures of the hessians. In effect, the different values for the signatures will signify different

topological types of transitions.

Remark. For the reader who is not familiar with this type of work, the *signature* of a matrix is the number of positive eigenvalues minus the number of negative eigenvalues that the matrix has. For example, if a matrix has the eigenvalues $-1, 1, -5, 3$, then the signature is zero, while if a matrix has the eigenvalues $-1, 2, 3, 4$, then the signature is two.

Regions 1 and 3

Consider regions 1 and 3, and denote $\mathrm{Hess}(\sigma_1)$ and $\mathrm{Hess}(\sigma_2)$ by H_1 and H_2, respectively. We calculate the signatures to be $\mathrm{sgn}(H_1) = 4$ and $\mathrm{sgn}(H_2) = 0$ in region 1 and vice versa for region 3. We are interested in visualizing the transitions in each of the submanifolds, so we begin with the double point and plane bitangency curves shown below.

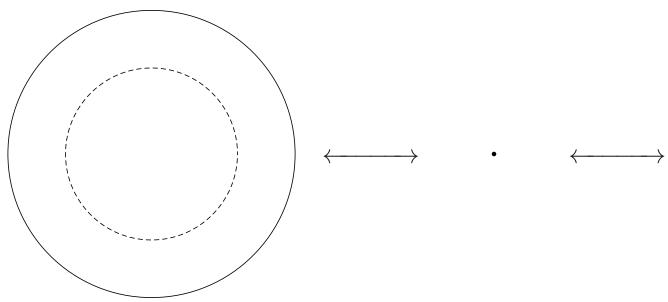


Figure 8: In regions 1 and 3, we visualize the transitions in the double point and plane bitangency curves. The double point curve is represented by the solid line, and the plane bitangency curve is represented by the dashed line.

To visualize the transition in the line bitangency manifold, consider the figure below.

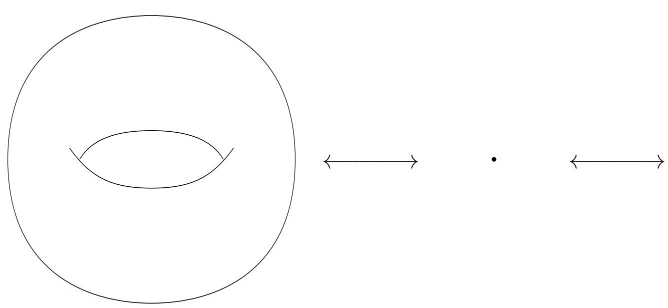


Figure 9: In regions 1 and 3, we visualize the transition in the line bitangency surface.

Region 2

Consider region 2. We calculate the signatures to be $\mathrm{sgn}(H_1) = \mathrm{sgn}(H_2) = 2$. We are interested in visualizing the transitions in each of the submanifolds, so we begin with the double point and plane bitangency curves shown below.

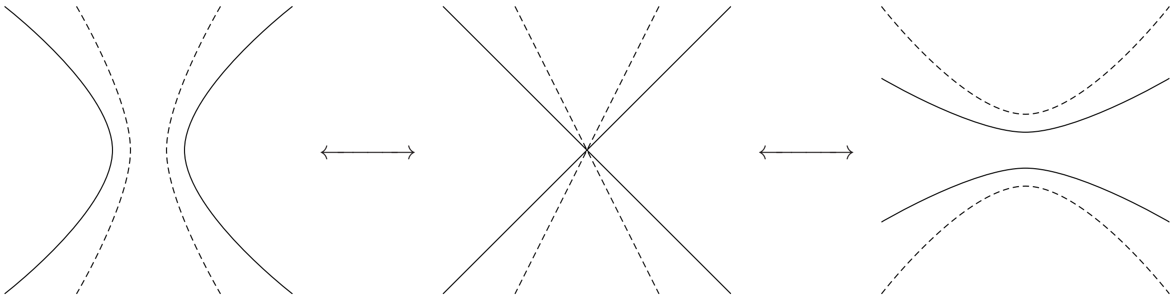


Figure 10: In region 2, we visualize the transitions in the double point and plane bitangency curves. The double point curve is represented by the solid line, and the plane bitangency curve is represented by the dashed line.

To visualize the transition in the line bitangency manifold, consider the figure below.

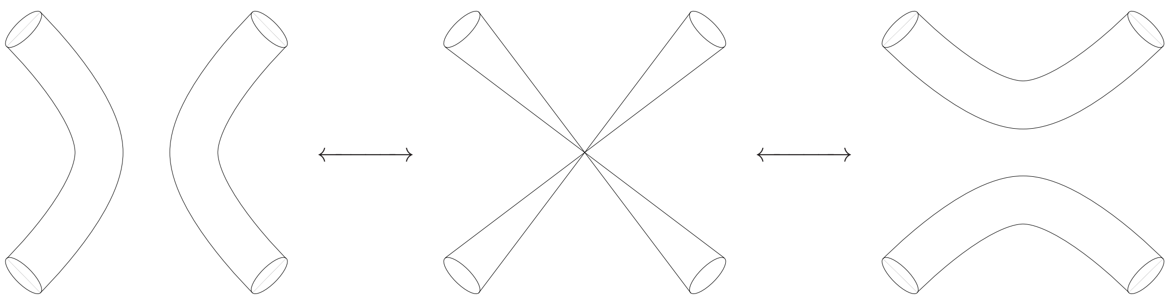


Figure 11: In region 2, we visualize the transition in the line bitangency surface.

Region 4

Consider region 4. We calculate the signatures to be $\mathrm{sgn}(H_1) = 2$ and $\mathrm{sgn}(H_2) = 0$. We are interested in visualizing the transitions in each of the submanifolds, so we begin with the double point and plane bitangency curves shown below.

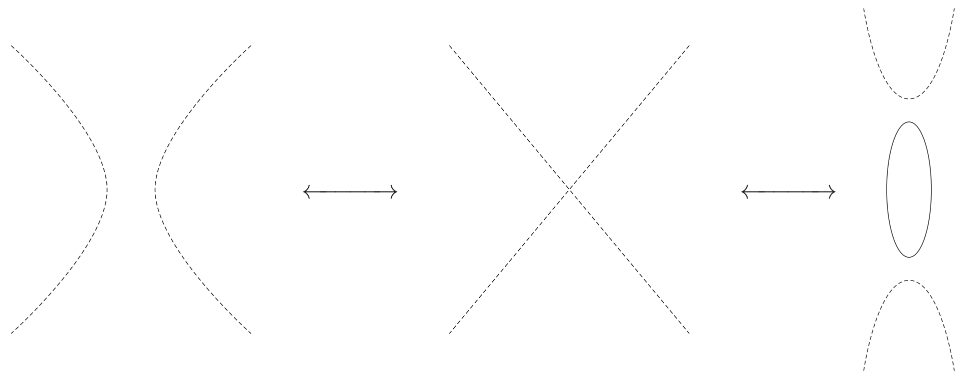


Figure 12: In regions 4 and 6, we visualize the transitions in the double point and plane bitangency curves. The double point curve is represented by the solid line, and the plane bitangency curve is represented by the dashed line.

To visualize the transition in the line bitangency manifold, consider the figure below.

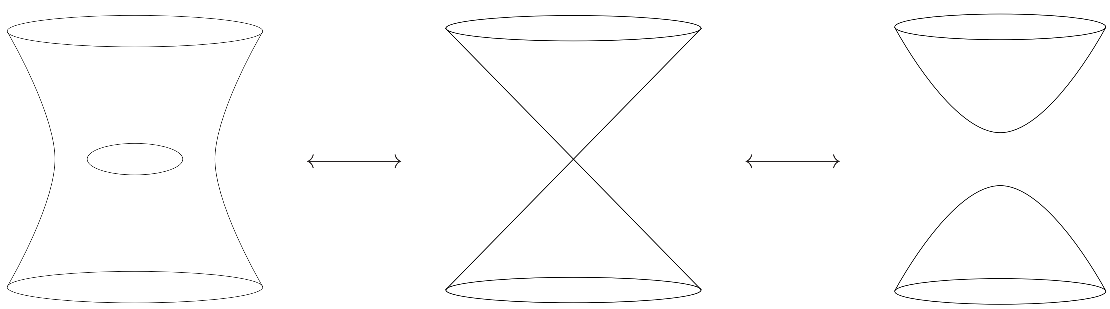


Figure 13: In regions 4 and 6, we visualize the transition in the line bitangency surface.

Region 5

Consider region 5. We calculate the signatures to be $\operatorname{sgn}(H_1) = \operatorname{sgn}(H_2) = 0$. We are interested in visualizing the transitions in each of the submanifolds, so we begin with the double point and plane bitangency curves shown below.

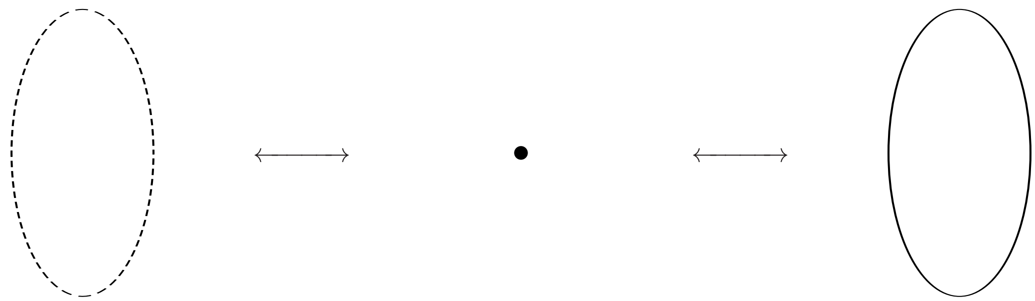


Figure 14: In region 5, we visualize the transitions in the double point and plane bitangency curves. The double point curve is represented by the solid line, and the plane bitangency curve is represented by the dashed line.

To visualize the transition in the line bitangency manifold, consider the figure below.

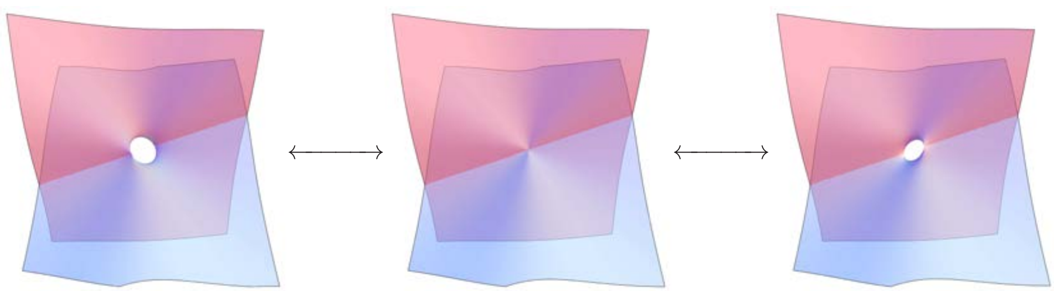


Figure 15: In region 5, we visualize the transition in the line bitangency surface.

3 The Immersion s is Hyperbolic

We now consider the second case where $\mathbf{s} : M \to \mathbb{R}^3$ is of the hyperbolic type. That is, we now assume that $\mathbf{s}(x, y) = (x, y, x^2 - y^2)$. As we did for the first case, we need to calculate the hessians of σ_1 and σ_2. This is done below.

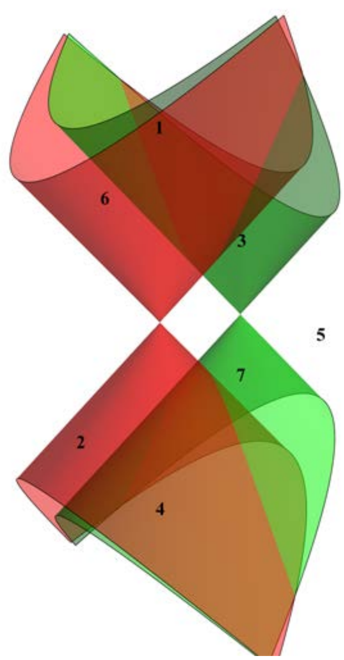
$$\text{Hess}(\sigma_1) = \begin{pmatrix} -2 & 0 & 2 & 0 \\ 0 & 2 & 0 & -2 \\ 2 & 0 & -2e & -2f \\ 0 & -2 & -2f & -2g \end{pmatrix} \qquad \text{Hess}(\sigma_2) = \begin{pmatrix} 2 & 0 & -2e & -2f \\ 0 & -2 & -2f & -2g \\ -2e & -2f & 2e & 2f \\ -2f & -2g & 2f & 2g \end{pmatrix}$$

Figure 16: Hessians for when $\mathbf{s}$ is of hyperbolic type.

We conclude that these Hessians will be singular if and only if one of the following two conditions are satisfied:

$$f^2 - eg = 0 \qquad f^2 - eg - e + g + 1 = 0.$$

Since we are only interested in non-degenerate singularities, we may safely avoid these conditions. We are therefore encouraged to consider the 7 regions shown below:



Fortunately, no new topological types of transitions occur when $\mathbf{s}$ is hyperbolic. Rather, we have the following table of correspondences:

Signature Pairs	**s** is elliptic	**s** is hyperbolic
(4,0)	Regions 1 and 3	N/A
(2,2)	Region 2	Regions 3 and 7
(2,0)	Regions 4 and 6	Regions 1, 2, 4, and 6
(0,0)	Region 5	Region 5

Figure 17: This table demonstrates the correspondences between the different transitions when **s** is elliptic and when **s** is hyperbolic according to topological type.

From the table above and our previous work for when **s** is of elliptic type, we can now visualize the different types of transitions when **s** is hyperbolic.

CONCLUSION

We conclude with our goals of future research and state some open questions. The first open question addresses singularities in Σ_{LB}. Recall that in a one-parameter family of immersion pairs, Σ_{LB} will have a singularity at (p, q) when the line bitangency is asymptotic to both surfaces, and when a specific relationship between the distance and the curvature of each surface is satisfied. In our prescribed Monge form, this relationship was for $\sin^2\theta = 4bfk^2$. We would like to have a better understanding of the geometric content of this relationship. In particular, we ask how we can visualize the significance of this relationsip, and why is it there? One future research goal is to investigate these questions in hopes of a complete geometric description of this singularity.

The second open question addresses asymptotic line bitangency submanifolds. As was demonstrated in this paper, these submanifolds are considerably more complicated than their counterparts, and because of this, we were unable to visualize the possible transitions in a one-parameter family of immersion pairs. In the future, we would like to obtain a full understanding of their transitions since they play such vital roles in the projections of Σ_{LB}.

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VITA

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