

2016

A Saddlepoint Approximation to Left-Tailed Hypothesis Tests of Variance for Non-normal Populations

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A Saddlepoint Approximation to Left-Tailed Hypothesis Tests of Variance for Non-
normal Populations

by

Tyler Luke Grimes

A Thesis submitted to the Department of Mathematics and Statistics
in partial fulfillment of the requirements for the degree of
Masters of Science in Mathematical Science with a concentration in Statistics

UNIVERSITY OF NORTH FLORIDA

COLLEGE OF ARTS AND SCIENCE

July, 2016

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DEDICATION

I dedicate this work to my parents, Alex and Suzy, for their endless love and support.

ACKNOWLEDGMENTS

I first want to extend my deepest gratitude to Dr. Sa. This has been an immensely rewarding journey that could not have been possible without your continual guidance, patience, and inspiration.

Thank you to Dr. Sen, who has been a mentor to me from the start of my graduate studies. I appreciate all of the knowledge you have shared, and thank you for your time and feedback while serving on my thesis committee.

Finally, thank you to Dr. Mohr for serving as a committee member and for all of your valuable input.

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ABSTRACT

When the variance of a single population needs to be assessed, the well-known chi-squared test of variance is often used but relies heavily on its normality assumption. For non-normal populations, few alternative tests have been developed to conduct left tailed hypothesis tests of variance. This thesis outlines a method for generating new test statistics using a saddlepoint approximation. Several novel test statistics are proposed. The type-I error rates and power of each test are evaluated using a Monte Carlo simulation study. One of the proposed test statistics, R_{gamma2} , controls type-I error rates better than existing tests, while having comparable power. The only observed limitation is for populations that are highly skewed with heavy-tails, for which all tests under consideration performed poorly.

CHAPTER 1: INTRODUCTION

1.1 Motivation

There are many real-world problems that require knowledge about the variability in a population. For example, in a quality control setting the variability of a product being produced, or of an input into a process, needs to be controlled. Likewise in a clinical trial, the amount of variation in treatment effects on the people in a population needs to be assessed. In these situations, low variability in the population is desired. To determine whether or not the variability is low, a hypothesis test can be conducted.

Variability can be measured in terms of variance. If evidence of low variability is desired, then a left-tailed test of variance can be conducted. The variance of the population is assumed to be at least some amount, say σ_0^2 . This is the hypothesized value under the null hypothesis, H_0 . The alternative hypothesis, H_a , is that the variance is smaller than σ_0^2 . A significance level is chosen, which indicates the acceptable rate of type-I errors; these errors occur when H_0 is falsely rejected. Finally, a test statistic is computed and a decision rule is followed to determine whether the null hypothesis should be rejected. If it is rejected, then there is sufficient evidence to conclude that the variance is smaller than σ_0^2 . Otherwise, the sample did not provide sufficient evidence to reject H_0 .

In this context, controlling type-I error rates is particularly important. A test statistic with inflated type-I errors will lead the researcher to falsely conclude that the population has small variance more often than is expected. In a quality control setting, this error could mean that a much larger proportion of products will fail to meet specifications than anticipated. In a financial market setting, underestimating the

variability will mean that more risk is being taken on than predicted. While these errors are inevitable, the researcher expects to be able to control their rate of occurrence by setting an appropriate significance level. On the other hand, if a type-II error occurs, that is, the test incorrectly fails to reject H_0 , the researcher will either collect more data and conduct another test, or acknowledge the lack of evidence for low variability and decide what to do from there. So, having low power may lead to a waste of money and resources, but this consequence is often more benign than making a type-I error. Therefore, when comparing and recommending a test statistic, controlling type-I error rates will take precedence over providing high power.

1.2 Background

This thesis focuses on left-tailed tests of variance for a single population. An attempt is made to find a test statistic that works well for a wide range of population distributions. The hypothesis test of interest is

$$H_0: \sigma^2 \geq \sigma_0^2$$

$$H_a: \sigma^2 < \sigma_0^2$$

with a significance level, α .

Suppose x_1, x_2, \dots, x_n is a random sample of size n from a population. If the population is normal, the well-known chi-squared test of variance is used. The test statistic is

$$\chi_T^2 = (n - 1) \frac{S^2}{\sigma_0^2}$$

where $S^2 = \frac{1}{n-1} \sum (x_i - \bar{x})^2$ is the sample variance and $\bar{x} = \frac{1}{n} \sum x_i$ is the sample mean.

The distribution of χ_T^2 under H_0 is $\chi_{df=n-1}^2$. Therefore, the decision rule is to reject H_0 if $\chi_T^2 < \chi_{\alpha, n-1}^2$, where $\chi_{\alpha, n-1}^2$ is the $(100 \cdot \alpha)^{th}$ percentile of the chi-square distribution with $n - 1$ degrees of freedom. The major problem with this test is its sensitivity to any departure from normality. Kendall (1994) proposed a robust chi-square statistic,

$$\chi_r^2 = (n - 1)\hat{d} \frac{S^2}{\sigma^2}$$

which has a $\chi_{df=[(n-1)\hat{d}]}^2$ distribution under H_0 , where

$$\hat{d} = \left(1 + \frac{\hat{\eta}}{2}\right)^{-1}$$

and

$$\hat{\eta} = \frac{\sum (x_i - \bar{x})^4 / n}{(\sum (x_i - \bar{x})^2 / n)^2}$$

is the sample kurtosis coefficient. The degrees of freedom, $df = [(n - 1)\hat{d}]$, is the smallest integer that is greater than or equal to $(n - 1)\hat{d}$. By introducing the factor \hat{d} into the original chi-square statistic, the robust chi-square allows flexibility to the tail behavior of the distribution. Lee & Sa (1998) investigates its performance for right-tailed hypothesis tests and find that it works well for heavy-tailed distributions. For most skewed distributions, it had inflated type-I error rates. The performance of χ_r^2 for left-tailed testing has not been extensively studied.

The next test requires the cumulant generating function and cumulants. Let $M_X(t) = E(e^{tX})$ denote the usual moment generating function (MGF) for a random variable X . The cumulant generating function (CGF) is defined as the log of the MGF.

That is, $K_X(t) = \log(M_X(t))$. The i^{th} cumulant is $\kappa_i = K_X^{(i)}(0)$, the i^{th} derivative of $K_X(t)$ evaluated at $t = 0$. Formulas for the first ten cumulants of a random variable are given in Kendall (1994), and are expressed in terms of the moments. The i^{th} sample cumulant, k_i , can be computed by plugging the sample moments, $m_i = \sum x^i/n$, into the formulas.

Long & Sa (2005) derived a statistic by inverting the Edgeworth expansion of the sample variance. This test statistic will be denoted by $Z6$. The approach incorporates the first six sample cumulants, which allows flexibility for both skewed and heavy-tailed distributions. For right-tailed tests, the decision rule is to reject H_0 if $Z6 > Z_{1-\alpha} + n^{-\frac{1}{2}} \left(\hat{B}_1 + \frac{\hat{B}_2(Z_{1-\alpha}^2 - 1)}{6} \right)$. This rule can be modified for a left-tailed test by reversing the inequality and using the percentile Z_α in place of $Z_{1-\alpha}$. Then, the decision rule for a left-tailed test is to reject H_0 if

$$Z6 < Z_\alpha + n^{-1/2} \left(\hat{B}_1 + \frac{\hat{B}_2(Z_\alpha^2 - 1)}{6} \right)$$

where,

$$Z6 = \frac{s^2 - \sigma_0^2}{\left(\frac{k_4 \sigma_0^2}{n s^2} + \frac{2 \sigma_0^4}{n - 1} \right)^{1/2}}$$

$$\hat{B}_1 = - \left(\frac{s^4}{k_4 + 2 s^4} \right)^{1/2}$$

$$\hat{B}_2 = \frac{k_6 + 12 k_4 s^2 + 4 k_3^2 + 8 (s^2)^3}{(k_4 + 2 s^4)^{3/2}}$$

and Z_α is the $(100 \cdot \alpha)^{th}$ percentile of the standard normal distribution. If $k_4 + 2s^4 < 0$ or $\frac{k_4\sigma_0^2}{ns^2} + \frac{2\sigma_0^4}{n-1} < 0$, set $k_4 = 0$. For sample sizes as small as $n = 20$, this procedure works well for right-tailed tests regardless if the population is skewed or heavy-tailed. The power curves were higher for heavy-tailed distributions, and lower, but still good, for skewed distributions. Inflated type-I error rates were noted for some skewed distributions when both sample size and alpha were small. The performance of Z6 was not studied comprehensively for left-tailed tests.

In this paper, several novel test statistics are proposed using results from Jensen (1995), who developed a large-deviation-adjusted signed log-likelihood ratio statistic. For this approach, the population is assumed to have a particular distribution from the exponential family. After expressing this distribution in its exponential form, the corresponding test statistic can be derived. This procedure is detailed in chapter 2. Monte Carlo simulations are used to assess the performance of each statistic. These test statistics have a known asymptotic distribution when the population has the same distribution that the test is derived from, and if any known nuisance parameters are included. However, in practice, nuisance parameters will be unknown, and the distribution of the population is likely to be unknown as well. The simulation study described in chapter 3 is used to determine how robust the tests are to violations in these assumptions.

The goal of this thesis is to assess and compare the performances of the proposed test statistics and the existing statistics χ_T^2 , χ_r^2 , and Z6, measured by their type-I-error rates and power. In chapter 4, the results from the simulation study are discussed, and a final recommendation is given in the conclusion of chapter 5.

CHAPTER 2: THE PROPOSED TESTS

Several novel test statistics are proposed in this thesis. This chapter provides the necessary background and summarizes the results used from Jensen (1995). Six statistics are derived, each from a particular base distribution. In addition, an adjusted test statistic, and a test that chooses an appropriate test statistic based on maximum likelihood, are also presented.

2.1 Exponential Family

Let x_1, x_2, \dots, x_n be n independent and identically distributed (iid) random variables from a distribution in the exponential family. A distribution is in the exponential family if its density can be written in the form of,

$$f(x) = e^{\theta t(x) - K(\theta)} h(x), \quad (2.1)$$

where $\theta = (\theta_1, \dots, \theta_p) \in \mathcal{R}^p$ are the parameters for the distribution. For some distributions, such as the Weibull, the distribution can only be written in the exponential form if some of its parameters are considered as known constants. For the Weibull distribution, its shape parameter must be considered as a constant. Otherwise, the distribution cannot be written in the form of (2.1) and thus cannot be considered as part of the exponential family.

2.2 Adjusted Signed Log-likelihood Ratio Statistic

The test statistic considered under the null hypothesis $H_0: \theta = \theta_0$ is the large-deviation-adjusted signed log-likelihood ratio, R_L , given by

$$R_L = R + \frac{1}{R} \log \left(\frac{U}{R} \right), \quad (2.2)$$

where,

$$R = \text{sign}(\hat{\theta} - \theta_0) \sqrt{2n} \{ (\hat{\theta} - \theta_0) \bar{t} - K(\hat{\theta}) + K(\theta_0) \}^{1/2} \quad (2.3)$$

$$U = \sqrt{n} (\hat{\theta} - \theta_0) \left(K''(\hat{\theta}) \right)^{1/2} \quad (2.4)$$

with $\bar{t} = \frac{1}{n} \sum_{i=1}^n t(x_i)$ and where $\hat{\theta}$ is determined by $K'(\hat{\theta}) = \bar{t}$ (Jensen 1995). Note, $t(\cdot)$ and $K(\cdot)$ come from the population density function as expressed in (2.1).

The Lugananni-Rice formula is a saddlepoint approximation to the tail probabilities of a distribution that has a simple structure and incorporates quantities related to statistics. For the distribution of R_L , the Lugananni-Rice expansion of R can be reformulated in terms of R_L such that

$$P(R_L \geq r_L) = \{1 - \Phi(r_L)\} \left(1 + O \left((1 + r_L) n^{-3/2} \right) \right)$$

where Φ is the cumulative distribution function (CDF) of the standard normal distribution (Jensen 1995). Hence R_L is asymptotically standard normally distributed with $P(R_L \leq r_L) \approx \Phi(r_L)$. The relative error of this normal approximation is $O(n^{-1})$ in a large-deviation region of r_L . This result also holds if the exponential family is of order $p > 1$ (Jensen 1995) and R_L is used to test the hypothesis $H_0: \theta_1 = \theta_{01}$.

2.3 Hypothesis Tests

Consider the left-tailed hypothesis test on the population variance,

$$H_0: \sigma^2 \geq \sigma_0^2$$

$$H_a: \sigma^2 < \sigma_0^2$$

In order to use the test statistic R_L , the hypothesis needs to be written in terms of the parameter θ . This is done for each test statistic derived later in this chapter. If the distribution has $p > 1$ parameters, the hypothesis will test $\theta_1 = \theta_{01}$, and treat the other $p - 1$ nuisance parameters $(\theta_2, \dots, \theta_p)$, as known. If the nuisance parameters truly are known, then the relative error of the normal approximation will still be on the order of n^{-1} . Otherwise, if the nuisance parameters are unknown and estimates are used instead, the error may be much higher. One goal of the simulation study is to determine the effects of using estimates in place of the presumed known parameters.

2.4 Proposed Test Statistics

Six base distributions are considered. These include the normal, chi-squared, exponential, gamma, Weibull, and lognormal. In this section, the corresponding test is derived for each of these base distributions. If a distribution has two parameters, one of them is assumed to be known. However, in practice this parameter will probably be unknown and will need to be estimated. For these situations, an estimator will be provided and is assessed in the simulation study.

2.4.1 Normal Distribution

First, consider the normal distribution with a known mean, μ , and unknown variance, σ^2 . The density for the normal can be written as $f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1(x-\mu)^2}{2\sigma^2}}$. This density can be expressed in the exponential form from (2.1) by,

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1(x-\mu)^2}{2\sigma^2}} \\
&= \exp\left\{\frac{1}{\sigma^2}\left(\frac{-(x-\mu)^2}{2}\right) + \ln\left(\frac{1}{\sqrt{\sigma^2}}\right)\right\} \frac{1}{\sqrt{2\pi}} \\
&= \exp\left\{\frac{1}{\sigma^2}\left(\frac{-x^2+2x\mu-\mu^2}{2}\right) - \left(-\frac{1}{2}\ln\left(\frac{1}{\sigma^2}\right)\right)\right\} \frac{1}{\sqrt{2\pi}} \\
&= \exp\left\{\frac{1}{\sigma^2}\left(-\frac{x^2}{2} + x\mu\right) - \frac{1}{2}\left(\frac{\mu^2}{\sigma^2} - \ln\left(\frac{1}{\sigma^2}\right)\right)\right\} \frac{1}{\sqrt{2\pi}} \\
&= e^{\theta t(x) - K(\theta)} h(x)
\end{aligned}$$

for $-\infty \leq x \leq \infty$, $-\infty \leq \mu \leq \infty$, and $\sigma^2 > 0$.

Note, since μ is assumed to be known, the parameter space is only one-dimensional; in this case it is \mathcal{R}^+ . From θ , t , and K , expressions for K' , K'' , \bar{t} , and $\hat{\theta}$ can be derived. A summary of these results follows.

$$\theta = \frac{1}{\sigma^2}$$

$$t(x) = -\frac{x^2}{2} + x\mu = -\frac{1}{2}(x - \mu)^2 + \frac{\mu^2}{2}$$

$$K(\theta) = \frac{1}{2}(\mu^2\theta - \ln(\theta))$$

$$K'(\theta) = \frac{1}{2}\left(\mu^2 - \frac{1}{\theta}\right)$$

$$K''(\theta) = \frac{1}{2\theta^2}$$

$$\bar{t} = \frac{1}{n} \sum_{i=1}^n \left(-\frac{1}{2}(x_i - \mu)^2 + \frac{\mu^2}{2} \right) = \frac{\mu^2}{2} - \frac{1}{2n} \sum_{i=1}^n (x_i - \mu)^2$$

Solving $K'(\hat{\theta}) = \bar{t}$ for $\hat{\theta}$,

$$\begin{aligned}
\frac{\mu^2}{2} - \frac{1}{2\hat{\theta}} &= \frac{\mu^2}{2} - \frac{1}{2n} \sum_{i=1}^n (x_i - \mu)^2 \\
\Rightarrow \frac{1}{\hat{\theta}} &= \frac{1}{n} \sum_{i=1}^n (x_i - \mu)^2 \\
\Rightarrow \hat{\theta} &= \frac{n}{\sum_{i=1}^n (x_i - \mu)^2}
\end{aligned}$$

Rewriting $t(x)$ by completing the square leads to a nice form for $\hat{\theta}$. It suggests the estimate $\hat{\theta} = 1/S^2$, in which case a factor of $n - 1$ is used rather than n . Since the hypothesis test is about variance, sample variance is preferred in the test statistic. And the simulation study confirms that this choice does, in fact, lead to a better result.

The appropriate θ_0 needs to be determined for the hypothesis test. Since $\theta = 1/\sigma^2$, $\theta_0 = 1/\sigma_0^2$. The original test is on $H_0: \sigma^2 \geq \sigma_0^2$. In terms of θ , the null is $\frac{1}{\theta} \geq \sigma_0^2$, which is $\theta \leq \frac{1}{\sigma_0^2}$. So the hypothesis becomes,

$$\begin{aligned}
H_0: \sigma^2 \geq \sigma_0^2 &\quad \Leftrightarrow \quad H_0: \theta \leq \theta_0 \\
H_a: \sigma^2 < \sigma_0^2 &\quad \Leftrightarrow \quad H_a: \theta > \theta_0
\end{aligned}$$

This is now a right tailed test in θ , but it still corresponds to a left tailed test with respect to variance. To obtain the test statistic, substitute everything into R and U from equation (2.3) and (2.4).

$$R = \text{sign}\left(\frac{1}{S^2} - \frac{1}{\sigma_0^2}\right) \sqrt{2n} \left\{ \left(\frac{1}{S^2} - \frac{1}{\sigma_0^2}\right) \frac{1}{2} (\hat{\mu}^2 - S^2) - \frac{1}{2} \left(\frac{\hat{\mu}^2}{S^2} - \ln\left(\frac{1}{S^2}\right)\right) + \frac{1}{2} \left(\frac{\hat{\mu}^2}{\sigma_0^2} - \ln\left(\frac{1}{\sigma_0^2}\right)\right) \right\}^{1/2}$$

$$U = \sqrt{n} \left(\frac{1}{S^2} - \frac{1}{\sigma_0^2}\right) \left(\frac{(S^2)^2}{2}\right)^{1/2}$$

And the test statistic is $R_{norm} = R + \frac{1}{R} \log \left(\frac{U}{R} \right)$. Note that R and U depend only on $\hat{\mu}$, S^2 , σ_0^2 , and n . The decision rule is to reject H_0 if $R_{norm} > Z_{1-\alpha}$. The maximum likelihood (MLE) estimator $\hat{\mu} = \frac{1}{n} \sum_{i=1}^n x_i = \bar{x}$ for μ is used, and $\hat{\theta}$ is computed using $1/S^2$, as discussed previously.

Another goal of the simulation study is to assess how well each test statistic performs for various distributions of the population. In this case, R_{norm} was derived from the assumption that the population is normally distributed. But, if R_{norm} is robust to this assumption, then it will work well even if the population is not normal. The hope is that at least one of the tests derived in this chapter will be robust against departures from its distribution assumption.

2.4.2 Chi-Squared Distribution

Consider a chi-squared distribution with unknown parameter $r > 0$. The density for the chi-square distribution can be written as $f(x) = \frac{1}{2^{(r/2)} \Gamma(r/2)} x^{(r/2-1)} e^{-x/2}$. This density

can be expressed in the exponential form from (2.1) by,

$$\begin{aligned} f(x) &= \frac{1}{2^{(r/2)} \Gamma(r/2)} x^{(r/2-1)} e^{-x/2} \\ &= \exp \left\{ -\ln \left(2^{(r/2)} \Gamma \left(\frac{r}{2} \right) \right) + \ln \left(x^{(r/2-1)} \right) - \frac{x}{2} \right\} \\ &= \exp \left\{ -\frac{r}{2} \ln(2) - \ln \left(\Gamma \left(\frac{r}{2} \right) \right) + \frac{r}{2} \ln(x) - \ln(x) - \frac{x}{2} \right\} \end{aligned}$$

$$\begin{aligned}
&= \exp \left\{ r \frac{\ln(x)}{2} - \left(\frac{r}{2} \ln(2) + \ln \left(\Gamma \left(\frac{r}{2} \right) \right) \right) \right\} \exp \left\{ -\ln(x) - \frac{x}{2} \right\} \\
&= e^{\theta t(x) - K(\theta)} h(x)
\end{aligned}$$

for $x > 0$ and $r > 0$.

A summary of $\theta, t, K, K', K'', \bar{t}$, and $\hat{\theta}$ follows. Note, these involve the log-gamma, digamma, and trigamma functions; each will be denoted by $\gamma(x) = \ln(\Gamma(x))$, $\psi(x) = \frac{d}{dx} \ln(\Gamma(x))$, and $\psi'(x) = \frac{d}{dx} \psi(x)$, respectively. The inverse of the digamma is also required and will be denoted by $\psi^{-1}(x)$.

$$\theta = r$$

$$t(x) = \frac{\ln(x)}{2}$$

$$K(\theta) = \frac{\theta}{2} \ln(2) + \gamma \left(\frac{\theta}{2} \right)$$

$$K'(\theta) = \frac{\ln(2)}{2} + \frac{1}{2} \psi \left(\frac{\theta}{2} \right)$$

$$K''(\theta) = \frac{1}{4} \psi' \left(\frac{\theta}{2} \right)$$

$$\bar{t} = \frac{1}{2n} \sum_{i=1}^n \ln(x_i)$$

$$\hat{\theta} = 2\psi^{-1}(2\bar{t} - \ln(2))$$

The variance of the chi-squared distribution is $\sigma^2 = 2r = 2\theta$. So $\theta_0 = \frac{\sigma_0^2}{2}$ and the null hypothesis of $\sigma^2 \geq \sigma_0^2$ is equivalent to $\theta \geq \frac{\sigma_0^2}{2}$.

$$H_0: \theta \geq \frac{\sigma_0^2}{2}$$

$$H_a: \theta < \frac{\sigma_0^2}{2}$$

Substituting everything into R and U ,

$$R = \text{sign}\left(\hat{\theta} - \frac{\sigma_0^2}{2}\right) \sqrt{2n} \left\{ \left(\hat{\theta} - \frac{\sigma_0^2}{2}\right) \frac{1}{2n} \sum_{i=1}^n \ln(x_i) - \left(\frac{\hat{\theta}}{2} \ln(2) + \gamma\left(\frac{\hat{\theta}}{2}\right)\right) + \left(\frac{\sigma_0^2}{4} \ln(2) + \gamma\left(\frac{\sigma_0^2}{4}\right)\right) \right\}^{1/2}$$

$$U = \sqrt{n} \left(\hat{\theta} - \frac{\sigma_0^2}{2}\right) \left(\frac{1}{4} \psi'\left(\frac{\hat{\theta}}{2}\right)\right)^{1/2}$$

and the test statistic is $R_{chisq} = R + \frac{1}{R} \log\left(\frac{U}{R}\right)$. The decision rule is to reject H_0 if

$$R_{chisq} < Z_\alpha.$$

2.4.3 Exponential Distribution

Consider the exponential distribution with rate parameter $\lambda > 0$. The density for the exponential distribution can be written as $f(x) = \lambda e^{-\lambda x}$. This density can be expressed in the exponential form from (2.1) by,

$$\begin{aligned} f(x) &= \lambda e^{-\lambda x} \\ &= e^{\ln \lambda - \lambda x} \\ &= e^{\lambda(-x) - (-\ln(\lambda))} \end{aligned}$$

for $x > 0$ and $\lambda > 0$. Therefore,

$$\theta = \lambda$$

$$t(x) = -x$$

$$K(\theta) = -\ln(\theta)$$

$$K'(\theta) = -\frac{1}{\theta}$$

$$K''(\theta) = \frac{1}{\theta^2}$$

$$\bar{t} = -\frac{1}{n} \sum_{i=1}^n x_i = -\bar{x}$$

$$\hat{\theta} = -\frac{1}{\bar{t}} = \frac{1}{\bar{x}}$$

Since the variance of the exponential distribution is $\sigma^2 = \frac{1}{\lambda^2} = \frac{1}{\theta^2}$, $\theta_0 = \left(\frac{1}{\sigma_0^2}\right)^{1/2}$.

The null hypothesis of $\sigma^2 \geq \sigma_0^2$ implies $1/\theta^2 \geq \sigma_0^2$, and hence $\theta \leq \left(\frac{1}{\sigma_0^2}\right)^{1/2}$. Therefore,

the original left-tailed test of $H_0: \sigma^2 \geq \sigma_0^2$ versus $H_a: \sigma^2 > \sigma_0^2$ becomes

$$H_0: \theta \leq \left(\frac{1}{\sigma_0^2}\right)^{1/2}$$

$$H_a: \theta > \theta_0$$

After substituting everything into R and U and some simplifying,

$$R = \text{sign} \left(\frac{1}{\bar{x}} - \left(\frac{1}{\sigma_0^2} \right)^{1/2} \right) \sqrt{2n} \left\{ \left(\frac{\bar{x}}{(\sigma_0^2)^{1/2}} - 1 \right) - \ln \left(\frac{\bar{x}}{(\sigma_0^2)^{1/2}} \right) \right\}^{1/2}$$

$$U = \sqrt{n} \left(1 - \frac{\bar{x}}{(\sigma_0^2)^{1/2}} \right)$$

and the test statistic is $R_{exp} = R + \frac{1}{R} \log \left(\frac{U}{R} \right)$. The decision rule is to reject H_0 if

$$R_{exp} > Z_{1-\alpha}.$$

2.4.4 Gamma Distribution

The gamma distribution has two parameters, a shape parameter r and a rate parameter λ . For this derivation, it is assumed that r is known. The density for the gamma distribution can be written as

$$\begin{aligned} f(x) &= \frac{\lambda^r}{\Gamma(r)} x^{r-1} e^{-x\lambda} \\ &= e^{-x\lambda} e^{\ln(\lambda^r)} \frac{x^{r-1}}{\Gamma(r)} \\ &= e^{-x\lambda - (-r \ln(\lambda))} \frac{x^{r-1}}{\Gamma(r)} \end{aligned}$$

for $x > 0, r > 0$, and $\lambda > 0$.

As with the normal distribution, the parameter space is again one-dimensional since only one parameter is considered to be unknown. A summary of $\theta, t, K, K', K'', \bar{t}$, and $\hat{\theta}$ follows.

$$\theta = \lambda$$

$$t(x) = -x$$

$$K(\theta) = -r \ln \theta$$

$$K'(\theta) = -\frac{r}{\theta}$$

$$K''(\theta) = \frac{r}{\theta^2}$$

$$\bar{t} = -\bar{x}$$

$$\hat{\theta} = \frac{r}{\bar{x}}$$

Since the variance of the gamma distribution is $\sigma^2 = \frac{r}{\lambda^2} = \frac{r}{\theta^2}$, $\theta_0 = \left(\frac{r}{\sigma_0^2}\right)^{1/2}$. The null hypothesis of $\sigma^2 \geq \sigma_0^2$ implies $\frac{r}{\theta^2} \geq \sigma_0^2$, and hence $\theta \leq \left(\frac{r}{\sigma_0^2}\right)^{1/2}$. The hypothesis becomes

$$H_0: \theta \leq \left(\frac{r}{\sigma_0^2}\right)^{1/2}$$

$$H_a: \theta > \theta_0$$

After substituting everything into R and U and some simplifying,

$$R = \text{sign}\left(\frac{r}{\bar{x}} - \left(\frac{r}{\sigma_0^2}\right)^{1/2}\right) \sqrt{2n} \left\{ \bar{x} \left(\frac{r}{\sigma_0^2}\right)^{1/2} - r - r \ln \left(\frac{\bar{x}}{(r\sigma_0^2)^{1/2}} \right) \right\}^{1/2}$$

$$U = \sqrt{n} \left(\frac{1}{r^{1/2}} - \frac{\bar{x}}{(\sigma_0^2)^{1/2}} \right)$$

and the test statistic is $R_{gamma} = R + \frac{1}{R} \log\left(\frac{U}{R}\right)$. The decision rule is to reject H_0 if

$$R_{gamma} > Z_{1-\alpha}.$$

The MLE estimate \hat{r}_{MLE} for r could be considered. However, there is no closed-form solution for that estimator, and an iterative procedure must be used to find it. In the simulation study, the estimator

$$\hat{r}_T = \frac{1}{4} L^{-1} \left(1 + \sqrt{1 + \frac{4}{3} L} \right)$$

where $L = \log(\bar{x}) - \overline{\log(x)}$ is used. This is Thom's approximation to \hat{r}_{MLE} (Johnson 1994). Preliminary results showed that this approximation is adequate, and it gives enough of a computational speed-up to be preferable.

2.4.5 Gamma Distribution with Adjustment

An adjustment was found which improves the performance of the test derived from the gamma distribution. Recall that the mean and variance of a gamma distribution are r/λ and r/λ^2 , respectively. Furthermore, if $r = 1$, then the gamma distribution is equivalent to an $\text{exponential}(\lambda)$ distribution. The idea is to remove the effects of the shape parameter, r , by dividing both \bar{x} and σ_0^2 by \hat{r} . We then proceed using the test statistic derived from the exponential distribution. The resulting test statistic for the hypothesis,

$$H_0: \sigma^2 \geq \sigma_0^2$$

$$H_a: \sigma^2 < \sigma_0^2$$

is,

$$R = \text{sign} \left(\frac{\hat{r}}{\bar{x}} - \left(\frac{\hat{r}}{\sigma_0^2} \right)^{1/2} \right) \sqrt{2n} \left\{ \left(\frac{\bar{x}}{(\hat{r}\sigma_0^2)^{1/2}} - 1 \right) - \ln \left(\frac{\bar{x}}{(\hat{r}\sigma_0^2)^{1/2}} \right) \right\}^{1/2}$$

$$U = \sqrt{n} \left(1 - \frac{\bar{x}}{(\hat{r}\sigma_0^2)^{1/2}} \right)$$

To discern this test from the non-adjusted one, denote it by $R_{\text{gamma}2} = R + \frac{1}{R} \log \left(\frac{U}{R} \right)$.

The decision rule is to reject H_0 if $R_{\text{gamma}2} > t_{1-\alpha/2,n}$, where $t_{\alpha,n}$ is the $(100 \cdot \alpha)^{th}$ percentile of the t distribution with n degrees of freedom. Notice that only half of the original significance level is used. This decision is based on preliminary findings of inflated type-I error rates, whereby dividing the significance level in half provided rates closer to the nominal level.

The scaling of both sample mean \bar{x} and null variance σ_0^2 by a factor of $1/\hat{r}$ deserves a comment. One typically expects variance to be scaled by a factor of $(1/\hat{r})^2$ if the mean is scaled by $1/\hat{r}$. During preliminary work, the scaling of σ_0^2 by $(1/\hat{r})^2$ was in fact tested, but the resulting test statistic performed poorly. The justification for using $1/\hat{r}$ is partially due to the fact that the mean and variance of a gamma distribution are r/λ and r/λ^2 , as noted previously. Dividing each of these by r would remove the effect of the scale parameter. In this light, dividing both \bar{x} and σ_0^2 by \hat{r} is sensible. The other part of the justification is empirical, as R_{gamma2} is shown to work well through the simulation study.

2.4.6 Weibull Distribution

The Weibull distribution has two parameters, a shape parameter r and a scale parameter β . If the shape parameter is known, then the Weibull belongs to the exponential family. Given r is known, the density for the Weibull can be written as

$$\begin{aligned}
 f(x) &= \frac{r}{\beta} \left(\frac{x}{\beta}\right)^{r-1} e^{-\left(\frac{x}{\beta}\right)^r} \\
 &= \exp \left\{ -\left(\frac{x}{\beta}\right)^r + \ln \left(\frac{r}{\beta} \left(\frac{x}{\beta}\right)^{r-1} \right) \right\} \\
 &= \exp \left\{ \frac{1}{\beta^r} (-x^r) + \ln \left(\frac{1}{\beta^r} \right) + \ln(rx^{r-1}) \right\} \\
 &= \exp \left\{ \frac{1}{\beta^r} (-x^r) - \left(-\ln \left(\frac{1}{\beta^r} \right) \right) \right\} r x^{r-1}
 \end{aligned}$$

for $x > 0$, $\beta > 0$, and $r > 0$. Then,

$$\theta = \frac{1}{\beta^r}$$

$$t(x) = -x^r$$

$$K(\theta) = -\ln(\theta)$$

$$K'(\theta) = -\frac{1}{\theta}$$

$$K''(\theta) = \frac{1}{\theta^2}$$

$$\bar{t} = -\frac{1}{n} \sum_{i=1}^n x_i^r$$

$$\hat{\theta} = -\frac{1}{\bar{t}} = \frac{n}{\sum_{i=1}^n x_i^r}$$

The variance of the Weibull distribution is $\sigma^2 = \beta^2 \left[\Gamma\left(1 + \frac{2}{r}\right) - \left(\Gamma\left(1 + \frac{1}{r}\right)\right)^2 \right]$. So

$$\theta_0 = \left(\frac{1}{\sigma_0^2} \left[\Gamma\left(1 + \frac{2}{r}\right) - \left(\Gamma\left(1 + \frac{1}{r}\right)\right)^2 \right] \right)^{r/2} \text{ and the null hypothesis of } \sigma^2 \geq \sigma_0^2 \text{ implies}$$

$\theta \leq \theta_0$. The hypothesis becomes

$$H_0: \theta \leq \left(\frac{1}{\sigma_0^2} \left[\Gamma\left(1 + \frac{2}{r}\right) - \left(\Gamma\left(1 + \frac{1}{r}\right)\right)^2 \right] \right)^{r/2}$$

$$H_a: \theta > \theta_0$$

Finally, substituting everything into R and U gives

$$R = \text{sign} \left(\frac{n}{\sum_{i=1}^n x_i^{\hat{r}}} - \theta_0 \right) \sqrt{2n} \left\{ \left(\theta_0 \frac{\sum_{i=1}^n x_i^{\hat{r}}}{n} - 1 \right) - \ln \left(\frac{\sum_{i=1}^n x_i^{\hat{r}}}{n} \right) - \ln(\theta_0) \right\}^{1/2}$$

$$U = \sqrt{n} \left(1 - \theta_0 \frac{\sum_{i=1}^n x_i^{\hat{r}}}{n} \right)$$

and the test statistic is $R_{weib} = R + \frac{1}{R} \log \left(\frac{U}{R} \right)$. The decision rule is to reject H_0 if $R_{weib} > Z_{1-\alpha}$. The MLE estimator \hat{r}_{MLE} for r is used. This estimator has no closed-form solution and is computed iteratively.

2.4.7 Log-normal Distribution

The log-normal distribution has two parameters, a location parameter μ and scale parameter τ . Note that if μ is assumed known, the resulting test statistic, θ_0 , is found related to σ_0^2 by the equation $\sigma_0^2 = (e^{-\theta_0} - 1)e^{2\mu - \theta_0}$, and hence, θ_0 will need to be solved for numerically. Instead of following this route, it will be assumed that τ is known. The density for the log-normal can be written as $f(x) = \frac{1}{x\sqrt{2\pi\tau^2}} e^{-\frac{(\ln x - \mu)^2}{2\tau^2}}$. This density can be expressed in the exponential form from (2.1) by,

$$\begin{aligned} f(x) &= \frac{1}{x\tau\sqrt{2\pi}} e^{-\frac{(\ln x - \mu)^2}{2\tau^2}} \\ &= \exp \left\{ \frac{-((\ln x)^2 - 2\mu \ln x + \mu^2)}{2\tau^2} \right\} \frac{1}{x\tau\sqrt{2\pi}} \\ &= \exp \left\{ \mu \left(\frac{\ln(x)}{\tau^2} \right) - \frac{\mu^2}{2\tau^2} \right\} \exp \left\{ -\frac{(\ln(x))^2}{2\tau^2} \right\} \frac{1}{x\tau\sqrt{2\pi}} \end{aligned}$$

for $x > 0, \mu > 0$, and $\tau > 0$. Then,

$$\theta = \mu$$

$$t(x) = \frac{\ln(x)}{\tau^2}$$

$$K(\theta) = \frac{\theta^2}{2\tau^2}$$

$$K'(\theta) = \frac{\theta}{\tau^2}$$

$$K''(\theta) = \frac{1}{\tau^2}$$

$$\bar{t} = \frac{1}{n\tau^2} \sum_{i=1}^n \ln(x_i)$$

$$\hat{\theta} = \frac{1}{n} \sum_{i=1}^n \ln(x_i)$$

The variance of the log-normal distribution is $\sigma^2 = (e^{\tau^2} - 1)e^{2\mu + \tau^2}$. So

$\theta_0 = \frac{1}{2} \ln \left(\frac{\sigma_0^2}{e^{\tau^2}} (e^{\tau^2} - 1)^{-1} \right)$ and the null hypothesis of $\sigma^2 \geq \sigma_0^2$ implies $\theta \geq \theta_0$.

$$H_0: \theta \geq \frac{1}{2} \ln \left(\frac{\sigma_0^2}{e^{\tau^2}} (e^{\tau^2} - 1)^{-1} \right)$$

$$H_a: \theta < \theta_0$$

Finally, substituting everything into R and U gives

$$R = \text{sign} \left(\frac{1}{n} \sum_{i=1}^n \ln(x_i) - \theta_0 \right) \sqrt{2n} \left\{ \left(\frac{1}{2\tau^2} \left(\frac{1}{n} \sum_{i=1}^n \ln(x_i) \right)^2 - \frac{\theta_0}{n\tau^2} \sum_{i=1}^n \ln(x_i) \right) + \frac{\theta_0^2}{2\tau^2} \right\}^{1/2}$$

$$U = \sqrt{n} \left(\frac{1}{n} \sum_{i=1}^n \ln(x_i) - \theta_0 \right) \frac{1}{\tau}$$

and the test statistic is $R_{lnorm} = R + \frac{1}{R} \log \left(\frac{U}{R} \right)$. The decision rule is to reject H_0 if

$R_{lnorm} < Z_\alpha$. The MLE estimator $\widehat{\tau}_{MLE}^2$ for τ^2 is used. This estimator has no closed form solution and is computed iteratively.

2.4.8 Maximum Likelihood

If the population being studied is known to follow a particular distribution, then an appropriate test corresponding to that distribution should be chosen. Usually, however, the population's distribution is unknown. In this case, the maximum likelihood is used to determine which distribution is most likely to represent the population.

Only two distributions are considered here, the normal and gamma; the reasoning for this is discussed at the end. Given a sample, the maximum likelihood is computed for each distribution. The test procedure corresponding to the distribution with the highest maximum likelihood is used. The test statistic resulting from this method will be denoted by R_{lh} .

Algorithm 1: Assume a sample of n observations $X = \{x_1, \dots, x_n\}$ and a significance level α . This procedure chooses between the two test statistics R_{norm} and R_{gamma2} , performs the test, and either rejects H_0 or fails to do so.

1. Compute the following two maximum likelihoods:

$$L_{norm} = \max_{\mu, \sigma^2} \left\{ \left(\frac{1}{2\pi\sigma^2} \right)^{\frac{n}{2}} \exp \left\{ \frac{-\sum_{i=1}^n (x_i - \mu)^2}{2\sigma^2} \right\} \right\}$$

$$L_{gamma} = \max_{r, \lambda} \frac{\lambda^r}{\Gamma(r)} \exp \left\{ -\lambda \sum_{i=1}^n x_i \right\} \left(\prod_{i=1}^n x_i \right)^{r-1}$$

2. If L_{norm} is largest, use R_{norm} for the test statistic and reject H_0 if $R_{norm} >$

$Z_{1-\alpha/2}$; otherwise, use R_{gamma2} and reject H_0 if $R_{gamma2} > t_{1-\alpha/2, n}$.

To correct for inflated type-I error rates, the critical value for R_{norm} uses only half of the original significance level. The default critical value for R_{gamma2} is already modified to control type-I errors, and it is left unchanged.

The decision to include only the normal distribution and gamma distribution in this procedure is based on preliminary simulations. When including all six distributions discussed in section 2.4, the maximum likelihood was often highest for either the Weibull or log-normal distribution, so these were incorrectly chosen a large proportion of the time. Since R_{weib} and R_{lnorm} do not perform well when the population is not Weibull or log-normal, respectively, the overall performance was poor. By excluding them, the performance is improved considerably. Furthermore, the chi-squared and exponential distributions are very rarely selected, so removing them simplifies the procedure while maintaining the same efficacy.

CHAPTER 3: SIMULATION

This chapter details the simulation study. The goal of the study is to examine the type-I error rates and the power for each of the proposed test procedures and for the existing statistics χ_T^2 , χ_r^2 , and $Z6$.

3.1 Distributions Examined

Ten different distributions are considered in this study. These include the normal distribution with mean $\mu = 0$ and 10 and standard deviation $\sigma = 0.01, 0.5, 1, 5, 10$, and 100; the Student's t distribution with degrees of freedom $d = 4, 6, 7, 12, 20$, and 30; the chi-squared distribution with degrees of freedom $r = 1, 2, 4, 6, 7$, and 10; the exponential distribution with rate parameter $\lambda = 0.01, 0.5, 1, 5, 20$ and 50; the gamma distribution with shape parameter $r = 0.5, 1, 5, 10, 50$ and 100 and rate parameter $\lambda = 0.5$ and 10; the Weibull distribution with shape parameter $r = 0.5, 1, 5, 20, 50$ and 100 and scale parameter $\beta = 0.5, 1$ and 10; the log-normal distribution with location parameter $\mu = 1$ and 10 and scale parameter $\tau = 0.01, 0.2, 0.4, 0.6, 0.8$ and 1; the Pareto distribution with shape parameter $r = 2.5, 5, 10, 20, 50$ and 100 and scale parameter $\beta = 0.5, 1$, and 10; the Beta distribution with the two shape parameters $r = 0.5, 1$, and 2 and $s = 0.5, 1$, and 2; and the Inverse-gamma distribution with shape parameter $r = 2.5, 5, 7.5, 10, 20$ and 50 and scale parameter $\beta = 0.5, 1$, and 10.

A simulation is run for each distribution with each combination of the parameter values listed. Thus, a total of 117 distributions with fixed parameter values are considered. Many of these distributions have similar shapes, most of them being right

skewed, but it is of interested to investigate whether the nuance differences between them is enough to hinder the performance of a test procedure.

3.2 Simulation Description

Simulations were run using GNU R 3.1.3. The functions provided in the base R package are used to generate random variables for each of the distributions listed in section 3.1, with exception to the Pareto distribution. Some external packages were required for other features of the simulation; these are discussed in section 3.3.

The eight tests described in section 2.4.1 – 2.4.8 were considered in the simulation study, along with the chi-squared test, robust chi-squared test, and Long & Sa's test, Z6, which are summarized in chapter 1.

Each simulation involves drawing m random samples of size n from a distribution $f(x)$ with fixed parameters θ . For each sample, a set of test statistics are calculated at an α -significance level to test the null hypothesis

$$H_0: \sigma^2 \geq \delta \sigma_0^2$$

where $\sigma^2 = \text{var}(X)$ is the variance of the distribution $f(x)$, σ_0^2 is the hypothesized variance, and δ is a constant factor. In these simulations, σ_0^2 is always set equal to the true value of the variance. The simulations with $\delta = 1$ are used to estimate the type-I error rate, and those with $\delta > 1$ are used to estimate power. A simulation is performed using significance levels $\alpha = 0.01, 0.05$ and 0.10 , sample sizes $n = 10, 20$ and 30 , and δ values of $\delta = 1, 2, 3$, and 4 . This means that for each of the 117 fixed distributions

detailed in section 3.1, there are 36 different simulations, each using a different combination of α , n , and δ .

Algorithm 2: Given some distribution $f(x)$ with fixed parameters θ , a sample size n , a simulation size m , an α -level, and a δ value, the goal is to estimate, for each test procedure, the rate of rejection of the null hypothesis $H_0: \sigma^2 \geq \delta\sigma_0^2$.

1. Generate n observations from $f(x)$.
 2. For each test procedure:

Conduct the hypothesis test and obtain a rejection or non-rejection decision.
 3. Repeat steps 1 – 2 m times.
 4. For each test statistic:

Calculate the proportion of samples that resulted in a rejection.
-

3.3 R Packages Used

The “moments” package is used to compute sample cumulants and estimate kurtosis through the `all.moments()`, `all.cumulants()`, and `kurtosis()` functions. Maximum likelihood estimates for the nuisance parameters of the log-normal and Weibull distributions are computed by `fitdistr()` from the “MASS” package. This function is also used to compute the maximum likelihood values used in R_{lh} . The “actuar” package provides the function `rpseudo()` for generating random samples from a Pareto distribution. The graphs in Appendix A and B were created with “ggplot2” and the “gridExtra” package.

CHAPTER 4: SIMULATION RESULTS

In this chapter the results of the simulation study are discussed. The asymptotic properties of the proposed test statistics are verified in section 4.1, followed by the type-I error rate comparison of all the test statistics in section 4.2, and the power study in section 4.3.

4.1 Verifying the Proposed Tests

The test statistics R_{norm} , R_{chisq} , R_{exp} , R_{gamma} , R_{weib} , and R_{lnorm} , which were derived assuming the population has a particular distribution and that any nuisance parameters are known, are expected to be asymptotically standard normal with relative error of at most $O(n^{-1})$. Note, the tests R_{gamma2} and R_{lh} are not considered here since they were not derived analytically and do not have any asymptotic guarantees. In this section, the performance of these six proposed test statistics is evaluated with the initial assumptions satisfied. The results suggest that the tests do, in fact, perform very well.

The tests are verified by simulating observations from the distribution that each test statistic is derived from. If a statistic involves a nuisance parameter, the true value of that parameter is used. The power of each test statistic is evaluated by simulating 10000 samples for different levels of δ . For large sample sizes, the type-I error rate and the nominal level α should agree, and the power of the test should approach 1 quickly as δ increases.

The type-I error rates for each test statistic behave as expected. The results are presented in tables 1 - 6 of appendix A. Only R_{norm} has a slightly inflated type-I error

rate for small sample sizes, but this effect diminishes with larger sample sizes. The other tests have appropriate rates even for the smallest sample size tested at $n = 10$. Graphs for the power curves are found in figures 1 – 6 in appendix A.

The power of each test also appears to be high, even for small sample sizes. The one exception is with R_{lnorm} ; as the scale parameter τ increases, the power of the test is significantly hindered. This phenomenon was investigated by comparing the sampling distribution of R_{lnorm} to the standard normal curve in figure 9 in appendix A. Three sampling distributions of R_{lnorm} are generated with $\delta = 1.5$ and sample sizes $n = 10, 100$, and 1000 . As the sample size increases, the sampling distribution of R_{lnorm} should shift away from the standard normal curve and into the critical region, so that the probability of rejecting H_0 increases. However, figure 9 shows that even with $n = 1000$ the distribution of R_{lnorm} is only slightly shifted, causing the test to have low power. A similar effect, although to a lesser degree, is observed for R_{gamma} and R_{Weib} ; as the shape parameter for their distributions decreases, the power of each test is diminished. The sampling distributions for these test statistics with $\delta = 1.5$ and sample sizes $n = 10, 100$, and 1000 are given in figures 7 and 8 in appendix A.

4.2 Type-I Error Rate Comparison

The test statistics are first compared based on the type-I error rate. If the type-I error is far above the nominal level, the test will be considered unviable. The goal is to determine which tests, if any, maintain an appropriate type-I error rate for a variety of distributions. The tables 7 - 16 in Appendix B compare the type-I errors for each test

statistic discussed in this paper using the ten distributions described in section 3.1 with samples of size $n = 10$ and a significance level $\alpha = 0.05$. For each distribution, only six different parameter values are shown, even though more may have been simulated. However, the results from any omitted distribution follows the same trend set by the six distributions shown.

The proposed test statistics R_{chisq} , R_{exp} , R_{gamma} , R_{weib} , and R_{lnorm} all tend to have inflated type-I error rates. In the case of R_{chisq} and R_{exp} , the type-I error rate is sometimes zero, but in these instances the power of the test is also at or near zero, even for large δ . The tests R_{norm} and R_{lh} have a relatively better performance, but R_{gamma2} provides the overall most stable type-I error rates of the proposed test statistics.

Appendix C provides a more complete comparison of the type-I error rates, using the ten distributions described in section 3.1 with sample sizes $n = 10, 20$, and 30 and significance levels $\alpha = 0.01, 0.05$, and 0.10 . The entries with a bold font have a type-I error rate that is 20% more than the nominal level. Of the proposed test statistics, only R_{lh} and R_{gamma2} had consistent results across all distributions considered, so only those two are included in these tables. They are accompanied by the two existing tests, χ_r^2 and $Z6$ for comparison. Overall, these four test statistics show two trends. First, they all have severely inflated type-I errors when the population is both skewed and heavy-tailed. Second, R_{gamma2} is the most conservative test statistic, while $Z6$ and R_{lh} tend to be the most inflated.

A short investigation into the cases where all four tests fail suggests that this is an effect from skewed and heavy-tailed distributions. In particular, the distributions that

cause all of the tests to fail include the chi-squared distribution with 1 degree of freedom, the gamma distribution when its shape parameter is 0.5 or less, the Weibull distribution when its shape parameter is 0.5 or less, the log-normal distribution when its scale parameter is 0.6 or higher, and the inverse-gamma distribution when its shape parameter is 7.5 or lower.

Appendix D presents some of these cases. For each distribution, three different parameter settings were chosen and their densities are graphed in a row. All of the left-most distributions lead to inflated type-I errors, while the distributions towards the right are more controlled. The excess kurtosis and skewness are given for each distribution. There appears to be a positive association between the kurtosis and type-I error rates. This cursory investigation suggests that a kurtosis below 10 is necessary for any test to be viable. However, low kurtosis is not sufficient for viability, considering the case of a log-normal(1, 0.6) population, which has a kurtosis of only 6.3 as shown in Appendix D, but from Appendix C it is clear that χ_r^2 , Z6 and R_{gamma2} all have a moderately inflated type-I error rate for this distribution. When considering skewness, almost identical results were found. A skewness of below 2 in absolute value appears to be necessary for viability, but is not sufficient. The log-normal(1, 0.6) again provides a counter-example with a skewness of only 0.285.

Another fault is observed for both χ_r^2 and Z6; if the population has a density that is monotonically decreasing, these two tests will have inflated type-I errors. The distributions with this property include the chi-squared distribution with 2 or fewer degrees of freedom, the gamma distribution with shape parameter 1 or less, the Weibull

with shape parameter of 1 or less, and the Pareto distribution with any parameter values. The one exception here is with the exponential distribution and a small sample ($n = 10$), for which the Z6 will not necessarily have an inflated type-I error.

4.3 Power Study

Figures 10 – 19 in appendix B provides some graphs of power curves, using the ten distributions described in section 3.1 with samples of size $n = 10$ and a significance level $\alpha = 0.05$. For populations with a normal, chi-squared, or exponential distribution, the tests R_{normal} , R_{chisq} , and R_{exp} , respectively, are viable and yield the highest power. On the other hand, the tests R_{gamma} , R_{weib} , and R_{lnorm} have a poor performance for their respective distributions. The remaining tests, χ_r^2 , Z6, R_{lh} , and R_{gamma2} , are compared across all ten distributions; in the cases where each test is in control, R_{lh} and χ_r^2 tend to provide the most power, followed by R_{gamma2} , with Z6 always trailing behind.

A more detailed comparison of power between χ_r^2 , Z6, R_{lh} , and R_{gamma2} can be conducted using the tables in Appendix E. These tables give the power at $\delta = 4$, and are otherwise set up in the same fashion as those in appendix C. The entries with bold font correspond to the cases with inflated type-I error rates.

There are two unique cases that stand out. First, for the normal distribution and t distribution, R_{lh} always provides the most power and has a distinct advantage when $\alpha = 0.01$. Secondly, for the beta distribution, the performance of χ_r^2 is the best overall. However, R_{lh} again had a distinct advantage when $\alpha = 0.01$, but only when the

distribution had a left skew, that is, for $\text{beta}(2, 1)$ and $\text{beta}(1, 0.5)$. Besides these two cases, the relative power among the four test statistics is fixed over the remaining distributions. However, the sample size and significance level affect how each test is ranked.

For $n = 10$, and when $\alpha = 0.01$, R_{lh} is the best with power around 0.20, χ_r^2 usually has about half of that, R_{gamma2} is slightly worse, and Z6 provides essentially zero power. When $\alpha = 0.05$, R_{lh} and χ_r^2 both have power around 0.50, R_{gamma2} is around 20% lower, and Z6 has about half the power as the leading two. When $\alpha = 0.10$, all four tests are comparable with power around 0.70, but R_{gamma2} tends to be about 10% lower than the rest.

For $n = 20$, and when $\alpha = 0.01$, both R_{lh} and Z6 have power around 0.60, and R_{gamma2} and χ_r^2 have about 20% of that. When $\alpha = 0.05$ or 0.10, all four statistics are comparable with power usually well above 0.80.

For $n = 30$, and when $\alpha = 0.01$, both R_{lh} and Z6 again have a small advantage over the other two. and when $\alpha = 0.05$ or 0.10, all four statistics are comparable with power usually well above 0.90.

CHAPTER 5: CONCLUSION

If the population has a known distribution with known nuisance parameters, the adjusted signed log-likelihood ratio statistic detailed in section (2.2) is a good test statistic to use, as discussed in section (4.1). From this approach, we found two test statistics that work well for a variety of non-normal distributions, the R_{gamma2} and R_{lh} . The R_{gamma2} test has the most controlled type-I error rate of all of the tests considered. For sample size $n = 10$, it provides moderate power, and for $n = 20$ and 30 , it has excellent power. The R_{lh} test controlled the type-I error about as well as $Z6$, while providing the best power at low significance levels and for small sample sizes. However, R_{lh} and $Z6$ are the least likely to control type-I error rates.

If the population is skewed with heavy-tails, all of the test statistics covered in this study are prone to inflated type-I error rates. However, the existing tests χ_r^2 and $Z6$ have an additional setback; they have uncontrolled type-I errors whenever the population density function is monotonically decreasing. These types of distributions can come up, for example, in a quality control setting if some time-to-event data follow an exponential distribution, or in economics where the Pareto distribution often models incomes and other financial data. We will refer to this type of distribution as having a “J”-shape.

In summary, if not much is known about the population except that it is unlikely to be highly skewed or heavy-tailed, then R_{gamma2} is the preferred test. It is most likely to control type-I errors while providing good power in all cases except for small sample sizes with low significance levels. As a rule of thumb, R_{gamma2} should not be used if the population has an excess kurtosis of more than 10 or an absolute skewness of more than

2, as suggested in section (4.2). The R_{lh} test provides remarkably good power when low significance levels are desired, however its results are less reliable, as it is more vulnerable to inflation when the population is heavy-tailed. If the population is not expected to have a “J”-shape distribution, then χ_r^2 is recommended for sample sizes around $n = 10$, while R_{gamma2} is recommended for larger samples.

REFERENCES

- Jensen, J. L. (1995) *Saddlepoint Approximations*. Oxford: Clarendon Print.
- Johnson, N. L., Kotz, S., & Balakrishnan, N. (1994). *Continuous Univariate Distributions* (Vol. 1). New York: John Wiley & Sons.
- Kendall, S. M. (1994). *Distribution Theory*. New York: Oxford University Press Inc.
- Lee, S. J., & Sa, P. (1998). Testing the variance of skewed distributions. *Communications in Statistics: Simulation and Computation*, 27(3), 807-822.
- Long, M. C., & Sa, P. (2005). Right-tailed testing of variance for non-normal distributions. *Journal of Modern Applied Statistical Methods*, 4(1), 187-213

APPENDIX A: VERIFYING PROPOSED TEST STATISTICS

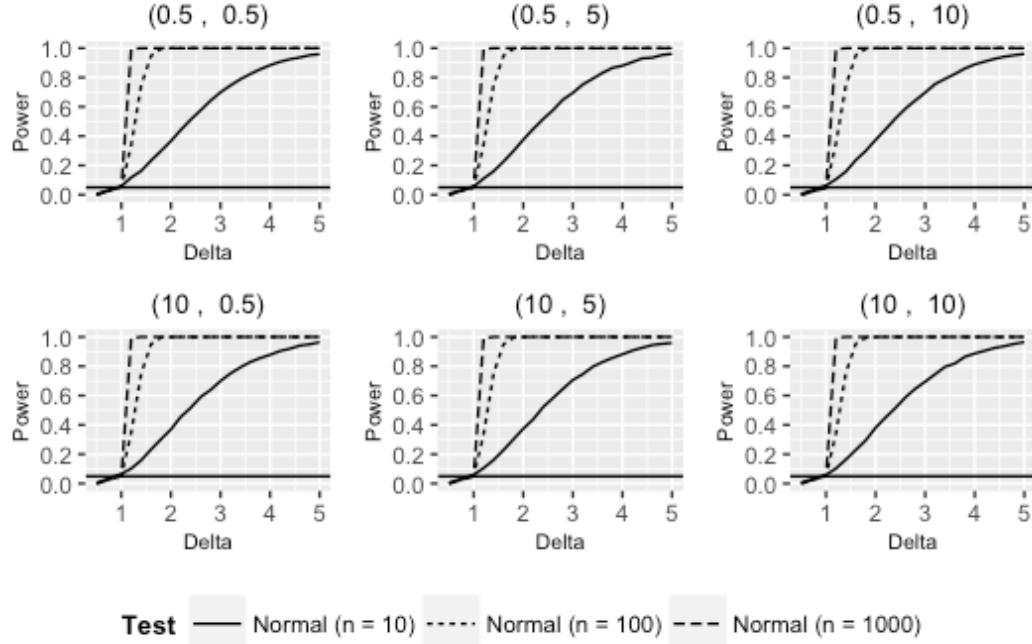


Figure 1. Power curve of R_{norm} for the left-tailed hypothesis test of $\sigma^2 = \delta\sigma_0^2$ when sampling from a normal distribution with parameters (μ, σ) where σ is known, using a significance level $\alpha = 0.05$ and sample size $n = 10, 100$, and 10000 . Based on 10000 simulations.

Distribution	(0.5, 0.5)	(0.5, 5)	(0.5, 10)	(10, 0.5)	(10, 5)	(10, 10)
Normal (n = 10)	0.0619	0.0656	0.0644	0.0666	0.0597	0.0585
Normal (n = 100)	0.0510	0.0517	0.0512	0.0456	0.0511	0.0508
Normal (n = 1000)	0.0466	0.0525	0.0503	0.0493	0.0529	0.0491

Table 1. Type-I Error rates of R_{norm} from 10000 simulations of samples of size $n = 10, 100$, and 1000 from a normal distribution with parameters (μ, σ) where σ is known. Entries with a bold font are more than 20% above the nominal level.

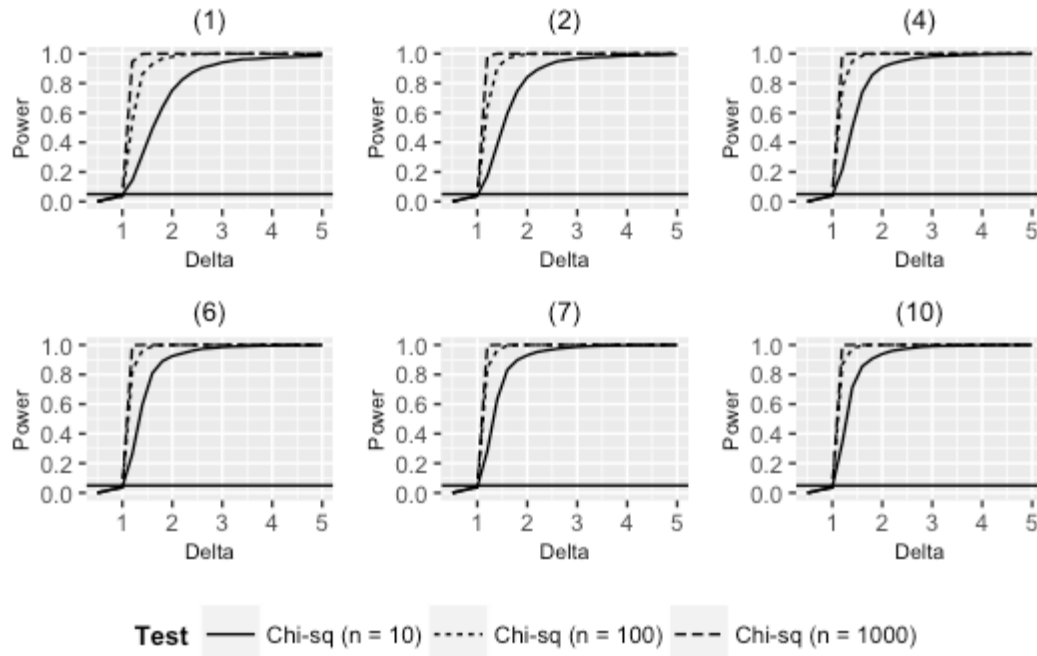


Figure 2. Power curve of R_{chisq} for the left-tailed hypothesis test of $\sigma^2 = \delta\sigma_0^2$ when sampling from a chi-squared distribution with parameter (df), using a significance level $\alpha = 0.05$ and sample size $n = 10, 100$, and 10000 . Based on 10000 simulations.

Distribution	(1)	(2)	(4)	(6)	(7)	(10)
Chi-squared ($n = 10$)	0.0439	0.0399	0.0439	0.0429	0.0404	0.0385
Chi-squared ($n = 100$)	0.0352	0.0371	0.0394	0.0331	0.0363	0.0357
Chi-squared ($n = 1000$)	0.0344	0.0402	0.0367	0.0373	0.0349	0.0354

Table 2. Type-I Error rates of R_{chisq} from 10000 simulations of samples of size $n = 10, 100$, and 1000 from a chi-squared distribution with parameters (df).

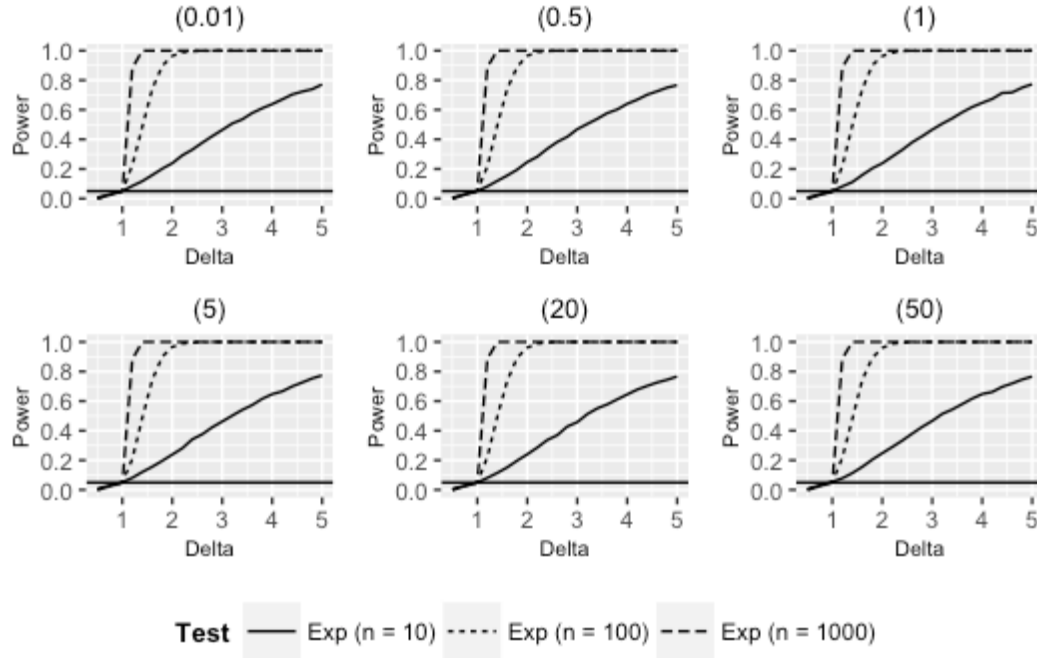


Figure 3. Power curve of R_{exp} for the left-tailed hypothesis test of $\sigma^2 = \delta\sigma_0^2$ when sampling from an exponential distribution with parameter (λ), using a significance level $\alpha = 0.05$ and sample size $n = 10, 100$, and 10000 . Based on 10000 simulations.

Distribution	(0.01)	(0.5)	(1)	(5)	(20)	(50)
Exponential (n = 10)	0.0486	0.0510	0.0464	0.0530	0.0509	0.0474
Exponential (n = 100)	0.0505	0.0513	0.0520	0.0455	0.0497	0.0515
Exponential (n = 1000)	0.0468	0.0474	0.0471	0.0503	0.0516	0.0507

Table 3. Type-I Error rates of R_{exp} from 10000 simulations of samples of size $n = 10, 100$, and 1000 from an exponential distribution with parameters (λ).

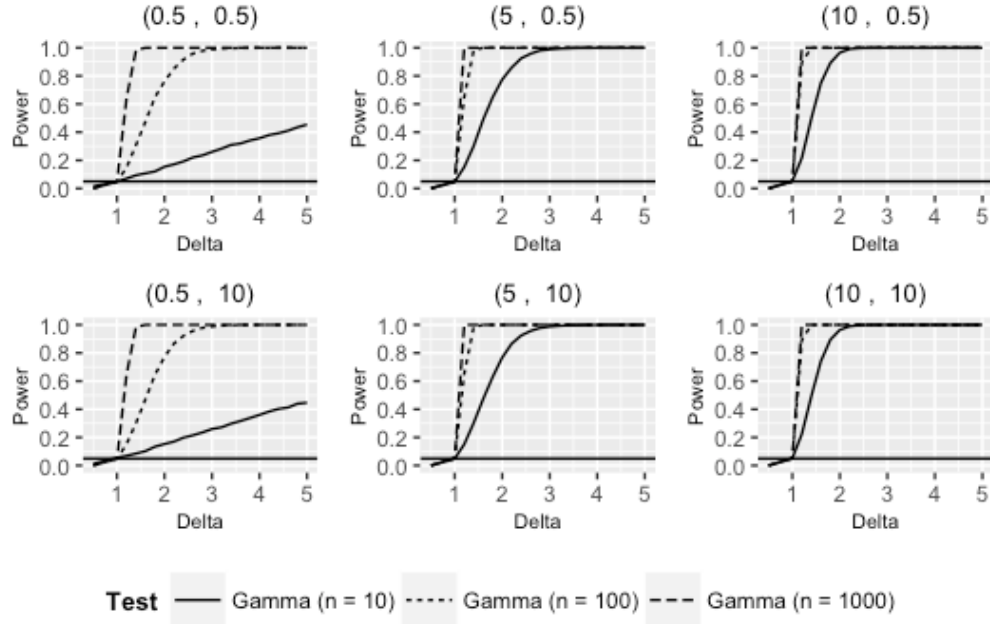


Figure 4. Power curve of R_{gamma} for the left-tailed hypothesis test of $\sigma^2 = \delta\sigma_0^2$ when sampling from a gamma distribution with parameters (r, β) where r is known, using a significance level $\alpha = 0.05$ and sample size $n = 10, 100$, and 10000 . Based on 10000 simulations.

Distribution	(0.5, 0.5)	(5, 0.5)	(10, 0.5)	(0.5, 10)	(5, 10)	(10, 10)
Gamma ($n = 10$)	0.0530	0.0480	0.0518	0.0503	0.0514	0.0515
Gamma ($n = 100$)	0.0478	0.0503	0.0468	0.0489	0.0537	0.0498
Gamma ($n = 1000$)	0.0481	0.0509	0.0488	0.0520	0.0451	0.0526

Table 4. Type-I Error rates of R_{gamma} from 10000 simulations of samples of size $n = 10, 100$, and 1000 from a gamma distribution with parameters (r, β) where r is known.

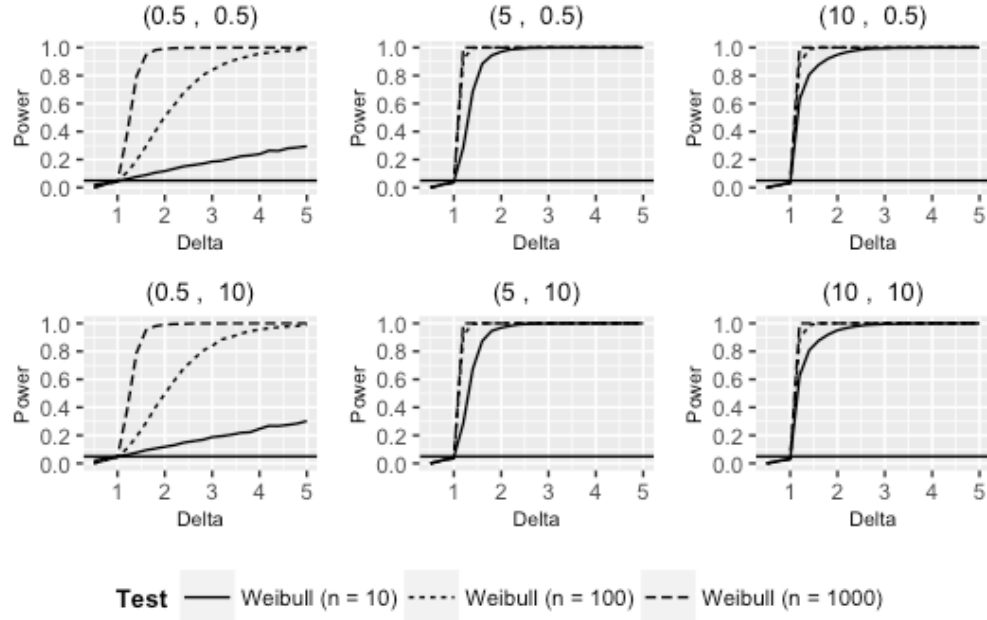


Figure 5. Power curve of R_{weib} for the left-tailed hypothesis test of $\sigma^2 = \delta\sigma_0^2$ when sampling from a Weibull distribution with parameters (r, β) where r is known, using a significance level $\alpha = 0.05$ and sample size $n = 10, 100$, and 10000 . Based on 10000 simulations.

Distribution	(0.5, 0.5)	(5, 0.5)	(20, 0.5)	(0.5, 10)	(5, 10)	(20, 10)
Weibull (n = 10)	0.0508	0.0403	0.0331	0.0467	0.0382	0.0343
Weibull (n = 100)	0.0485	0.0400	0.0319	0.0455	0.0366	0.0275
Weibull (n = 1000)	0.0467	0.0353	0.0308	0.0420	0.0349	0.0309

Table 5. Type-I Error rates of R_{weib} from 10000 simulations of samples of size $n = 10, 100$, and 1000 from a Weibull distribution with parameters (r, β) where r is known.

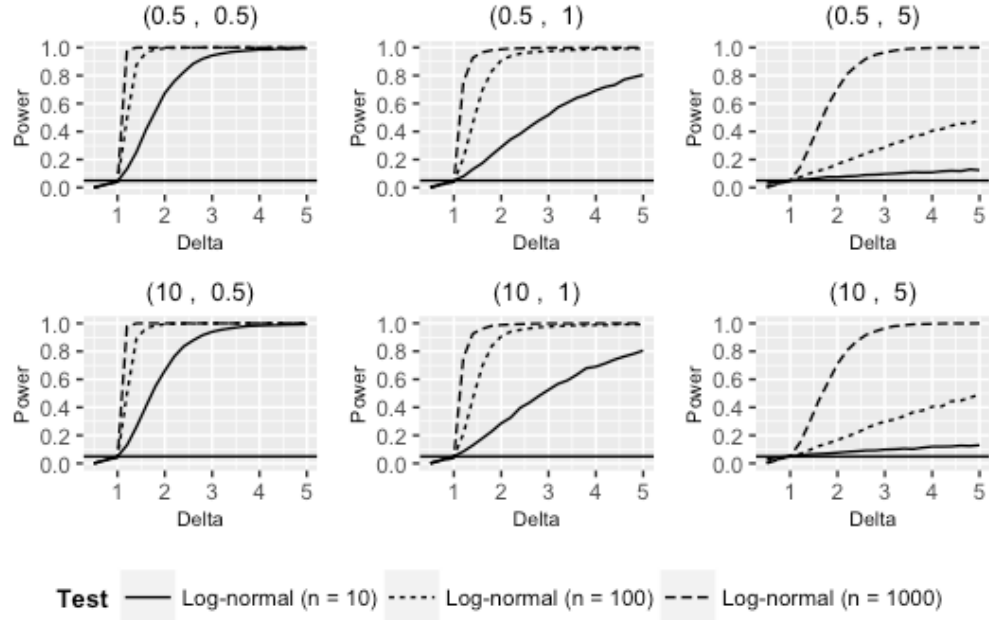


Figure 6. Power curve of R_{lnorm} for the left-tailed hypothesis test of $\sigma^2 = \delta\sigma_0^2$ when sampling from a log-normal distribution with parameters (μ, τ) where τ^2 is known, using a significance level $\alpha = 0.05$ and sample size $n = 10, 100$, and 10000 . Based on 10000 simulations.

Distribution	(0.5, 0.5)	(0.5, 1)	(0.5, 5)	(10, 0.5)	(10, 1)	(10, 5)
Log-normal (n = 10)	0.0474	0.0537	0.0525	0.0464	0.0479	0.0459
Log-normal (n = 100)	0.0420	0.0437	0.0516	0.0456	0.0448	0.0458
Log-normal (n = 1000)	0.0402	0.0406	0.0466	0.0396	0.0376	0.0483

Table 6. Type-I Error rates of R_{lnorm} from 10000 simulations of samples of size $n = 10, 100$, and 1000 from a normal distribution with parameters (μ, τ) where τ^2 is known.

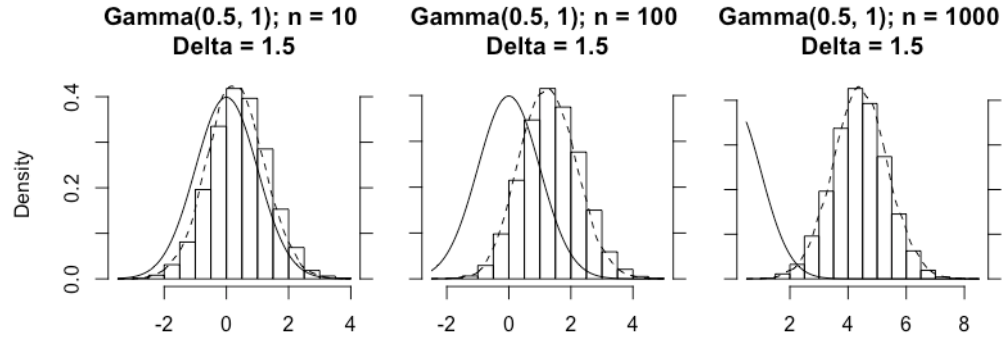


Figure 7. Sampling distribution of R_{gamma} under $H_0: \sigma^2 = 1.5\sigma_0^2$ from a $gamma(0.5, 1)$ distribution and sample sizes $n = 10, 100$, and 1000 . The dotted line marks the empirical density of the test statistic, and the solid line is the density of a $normal(0, 1)$ distribution.

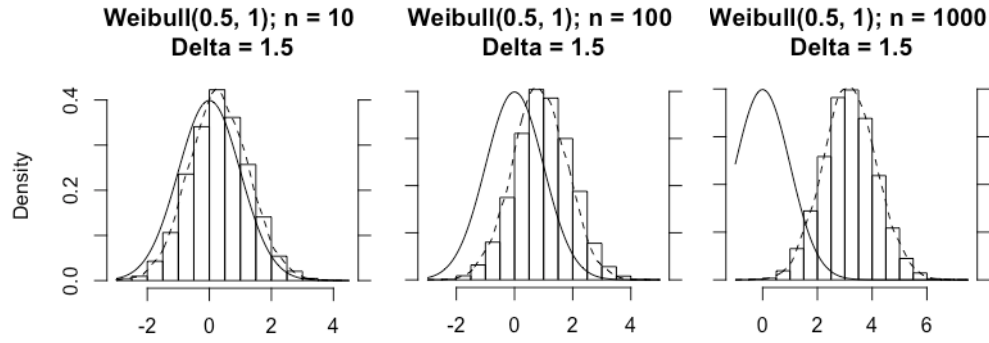


Figure 8. Sampling distribution of R_{weib} under $H_0: \sigma^2 = 1.5\sigma_0^2$ from a $Weibull(0.5, 1)$ distribution and sample sizes $n = 10, 100$, and 1000 . The dotted line marks the empirical density of the test statistic, and the solid line is the density of a $normal(0, 1)$ distribution.

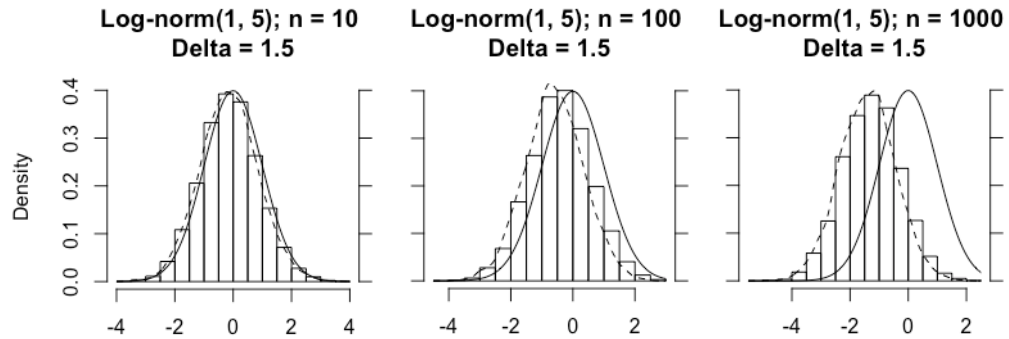


Figure 9. Sampling distribution of R_{lnorm} under $H_0: \sigma^2 = 1.5\sigma_0^2$ from a $log-normal(1, 5)$ distribution and sample sizes $n = 10, 100$, and 1000 . The dotted line marks the empirical density of the test statistic, and the solid line is the density of a $normal(0, 1)$ distribution.

APPENDIX B: POWER CURVES AND TYPE-I ERROR RATES

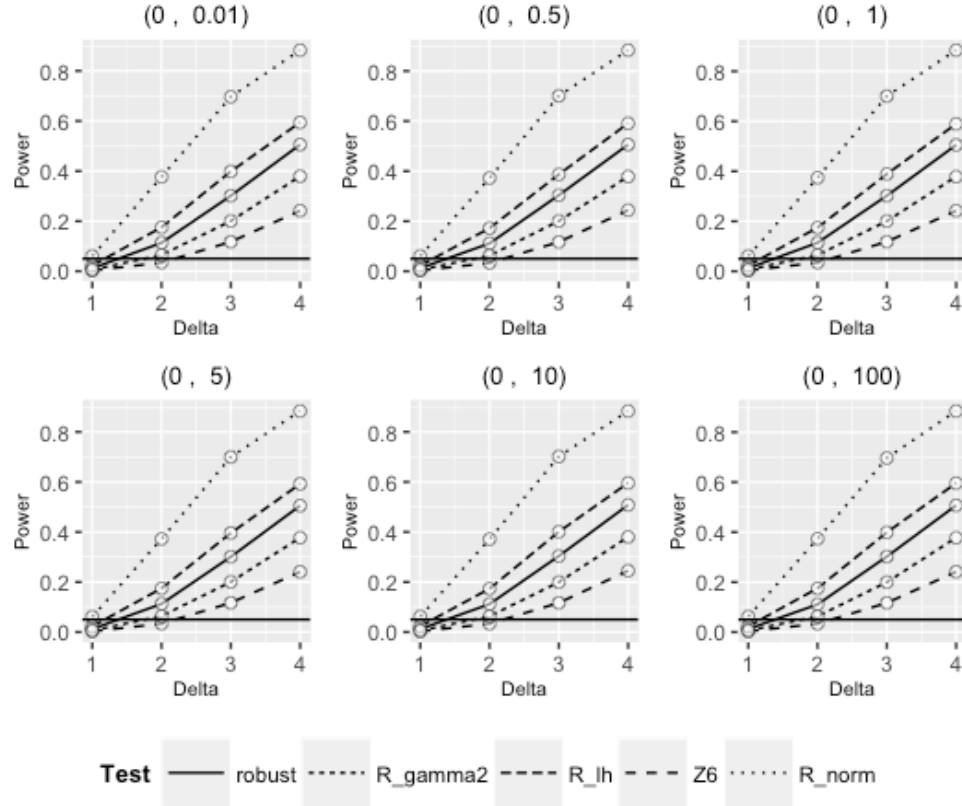


Figure 10. Power study for the left-tailed hypothesis test of $\sigma^2 = \delta\sigma_0^2$ from normal distributions with parameters (μ, σ) using a significance level $\alpha = 0.05$ and sample size $n = 10$. Based on 10^5 simulations.

Tests	(0, 0.01)	(0, 0.5)	(0, 1)	(0, 5)	(0, 10)	(0, 100)
chisq	0.0493	0.0497	0.0504	0.0500	0.0506	0.0508
robust	0.0117	0.0122	0.0124	0.0122	0.0119	0.0122
Z6	0.0027	0.0028	0.0029	0.0028	0.0027	0.0029
R_lh	0.0226	0.0227	0.0231	0.0226	0.0233	0.0231
R_gamma2	0.0058	0.0058	0.0060	0.0058	0.0057	0.0059
R_normal	0.0606	0.0607	0.0610	0.0611	0.0622	0.0622
R_chisq	0.0000	0.0000	0.0000	0.0000	0.0000	0.5630
R_exp	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
R_gamma	0.5635	0.5599	0.5592	0.5625	0.5606	0.5605
R_weib	0.4175	0.4135	0.4138	0.4156	0.4135	0.4151
R_lnorm	0.5431	0.5389	0.5381	0.5416	0.5401	0.5398

Table 7. Type-I Error rates from 10^5 simulations of samples of size $n = 10$ from a normal distribution with parameters (μ, σ) and a significance level $\alpha = 0.05$. Entries with a bold font are more than 20% above the nominal level.

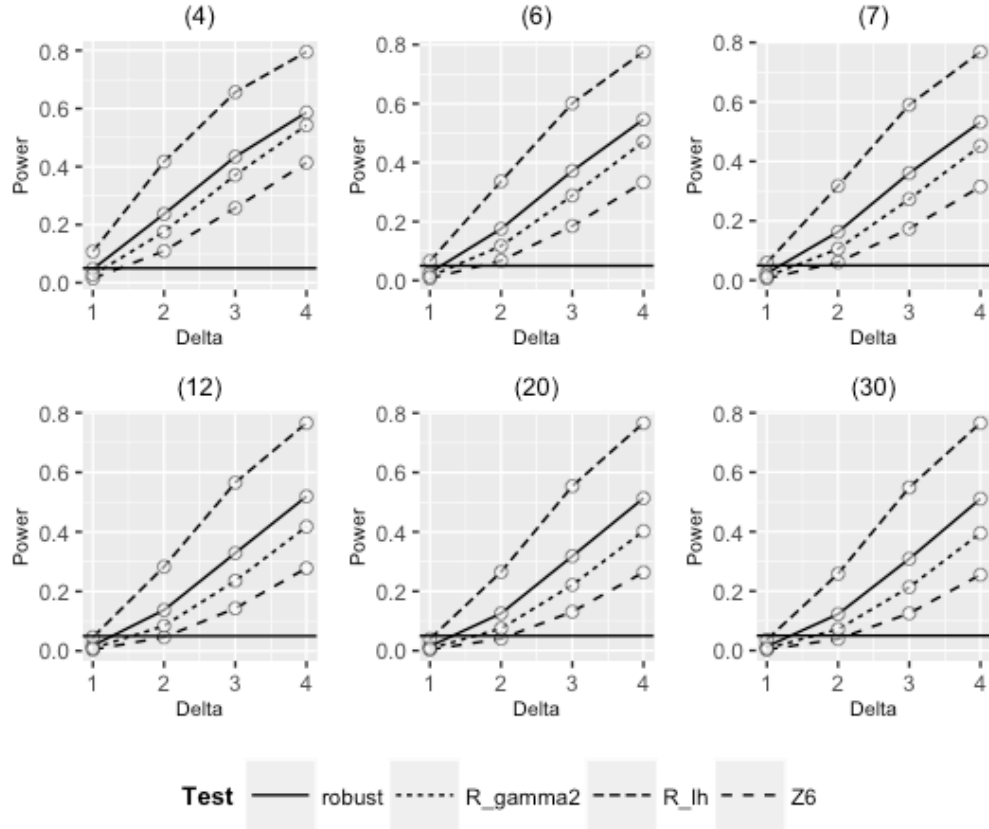


Figure 11. Power study for the left-tailed hypothesis test of $\sigma^2 = \delta\sigma_0^2$ from t distributions with parameter (df) using a significance level $\alpha = 0.05$ and sample size $n = 10$. Based on 10^5 simulations.

Tests	(4)	(6)	(7)	(12)	(20)	(30)
chisq	0.1463	0.0944	0.0852	0.0676	0.0597	0.0551
robust	0.0472	0.0264	0.0234	0.0178	0.0148	0.0134
Z6	0.0146	0.0068	0.0059	0.0044	0.0036	0.0030
R_lh	0.1072	0.0655	0.0590	0.0461	0.0393	0.0366
R_gamma2	0.0273	0.0136	0.0121	0.0089	0.0071	0.0065
R_normal	0.1694	0.1123	0.1015	0.0810	0.0724	0.0676
R_chisq	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
R_exp	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
R_gamma	0.7229	0.6822	0.6777	0.6632	0.6566	0.6524
R_weib	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
R_lnorm	0.6692	0.6164	0.6096	0.5865	0.5754	0.5692

Table 8. Type-I Error rates from 10^5 simulations of samples of size $n = 10$ from a t distribution with parameter (df) and a significance level $\alpha = 0.05$. Entries with a bold font are more than 20% above the nominal level.

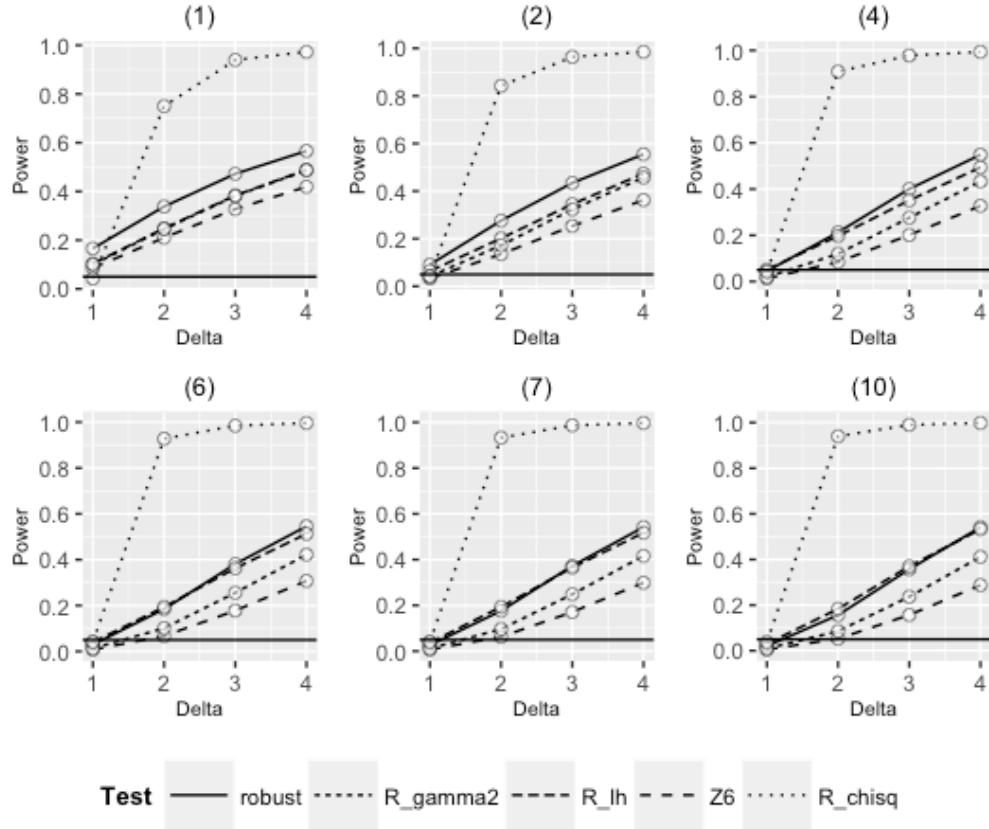


Figure 12. Power study for the left-tailed hypothesis test of $\sigma^2 = \delta \sigma_0^2$ from chi-squared distributions with parameter (df) using a significance level $\alpha = 0.05$ and sample size $n = 10$. Based on 10^5 simulations.

Tests	(1)	(2)	(4)	(6)	(7)	(10)
chisq	0.3154	0.2007	0.1272	0.0967	0.0906	0.0763
robust	0.1659	0.0907	0.0446	0.0296	0.0262	0.0205
Z6	0.0858	0.0328	0.0130	0.0077	0.0068	0.0049
R_lh	0.1035	0.0641	0.0514	0.0399	0.0370	0.0306
R_gamma2	0.0998	0.0416	0.0192	0.0128	0.0115	0.0093
R_normal	0.3361	0.2232	0.1468	0.1146	0.1076	0.0916
R_chisq	0.0427	0.0423	0.0416	0.0417	0.0412	0.0389
R_exp	0.3371	0.0492	0.0002	0.0000	0.0000	0.0000
R_gamma	0.1360	0.1859	0.2537	0.2963	0.3125	0.3510
R_weib	0.1254	0.2171	0.3028	0.3394	0.3529	0.3784
R_lnorm	0.0254	0.0767	0.1592	0.2148	0.2376	0.2885

Table 9. Type-I Error rates from 10^5 simulations of samples of size $n = 10$ from a chi-squared distribution with parameter (df) and a significance level $\alpha = 0.05$. Entries with a bold font are more than 20% above the nominal level.

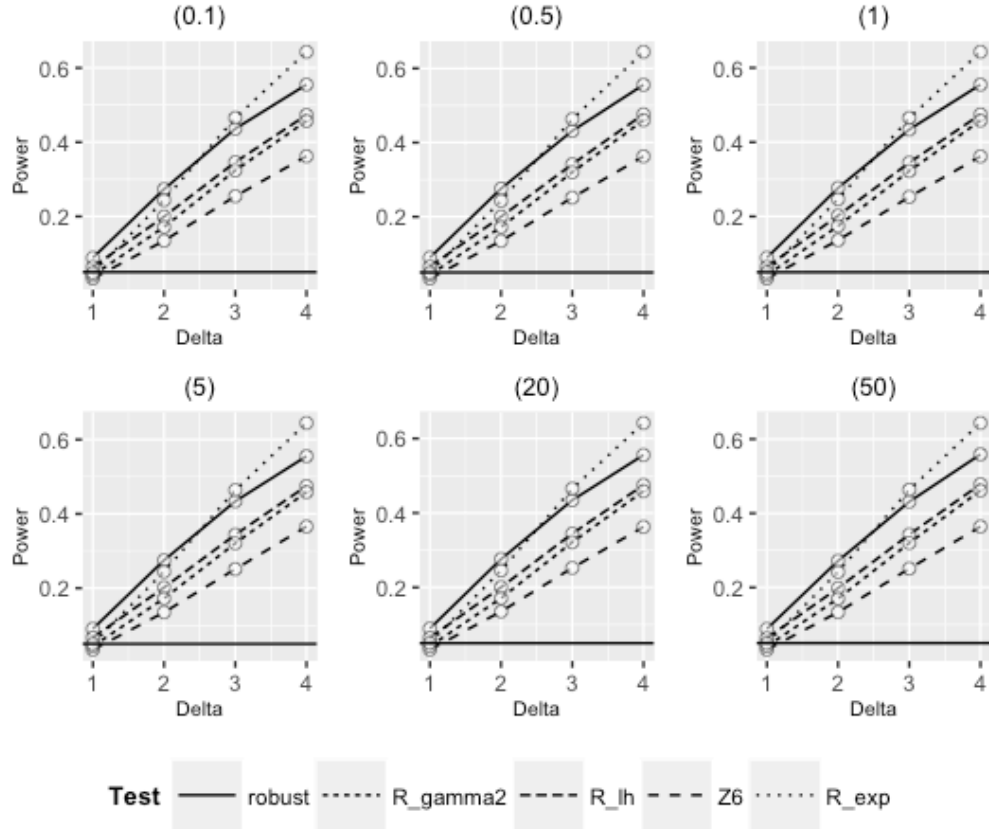


Figure 13. Power study for the left-tailed hypothesis test of $\sigma^2 = \delta\sigma_0^2$ from exponential distributions with parameter (λ) using a significance level $\alpha = 0.05$ and sample size $n = 10$. Based on 10^5 simulations.

Tests	(0.1)	(0.5)	(1)	(5)	(20)	(50)
chisq	0.2011	0.2020	0.2003	0.2042	0.2038	0.2017
robust	0.0886	0.0899	0.0891	0.0906	0.0895	0.0883
Z6	0.0328	0.0337	0.0331	0.0345	0.0328	0.0324
R_lh	0.0636	0.0658	0.0640	0.0669	0.0646	0.0632
R_gamma2	0.0414	0.0434	0.0419	0.0436	0.0416	0.0416
R_normal	0.2235	0.2231	0.2211	0.2261	0.2254	0.2237
R_chisq	0.6458	0.0417	0.0000	0.0000	0.0000	0.0000
R_exp	0.0481	0.0493	0.0500	0.0504	0.0505	0.0498
R_gamma	0.1857	0.1860	0.1845	0.1892	0.1873	0.1854
R_weib	0.2181	0.2160	0.2152	0.2200	0.2177	0.2176
R_lnorm	0.0768	0.0781	0.0758	0.0796	0.0772	0.0766

Table 10. Type-I Error rates from 10^5 simulations of samples of size $n = 10$ from an exponential distribution with parameter (λ) and a significance level $\alpha = 0.05$. Entries with a bold font are more than 20% above the nominal level.

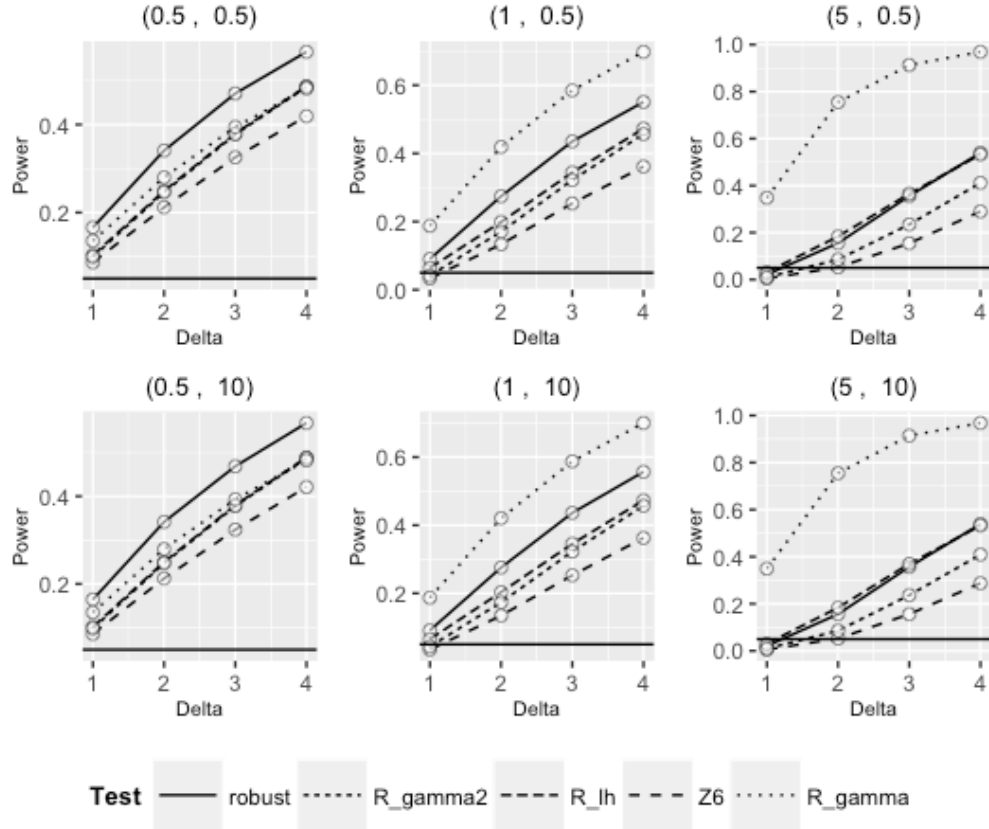


Figure 14. Power study for the left-tailed hypothesis test of $\sigma^2 = \delta\sigma_0^2$ from gamma distributions with parameters (shape, scale) using a significance level $\alpha = 0.05$ and sample size $n = 10$. Based on 10^5 simulations.

Tests	(0.5, 0.5)	(1, 0.5)	(5, 0.5)	(0.5, 10)	(1, 10)	(5, 10)
chisq	0.3131	0.2046	0.0764	0.3118	0.2034	0.0752
robust	0.1660	0.0903	0.0214	0.1645	0.0911	0.0210
Z6	0.0865	0.0330	0.0047	0.0859	0.0345	0.0054
R_lh	0.1030	0.0640	0.0315	0.1018	0.0652	0.0309
R_gamma2	0.0991	0.0414	0.0095	0.0983	0.0433	0.0095
R_normal	0.3341	0.2270	0.0915	0.3327	0.2250	0.0904
R_chisq	0.0435	0.0426	0.0401	0.0000	0.0000	0.0000
R_exp	0.3363	0.0494	0.0000	0.3328	0.0508	0.0000
R_gamma	0.1356	0.1876	0.3480	0.1351	0.1869	0.3494
R_weib	0.1251	0.2193	0.3755	0.1251	0.2193	0.3776
R_lnorm	0.0253	0.0775	0.2861	0.0255	0.0770	0.2870

Table 11. Type-I Error rates from 10^5 simulations of samples of size $n = 10$ from a gamma distribution with parameters (shape, scale) and a significance level $\alpha = 0.05$. Entries with a bold font are more than 20% above the nominal level.

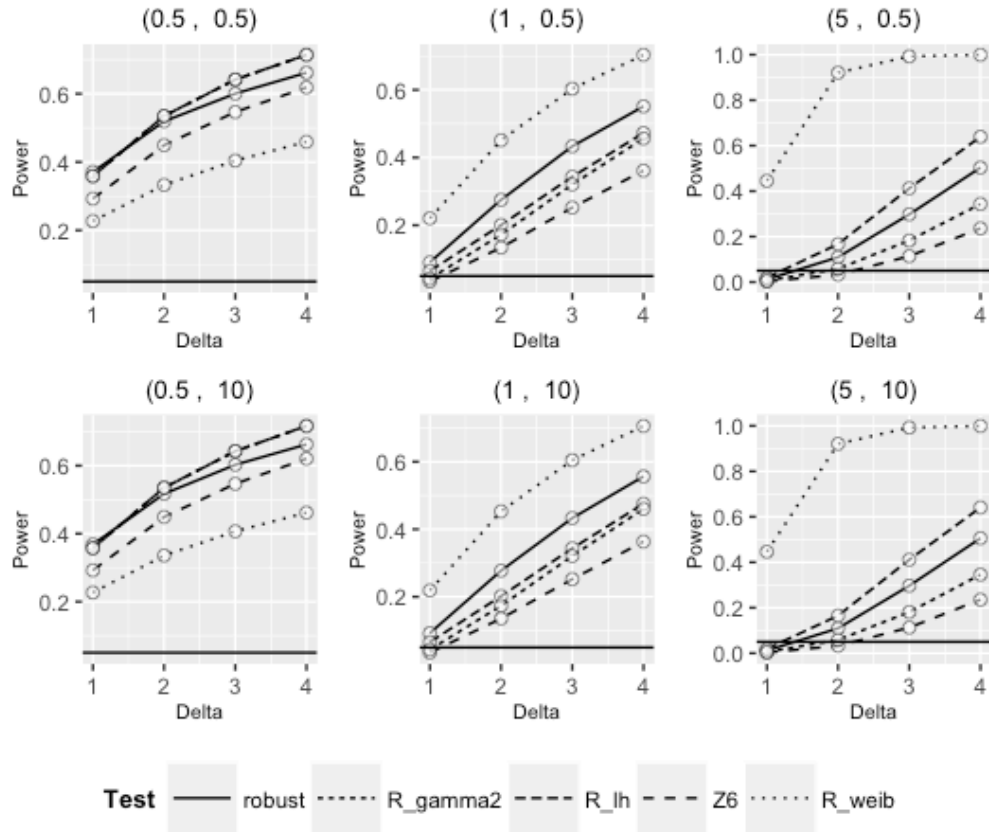


Figure 15. Power study for the left-tailed hypothesis test of $\sigma^2 = \delta\sigma_0^2$ from Weibull distributions with parameters (shape, scale) using a significance level $\alpha = 0.05$ and sample size $n = 10$. Based on 10^5 simulations.

Tests	(0.5, 0.5)	(1, 0.5)	(5, 0.5)	(0.5, 10)	(1, 10)	(5, 10)
chisq	0.5574	0.2034	0.0475	0.5553	0.2037	0.0469
robust	0.3708	0.0905	0.0113	0.3683	0.0924	0.0112
Z6	0.2926	0.0338	0.0026	0.2921	0.0347	0.0024
R_lh	0.3586	0.0650	0.0192	0.3574	0.0658	0.0192
R_gamma2	0.3585	0.0424	0.0052	0.3572	0.0435	0.0048
R_normal	0.5740	0.2255	0.0582	0.5709	0.2262	0.0578
R_chisq	0.7768	0.0000	0.0000	0.7791	0.6449	0.0000
R_exp	0.7015	0.0508	0.0000	0.6988	0.0512	0.0000
R_gamma	0.3032	0.1884	0.4170	0.3023	0.1880	0.4163
R_weib	0.2269	0.2204	0.4452	0.2268	0.2188	0.4460
R_lnorm	0.0448	0.0782	0.3812	0.0453	0.0781	0.3797

Table 12. Type-I Error rates from 10^5 simulations of samples of size $n = 10$ from a Weibull distribution with parameters (shape, scale) and a significance level $\alpha = 0.05$. Entries with a bold font are more than 20% above the nominal level.

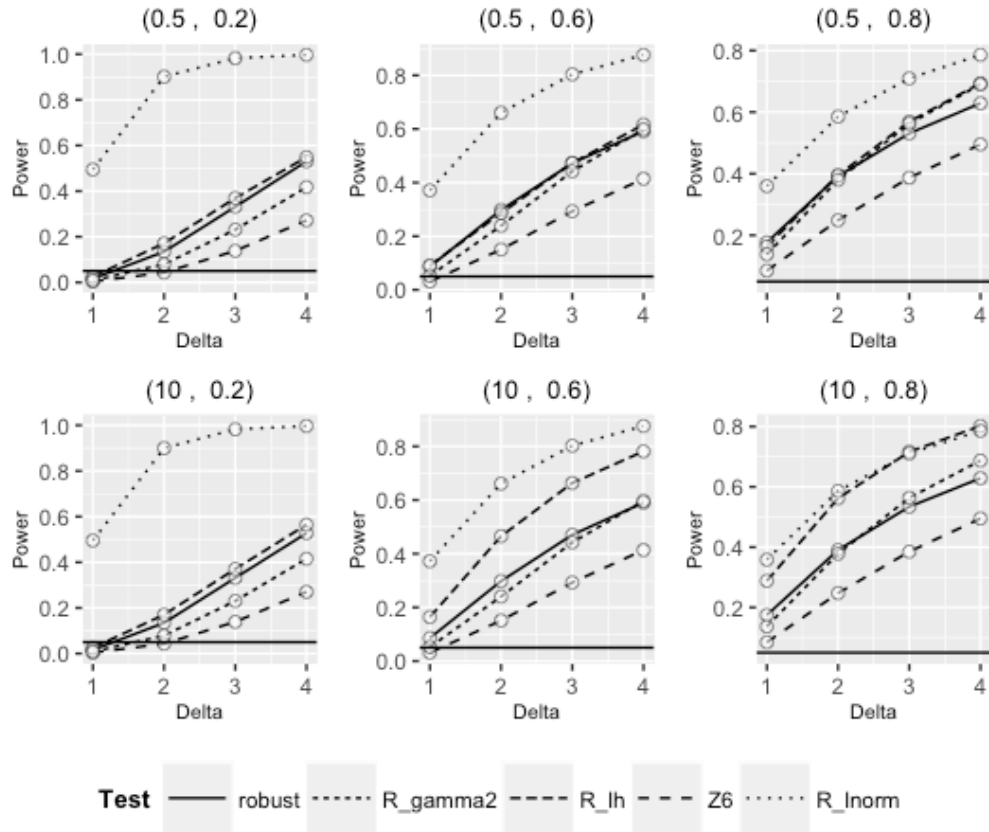


Figure 16. Power study for the left-tailed hypothesis test of $\sigma^2 = \delta\sigma_0^2$ from log-normal distributions with parameters (μ, τ) using a significance level $\alpha = 0.05$ and sample size $n = 10$. Based on 10^5 simulations.

Tests	(1, 0.2)	(1, 0.6)	(1, 0.8)	(10, 0.2)	(10, 0.6)	(10, 0.8)
chisq	0.0646	0.2112	0.3383	0.0635	0.2082	0.3361
robust	0.0167	0.0866	0.1772	0.0163	0.0858	0.1747
Z6	0.0041	0.0312	0.0852	0.0041	0.0315	0.0844
R_lh	0.0254	0.0912	0.1640	0.0254	0.1640	0.2881
R_gamma2	0.0083	0.0524	0.1383	0.0080	0.0522	0.1368
R_normal	0.0776	0.2362	0.3646	0.0766	0.2330	0.3604
R_chisq	0.0000	0.0000	0.0010	0.5829	0.6762	0.7218
R_exp	0.0000	0.0000	0.0127	0.0000	0.0000	0.0126
R_gamma	0.5161	0.4545	0.4831	0.5166	0.4565	0.4819
R_weib	0.4174	0.4539	0.4823	0.4188	0.4562	0.4825
R_inorm	0.4944	0.3709	0.3596	0.4950	0.3723	0.3584

Table 13. Type-I Error rates from 10^5 simulations of samples of size $n = 10$ from a log-normal distribution with parameters (μ, τ) and a significance level $\alpha = 0.05$. Entries with a bold font are more than 20% above the nominal level.

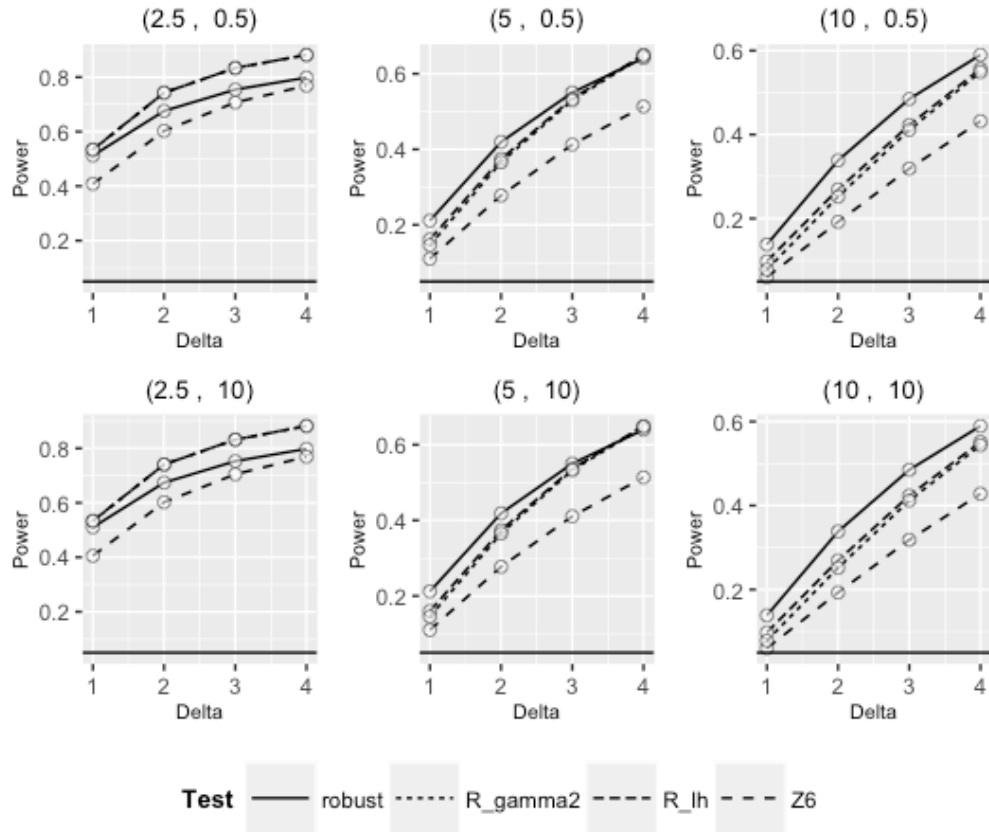


Figure 17. Power study for the left-tailed hypothesis test of $\sigma^2 = \delta\sigma_0^2$ from Pareto distributions with parameters (shape, scale) using a significance level $\alpha = 0.05$ and sample size $n = 10$. Based on 10^5 simulations.

Tests	(2.5, 1)	(5, 1)	(10, 1)	(2.5, 10)	(5, 10)	(10, 10)
chisq	0.7043	0.3830	0.2792	0.7044	0.3823	0.2796
robust	0.5122	0.2111	0.1382	0.5102	0.2119	0.1386
Z6	0.4082	0.1099	0.0604	0.4056	0.1098	0.0605
R_lh	0.5344	0.1613	0.0982	0.5339	0.1599	0.0979
R_gamma2	0.5331	0.1462	0.0782	0.5327	0.1454	0.0783
R_normal	0.7200	0.4073	0.3028	0.7198	0.4068	0.3031
R_chisq	0.0000	0.0000	0.0000	0.8794	0.7331	0.0000
R_exp	0.7584	0.2321	0.1116	0.7585	0.2330	0.1107
R_gamma	0.6283	0.3210	0.2424	0.6280	0.3199	0.2433
R_weib	0.5990	0.3343	0.2663	0.5971	0.3335	0.2671
R_inorm	0.2930	0.1311	0.0978	0.2911	0.1300	0.0996

Table 14. Type-I Error rates from 10^5 simulations of samples of size $n = 10$ from a Pareto distribution with parameters (shape, scale) and a significance level $\alpha = 0.05$. Entries with a bold font are more than 20% above the nominal level.

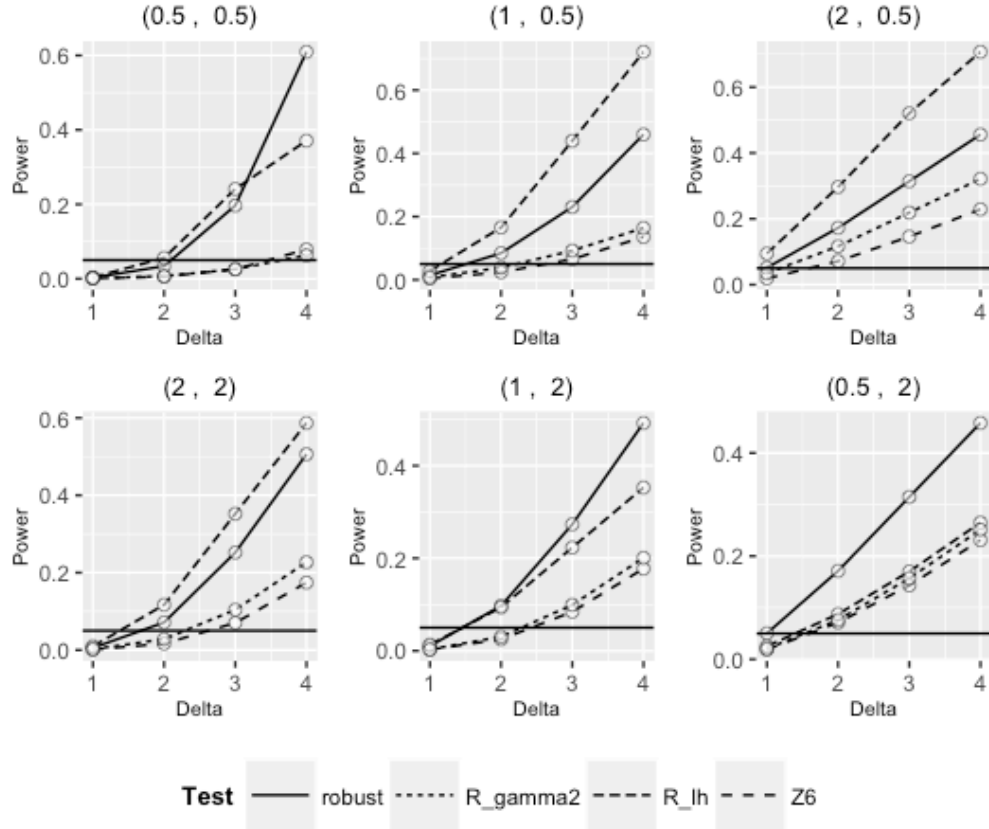


Figure 18. Power study for the left-tailed hypothesis test of $\sigma^2 = \delta\sigma_0^2$ from beta distributions with parameters (a, b) using a significance level $\alpha = 0.05$ and sample size $n = 10$. Based on 10^5 simulations.

Tests	(0.5, 0.5)	(1, 0.5)	(2, 0.5)	(2, 2)	(1, 2)	(0.5, 2)
chisq	0.0110	0.0423	0.1228	0.0247	0.0396	0.1218
robust	0.0028	0.0144	0.0516	0.0054	0.0104	0.0502
Z6	0.0006	0.0042	0.0188	0.0010	0.0026	0.0187
R_lh	0.0037	0.0282	0.0945	0.0094	0.0132	0.0253
R_gamma2	0.0009	0.0076	0.0346	0.0020	0.0030	0.0203
R_normal	0.0139	0.0495	0.1380	0.0318	0.0486	0.1366
R_chisq	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
R_exp	0.0024	0.0000	0.0000	0.0000	0.0008	0.0988
R_gamma	0.0264	0.1470	0.3582	0.1854	0.0814	0.0492
R_weib	0.0617	0.3135	0.5357	0.2794	0.1226	0.0549
R_lnorm	0.0147	0.1072	0.3116	0.1361	0.0418	0.0109

Table 15. Type-I Error rates from 10^5 simulations of samples of size $n = 10$ from a beta distribution with parameters (a, b) and a significance level $\alpha = 0.05$. Entries with a bold font are more than 20% above the nominal level.

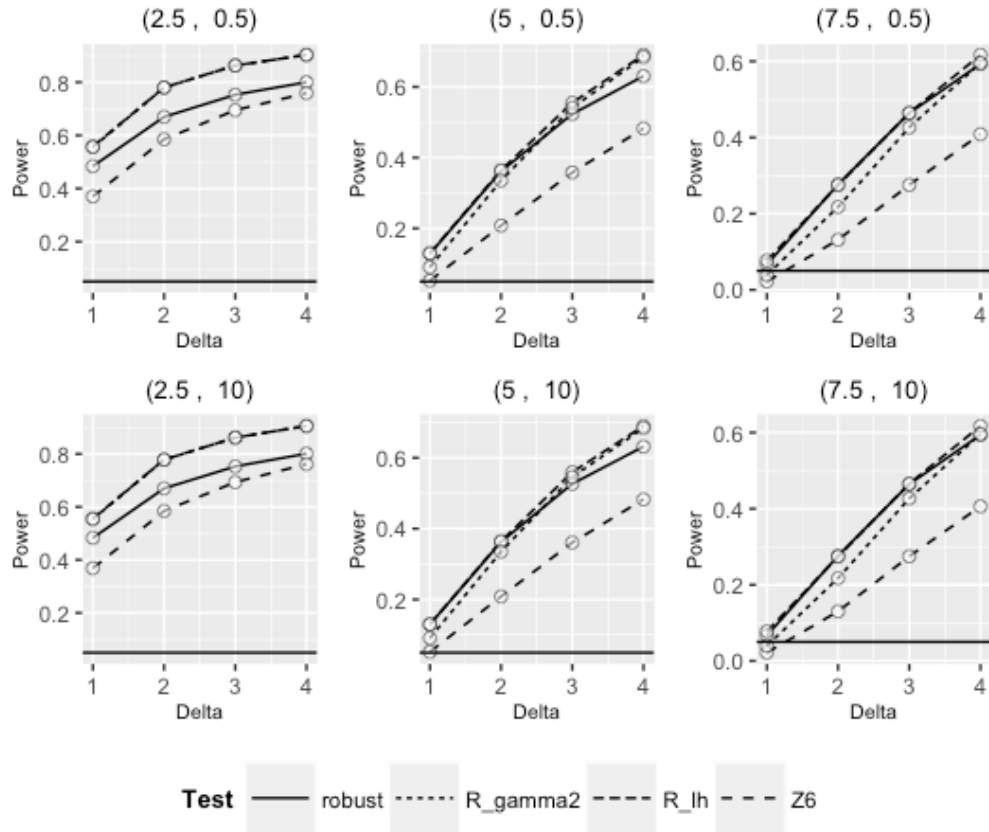


Figure 19. Power study for the left-tailed hypothesis test of $\sigma^2 = \delta\sigma_0^2$ from inverse-gamma distributions with parameters (shape, scale) using a significance level $\alpha = 0.05$ and sample size $n = 10$. Based on 10^5 simulations.

Tests	(2.5, 1)	(5, 1)	(7.5, 1)	(2.5, 10)	(5, 10)	(7.5, 10)
chisq	0.6839	0.2797	0.1800	0.6832	0.2801	0.1809
robust	0.4837	0.1283	0.0667	0.4831	0.1294	0.0669
Z6	0.3699	0.0524	0.0221	0.3686	0.0521	0.0218
R_lh	0.5580	0.1298	0.0768	0.5557	0.1299	0.0775
R_gamma2	0.5557	0.0905	0.0393	0.5536	0.0910	0.0396
R_normal	0.7010	0.3068	0.2042	0.7006	0.3069	0.2046
R_chisq	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
R_exp	0.2600	0.0000	0.0000	0.2593	0.0000	0.0000
R_gamma	0.8050	0.6016	0.5586	0.8051	0.5999	0.5593
R_weib	0.7721	0.5518	0.4968	0.7716	0.5515	0.4996
R_lnorm	0.7655	0.5602	0.5262	0.7658	0.5600	0.5270

Table 16. Type-I Error rates from 10^5 simulations of samples of size $n = 10$ from an inverse-gamma distribution with parameters (shape, scale) and a significance level $\alpha = 0.05$. Entries with a bold font are more than 20% above the nominal level.

APPENDIX C: EXPANDED TYPE-I ERROR COMPARISONS

Type-I Error rates ($n = 10$)							
<i>Normal</i> (μ, σ)							Nominal level
robust	(0, 0.01)	(0, 0.5)	(0, 1)	(0, 5)	(0, 10)	(0, 100)	(0.01)
	0.0007	0.0008	0.0007	0.0007	0.0008	0.0007	(0.05)
	0.0121	0.0126	0.0120	0.0115	0.0119	0.0117	(0.10)
Z6	0.0401	0.0396	0.0400	0.0391	0.0398	0.0406	(0.01)
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	(0.05)
	0.0027	0.0026	0.0030	0.0028	0.0029	0.0026	(0.10)
R_lh	0.0288	0.0288	0.0288	0.0297	0.0297	0.0297	(0.01)
	0.0049	0.0050	0.0052	0.0053	0.0049	0.0052	(0.05)
	0.0227	0.0229	0.0226	0.0233	0.0232	0.0231	(0.10)
R_gamma2	0.0466	0.0459	0.0459	0.0460	0.0471	0.0468	(0.01)
	0.0002	0.0001	0.0002	0.0002	0.0001	0.0002	(0.05)
	0.0056	0.0057	0.0057	0.0057	0.0057	0.0057	(0.10)
	0.0200	0.0202	0.0208	0.0200	0.0202	0.0210	
<i>t</i> (<i>df</i>)							Nominal level
robust	(4)	(6)	(7)	(12)	(20)	(30)	(0.01)
	0.0045	0.0022	0.0018	0.0013	0.0010	0.0009	(0.05)
	0.0479	0.0267	0.0242	0.0172	0.0151	0.0139	(0.10)
Z6	0.1195	0.0759	0.0684	0.0537	0.0470	0.0438	(0.01)
	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	(0.05)
	0.0147	0.0070	0.0060	0.0043	0.0033	0.0033	(0.10)
R_lh	0.0971	0.0596	0.0544	0.0399	0.0348	0.0330	(0.01)
	0.0357	0.0187	0.0160	0.0113	0.0100	0.0090	(0.05)
	0.1053	0.0664	0.0573	0.0445	0.0390	0.0376	(0.10)
R_gamma2	0.1710	0.1117	0.1013	0.0795	0.0709	0.0686	(0.01)
	0.0015	0.0005	0.0004	0.0003	0.0002	0.0003	(0.05)
	0.0276	0.0148	0.0119	0.0082	0.0071	0.0070	(0.10)
	0.0756	0.0429	0.0384	0.0278	0.0245	0.0237	

(Continued) Type-I Error rates ($n = 10$)

<i>Chi-squared</i>							Nominal
(<i>df</i>)	(1)	(2)	(4)	(6)	(7)	(10)	level
robust	0.0574	0.0180	0.0046	0.0022	0.0021	0.0017	(0.01)
	0.1630	0.0892	0.0440	0.0296	0.0265	0.0207	(0.05)
	0.2552	0.1723	0.1038	0.0820	0.0736	0.0625	(0.10)
Z6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	(0.01)
	0.0870	0.0330	0.0123	0.0078	0.0070	0.0052	(0.05)
	0.2374	0.1464	0.0845	0.0625	0.0573	0.0463	(0.10)
R_lh	0.0284	0.0196	0.0147	0.0101	0.0087	0.0072	(0.01)
	0.1036	0.0662	0.0491	0.0412	0.0373	0.0314	(0.05)
	0.1662	0.1137	0.0921	0.0754	0.0735	0.0632	(0.10)
R_gamma2	0.0242	0.0042	0.0010	0.0005	0.0005	0.0004	(0.01)
	0.0999	0.0431	0.0185	0.0135	0.0113	0.0094	(0.05)
	0.1629	0.0896	0.0516	0.0378	0.0369	0.0302	(0.10)
<i>Exponential</i>							Nominal
(λ)	(0.1)	(0.5)	(1)	(5)	(20)	(50)	level
robust	0.0186	0.0184	0.0178	0.0181	0.0181	0.0174	(0.01)
	0.0895	0.0909	0.0885	0.0898	0.0899	0.0911	(0.05)
	0.1698	0.1711	0.1699	0.1708	0.1689	0.1708	(0.10)
Z6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	(0.01)
	0.0336	0.0339	0.0340	0.0331	0.0333	0.0322	(0.05)
	0.1462	0.1461	0.1458	0.1456	0.1470	0.1466	(0.10)
R_lh	0.0207	0.0200	0.0197	0.0203	0.0205	0.0194	(0.01)
	0.0653	0.0648	0.0657	0.0636	0.0650	0.0652	(0.05)
	0.1139	0.1126	0.1126	0.1123	0.1157	0.1124	(0.10)
R_gamma2	0.0046	0.0046	0.0043	0.0048	0.0046	0.0041	(0.01)
	0.0424	0.0432	0.0426	0.0412	0.0423	0.0434	(0.05)
	0.0903	0.0890	0.0898	0.0886	0.0916	0.0895	(0.10)
<i>Gamma</i>							Nominal
(<i>shape, rate</i>)	(0.5, 10)	(1, 10)	(5, 10)	(10, 10)	(50, 10)	(100, 10)	level
robust	0.0581	0.0184	0.0014	0.0012	0.0007	0.0008	(0.01)
	0.1663	0.0914	0.0211	0.0160	0.0126	0.0124	(0.05)
	0.2546	0.1696	0.0627	0.0500	0.0425	0.0410	(0.10)
Z6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	(0.01)
	0.0869	0.0345	0.0047	0.0037	0.0029	0.0032	(0.05)
	0.2392	0.1472	0.0473	0.0364	0.0299	0.0290	(0.10)
R_lh	0.0283	0.0200	0.0075	0.0052	0.0048	0.0053	(0.01)
	0.1050	0.0650	0.0320	0.0238	0.0217	0.0220	(0.05)
	0.1658	0.1144	0.0630	0.0523	0.0443	0.0465	(0.10)
R_gamma2	0.0245	0.0043	0.0003	0.0002	0.0002	0.0002	(0.01)
	0.1016	0.0422	0.0099	0.0069	0.0059	0.0053	(0.05)
	0.1621	0.0905	0.0302	0.0260	0.0214	0.0205	(0.10)

(Continued) Type-I Error rates ($n = 10$)

<i>Weibull</i>								Nominal
(shape, scale)	(0.5, 1)	(1, 1)	(5, 1)	(20, 1)	(50, 1)	(100, 1)		level
robust	0.2029	0.0174	0.0007	0.0015	0.0021	0.0022		(0.01)
	0.3685	0.0907	0.0115	0.0229	0.0267	0.0278		(0.05)
	0.4677	0.1704	0.0378	0.0634	0.0727	0.0792		(0.10)
Z6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000		(0.01)
	0.2912	0.0335	0.0023	0.0056	0.0064	0.0076		(0.05)
	0.4791	0.1455	0.0275	0.0490	0.0586	0.0594		(0.10)
R_lh	0.1808	0.0203	0.0041	0.0113	0.0172	0.0183		(0.01)
	0.3574	0.0662	0.0204	0.0452	0.0599	0.0664		(0.05)
	0.4462	0.1131	0.0424	0.0828	0.1033	0.1117		(0.10)
R_gamma2	0.1803	0.0044	0.0002	0.0004	0.0006	0.0006		(0.01)
	0.3573	0.0429	0.0054	0.0111	0.0131	0.0139		(0.05)
	0.4461	0.0890	0.0188	0.0356	0.0422	0.0444		(0.10)
<i>Log-normal</i>								Nominal
(μ, τ)	(1, 0.01)	(1, 0.2)	(1, 0.4)	(1, 0.6)	(1, 0.8)	(1, 1)		level
robust	0.0009	0.0010	0.0030	0.0120	0.0459	0.1133		(0.01)
	0.0122	0.0155	0.0352	0.0862	0.1764	0.2861		(0.05)
	0.0400	0.0518	0.0933	0.1793	0.2894	0.3978		(0.10)
Z6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000		(0.01)
	0.0025	0.0041	0.0095	0.0317	0.0843	0.1756		(0.05)
	0.0302	0.0377	0.0745	0.1517	0.2652	0.3878		(0.10)
R_lh	0.0079	0.0054	0.0117	0.0284	0.0504	0.0930		(0.01)
	0.0327	0.0246	0.0480	0.0909	0.1614	0.2844		(0.05)
	0.0613	0.0523	0.0897	0.1592	0.2620	0.4004		(0.10)
R_gamma2	0.0002	0.0002	0.0008	0.0032	0.0194	0.0738		(0.01)
	0.0056	0.0075	0.0181	0.0520	0.1349	0.2756		(0.05)
	0.0205	0.0258	0.0524	0.1209	0.2426	0.3951		(0.10)
<i>Pareto</i>								Nominal
(shape, scale)	(2.5, 1)	(5, 1)	(10, 1)	(20, 1)	(50, 1)	(100, 1)		level
robust	0.2893	0.0692	0.0348	0.0254	0.0205	0.0191		(0.01)
	0.5146	0.2115	0.1366	0.1112	0.0976	0.0951		(0.05)
	0.6253	0.3256	0.2366	0.1986	0.1813	0.1764		(0.10)
Z6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000		(0.01)
	0.4059	0.1102	0.0603	0.0448	0.0372	0.0364		(0.05)
	0.6302	0.3039	0.2103	0.1766	0.1562	0.1527		(0.10)
R_lh	0.2602	0.0476	0.0297	0.0233	0.0216	0.0208		(0.01)
	0.5362	0.1612	0.0984	0.0798	0.0714	0.0686		(0.05)
	0.6466	0.2536	0.1646	0.1357	0.1227	0.1177		(0.10)
R_gamma2	0.2542	0.0289	0.0105	0.0066	0.0053	0.0047		(0.01)
	0.5349	0.1457	0.0784	0.0574	0.0492	0.0467		(0.05)
	0.6462	0.2429	0.1463	0.1146	0.1002	0.0952		(0.10)

(Continued) Type-I Error rates ($n = 10$)

<i>Beta</i> (<i>a, b</i>)	(0.5, 1)	(1, 1)	(2, 1)	(1, 0.5)	(2, 2)	(1, 2)	Nominal level
robust	0.0024	0.0001	0.0008	0.0024	0.0004	0.0006	(0.01)
	0.0152	0.0032	0.0108	0.0147	0.0053	0.0106	(0.05)
	0.0371	0.0132	0.0344	0.0357	0.0206	0.0334	(0.10)
Z6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	(0.01)
	0.0043	0.0006	0.0026	0.0041	0.0011	0.0026	(0.05)
	0.0268	0.0084	0.0235	0.0270	0.0141	0.0244	(0.10)
R_lh	0.0022	0.0011	0.0046	0.0102	0.0018	0.0031	(0.01)
	0.0073	0.0061	0.0199	0.0290	0.0100	0.0135	(0.05)
	0.0160	0.0137	0.0419	0.0486	0.0221	0.0274	(0.10)
R_gamma2	0.0006	0.0000	0.0002	0.0010	0.0002	0.0001	(0.01)
	0.0040	0.0010	0.0049	0.0076	0.0020	0.0029	(0.05)
	0.0102	0.0040	0.0154	0.0176	0.0080	0.0105	(0.10)
<i>Inv-gamma</i> (<i>shape, scale</i>)	(2.5, 1)	(5, 1)	(7.5, 1)	(10, 1)	(20, 1)	(50, 1)	Nominal level
robust	0.2579	0.0213	0.0076	0.0040	0.0020	0.0010	(0.01)
	0.4844	0.1284	0.0676	0.0465	0.0242	0.0155	(0.05)
	0.6062	0.2398	0.1530	0.1144	0.0709	0.0519	(0.10)
Z6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	(0.01)
	0.3678	0.0518	0.0215	0.0140	0.0065	0.0039	(0.05)
	0.6076	0.2112	0.1263	0.0926	0.0552	0.0379	(0.10)
R_lh	0.2542	0.0412	0.0234	0.0157	0.0086	0.0068	(0.01)
	0.5596	0.1292	0.0766	0.0577	0.0352	0.0267	(0.05)
	0.6805	0.2182	0.1402	0.1082	0.0699	0.0545	(0.10)
R_gamma2	0.2405	0.0072	0.0022	0.0012	0.0006	0.0003	(0.01)
	0.5575	0.0900	0.0391	0.0248	0.0119	0.0073	(0.05)
	0.6797	0.1873	0.1002	0.0707	0.0390	0.0269	(0.10)

Table 17. Type-I Error rates from 10^5 simulations of samples of size $n = 10$ and significance levels $\alpha = 0.01, 0.05$, and 0.10 . Entries with bold font are more than 20% above the nominal level.

Type-I Error rates ($n = 20$)

<i>Normal</i>							Nominal
(μ, σ)	(0, 0.01)	(0, 0.5)	(0, 1)	(0, 5)	(0, 10)	(0, 100)	level
robust	0.0004	0.0003	0.0004	0.0005	0.0005	0.0004	(0.01)
	0.0083	0.0084	0.0080	0.0081	0.0082	0.0078	(0.05)
	0.0320	0.0313	0.0320	0.0306	0.0305	0.0312	(0.10)
Z6	0.0011	0.0010	0.0010	0.0011	0.0010	0.0009	(0.01)
	0.0231	0.0228	0.0231	0.0230	0.0235	0.0229	(0.05)
	0.0655	0.0651	0.0657	0.0647	0.0637	0.0645	(0.10)
R_lh	0.0038	0.0036	0.0034	0.0035	0.0035	0.0036	(0.01)
	0.0192	0.0190	0.0179	0.0178	0.0179	0.0187	(0.05)
	0.0400	0.0387	0.0394	0.0406	0.0395	0.0389	(0.10)
R_gamma2	0.0003	0.0003	0.0002	0.0002	0.0003	0.0004	(0.01)
	0.0058	0.0058	0.0055	0.0052	0.0051	0.0053	(0.05)
	0.0176	0.0176	0.0180	0.0189	0.0174	0.0182	(0.10)
<i>t</i>							Nominal
(df)	(4)	(6)	(7)	(12)	(20)	(30)	level
robust	0.0058	0.0022	0.0014	0.0007	0.0006	0.0004	(0.01)
	0.0532	0.0251	0.0206	0.0137	0.0114	0.0101	(0.05)
	0.1223	0.0706	0.0626	0.0448	0.0386	0.0366	(0.10)
Z6	0.0135	0.0048	0.0037	0.0022	0.0015	0.0014	(0.01)
	0.1065	0.0580	0.0505	0.0349	0.0297	0.0270	(0.05)
	0.1954	0.1276	0.1137	0.0896	0.0779	0.0732	(0.10)
R_lh	0.0473	0.0216	0.0173	0.0112	0.0084	0.0077	(0.01)
	0.1304	0.0705	0.0610	0.0435	0.0367	0.0342	(0.05)
	0.1924	0.1156	0.1044	0.0780	0.0669	0.0622	(0.10)
R_gamma2	0.0056	0.0017	0.0013	0.0007	0.0006	0.0004	(0.01)
	0.0457	0.0195	0.0158	0.0096	0.0080	0.0068	(0.05)
	0.0972	0.0502	0.0420	0.0285	0.0235	0.0211	(0.10)
<i>Chi-squared</i>							Nominal
(df)	(1)	(2)	(4)	(6)	(7)	(10)	level
robust	0.0481	0.0179	0.0050	0.0026	0.0018	0.0012	(0.01)
	0.1370	0.0820	0.0428	0.0294	0.0262	0.0187	(0.05)
	0.2179	0.1510	0.0994	0.0777	0.0706	0.0572	(0.10)
Z6	0.0589	0.0254	0.0093	0.0051	0.0046	0.0026	(0.01)
	0.1931	0.1288	0.0816	0.0611	0.0560	0.0447	(0.05)
	0.2764	0.2108	0.1572	0.1301	0.1226	0.1035	(0.10)
R_lh	0.0425	0.0147	0.0135	0.0114	0.0104	0.0068	(0.01)
	0.0915	0.0448	0.0363	0.0331	0.0321	0.0277	(0.05)
	0.1279	0.0698	0.0566	0.0530	0.0534	0.0494	(0.10)
R_gamma2	0.0588	0.0185	0.0058	0.0045	0.0032	0.0021	(0.01)
	0.1279	0.0619	0.0272	0.0188	0.0201	0.0150	(0.05)
	0.1747	0.1053	0.0578	0.0432	0.0449	0.0364	(0.10)

(Continued) Type-I Error rates ($n = 20$)

<i>Exponential</i>							Nominal
(λ)	(0.1)	(0.5)	(1)	(5)	(20)	(50)	level
robust	0.0189	0.0186	0.0178	0.0180	0.0179	0.0191	(0.01)
	0.0815	0.0815	0.0826	0.0816	0.0825	0.0829	(0.05)
	0.1531	0.1521	0.1516	0.1518	0.1495	0.1520	(0.10)
Z6	0.0245	0.0248	0.0245	0.0249	0.0242	0.0253	(0.01)
	0.1296	0.1282	0.1272	0.1289	0.1291	0.1290	(0.05)
	0.2126	0.2106	0.2122	0.2140	0.2111	0.2137	(0.10)
R_lh	0.0139	0.0147	0.0145	0.0152	0.0152	0.0151	(0.01)
	0.0548	0.0563	0.0554	0.0543	0.0542	0.0535	(0.05)
	0.0984	0.0985	0.0962	0.0953	0.0962	0.0983	(0.10)
R_gamma2	0.0095	0.0095	0.0094	0.0103	0.0100	0.0104	(0.01)
	0.0495	0.0507	0.0498	0.0489	0.0487	0.0482	(0.05)
	0.0932	0.0933	0.0912	0.0899	0.0909	0.0928	(0.10)
<i>Gamma</i>							Nominal
(shape, rate)	(0.5, 10)	(1, 10)	(5, 10)	(10, 10)	(50, 10)	(100, 10)	level
robust	0.0464	0.0182	0.0012	0.0006	0.0005	0.0004	(0.01)
	0.1374	0.0810	0.0187	0.0126	0.0089	0.0088	(0.05)
	0.2156	0.1536	0.0571	0.0436	0.0345	0.0318	(0.10)
Z6	0.0600	0.0246	0.0028	0.0018	0.0011	0.0011	(0.01)
	0.1943	0.1288	0.0444	0.0339	0.0241	0.0246	(0.05)
	0.2753	0.2147	0.1049	0.0862	0.0698	0.0665	(0.10)
R_lh	0.0398	0.0147	0.0075	0.0052	0.0037	0.0036	(0.01)
	0.1083	0.0553	0.0313	0.0238	0.0189	0.0187	(0.05)
	0.1600	0.0978	0.0604	0.0486	0.0399	0.0389	(0.10)
R_gamma2	0.0396	0.0104	0.0008	0.0006	0.0004	0.0002	(0.01)
	0.1082	0.0499	0.0108	0.0075	0.0057	0.0062	(0.05)
	0.1600	0.0926	0.0289	0.0227	0.0193	0.0182	(0.10)
<i>Weibull</i>							Nominal
(shape, scale)	(0.5, 1)	(1, 1)	(5, 1)	(20, 1)	(50, 1)	(100, 1)	level
robust	0.1696	0.0182	0.0004	0.0013	0.0018	0.0021	(0.01)
	0.3168	0.0810	0.0079	0.0186	0.0255	0.0265	(0.05)
	0.4098	0.1512	0.0289	0.0588	0.0706	0.0751	(0.10)
Z6	0.2432	0.0257	0.0009	0.0029	0.0045	0.0048	(0.01)
	0.4038	0.1280	0.0212	0.0457	0.0560	0.0600	(0.05)
	0.4824	0.2122	0.0628	0.1084	0.1247	0.1314	(0.10)
R_lh	0.2964	0.0144	0.0036	0.0111	0.0159	0.0195	(0.01)
	0.4392	0.0546	0.0195	0.0452	0.0587	0.0673	(0.05)
	0.5082	0.0957	0.0392	0.0832	0.1021	0.1118	(0.10)
R_gamma2	0.2964	0.0098	0.0002	0.0011	0.0015	0.0017	(0.01)
	0.4392	0.0493	0.0044	0.0131	0.0178	0.0205	(0.05)
	0.5082	0.0907	0.0139	0.0364	0.0459	0.0501	(0.10)

(Continued) Type-I Error rates ($n = 20$)

<i>Log-normal</i>							Nominal
(μ, τ)	(1, 0.01)	(1, 0.2)	(1, 0.4)	(1, 0.6)	(1, 0.8)	(1, 1)	level
robust	0.0004	0.0006	0.0036	0.0176	0.0596	0.1248	(0.01)
	0.0088	0.0135	0.0378	0.0940	0.1765	0.2716	(0.05)
	0.0313	0.0442	0.0913	0.1759	0.2727	0.3739	(0.10)
Z6	0.0011	0.0019	0.0071	0.0291	0.0814	0.1644	(0.01)
	0.0239	0.0336	0.0741	0.1522	0.2450	0.3561	(0.05)
	0.0659	0.0866	0.1523	0.2451	0.3416	0.4478	(0.10)
R_lh	0.0047	0.0050	0.0131	0.0310	0.0834	0.2216	(0.01)
	0.0234	0.0245	0.0490	0.1057	0.2340	0.4195	(0.05)
	0.0455	0.0487	0.0931	0.1774	0.3335	0.5170	(0.10)
R_gamma2	0.0002	0.0006	0.0024	0.0164	0.0782	0.2212	(0.01)
	0.0059	0.0091	0.0269	0.0918	0.2316	0.4194	(0.05)
	0.0182	0.0257	0.0662	0.1658	0.3318	0.5169	(0.10)
<i>Pareto</i>							Nominal
$(shape, scale)$	(2.5, 1)	(5, 1)	(10, 1)	(20, 1)	(50, 1)	(100, 1)	level
robust	0.3216	0.0798	0.0390	0.0268	0.0206	0.0199	(0.01)
	0.5095	0.2039	0.1318	0.1036	0.0904	0.0853	(0.05)
	0.6097	0.3026	0.2163	0.1795	0.1625	0.1598	(0.10)
Z6	0.4247	0.0998	0.0497	0.0356	0.0287	0.0261	(0.01)
	0.6059	0.2762	0.1896	0.1562	0.1372	0.1330	(0.05)
	0.6768	0.3744	0.2813	0.2443	0.2246	0.2177	(0.10)
R_lh	0.5451	0.0812	0.0316	0.0208	0.0170	0.0159	(0.01)
	0.7165	0.2163	0.1074	0.0773	0.0635	0.0589	(0.05)
	0.7782	0.3069	0.1725	0.1291	0.1081	0.1018	(0.10)
R_gamma2	0.5451	0.0797	0.0282	0.0165	0.0127	0.0111	(0.01)
	0.7165	0.2154	0.1050	0.0735	0.0579	0.0540	(0.05)
	0.7782	0.3064	0.1703	0.1254	0.1035	0.0969	(0.10)
<i>Beta</i>							Nominal
(a, b)	(0.5, 1)	(1, 1)	(2, 1)	(1, 0.5)	(2, 2)	(1, 2)	level
robust	0.0008	0.0000	0.0004	0.0006	0.0001	0.0003	(0.01)
	0.0062	0.0009	0.0062	0.0058	0.0023	0.0062	(0.05)
	0.0186	0.0057	0.0221	0.0188	0.0116	0.0229	(0.10)
Z6	0.0009	0.0000	0.0007	0.0010	0.0001	0.0006	(0.01)
	0.0125	0.0033	0.0152	0.0122	0.0081	0.0158	(0.05)
	0.0360	0.0182	0.0460	0.0355	0.0302	0.0478	(0.10)
R_lh	0.0003	0.0003	0.0036	0.0048	0.0009	0.0017	(0.01)
	0.0019	0.0029	0.0175	0.0179	0.0069	0.0086	(0.05)
	0.0046	0.0084	0.0360	0.0340	0.0171	0.0163	(0.10)
R_gamma2	0.0001	0.0000	0.0002	0.0003	0.0000	0.0001	(0.01)
	0.0011	0.0003	0.0023	0.0025	0.0008	0.0012	(0.05)
	0.0028	0.0008	0.0069	0.0050	0.0031	0.0041	(0.10)

(Continued) Type-I Error rates ($n = 20$)

<i>Inv-gamma</i> (<i>shape, scale</i>)	(2.5, 1)	(5, 1)	(7.5, 1)	(10, 1)	(20, 1)	(50, 1)	Nominal level
robust	0.3083	0.0376	0.0113	0.0053	0.0014	0.0008	(0.01)
	0.4985	0.1441	0.0759	0.0520	0.0234	0.0130	(0.05)
	0.5962	0.2424	0.1570	0.1179	0.0669	0.0440	(0.10)
Z6	0.3976	0.0559	0.0222	0.0112	0.0039	0.0018	(0.01)
	0.5950	0.2174	0.1330	0.0980	0.0546	0.0331	(0.05)
	0.6682	0.3195	0.2276	0.1851	0.1214	0.0854	(0.10)
R_lh	0.5816	0.0552	0.0257	0.0168	0.0085	0.0051	(0.01)
	0.7595	0.1818	0.0901	0.0643	0.0356	0.0234	(0.05)
	0.8161	0.2818	0.1599	0.1176	0.0684	0.0482	(0.10)
R_gamma2	0.5816	0.0430	0.0106	0.0047	0.0012	0.0005	(0.01)
	0.7595	0.1744	0.0724	0.0435	0.0174	0.0089	(0.05)
	0.8161	0.2767	0.1453	0.0964	0.0449	0.0268	(0.10)

Table 18. Type-I Error rates from 10^5 simulations of samples of size $n = 20$ and significance levels $\alpha = 0.01, 0.05$, and 0.10 . Entries with bold font are more than 20% above the nominal level.

Type-I Error rates ($n = 30$)

<i>Normal</i>							Nominal
(μ, σ)	(0, 0.01)	(0, 0.5)	(0, 1)	(0, 5)	(0, 10)	(0, 100)	level
robust	0.0003	0.0002	0.0003	0.0002	0.0003	0.0003	(0.01)
	0.0069	0.0070	0.0068	0.0073	0.0071	0.0073	(0.05)
	0.0272	0.0276	0.0280	0.0276	0.0277	0.0278	(0.10)
Z6	0.0035	0.0033	0.0037	0.0033	0.0032	0.0032	(0.01)
	0.0321	0.0330	0.0328	0.0321	0.0324	0.0321	(0.05)
	0.0764	0.0760	0.0767	0.0761	0.0763	0.0777	(0.10)
R_lh	0.0032	0.0036	0.0033	0.0032	0.0034	0.0034	(0.01)
	0.0179	0.0178	0.0173	0.0180	0.0173	0.0174	(0.05)
	0.0374	0.0380	0.0388	0.0383	0.0375	0.0377	(0.10)
R_gamma2	0.0002	0.0003	0.0003	0.0003	0.0003	0.0004	(0.01)
	0.0050	0.0050	0.0048	0.0050	0.0053	0.0048	(0.05)
	0.0156	0.0164	0.0165	0.0163	0.0158	0.0163	(0.10)
<i>t</i>							Nominal
(df)	(4)	(6)	(7)	(12)	(20)	(30)	level
robust	0.0069	0.0019	0.0014	0.0006	0.0005	0.0006	(0.01)
	0.0550	0.0253	0.0199	0.0124	0.0101	0.0095	(0.05)
	0.1264	0.0694	0.0601	0.0428	0.0356	0.0327	(0.10)
Z6	0.0377	0.0154	0.0114	0.0063	0.0046	0.0046	(0.01)
	0.1359	0.0768	0.0676	0.0480	0.0411	0.0381	(0.05)
	0.2166	0.1427	0.1301	0.1013	0.0907	0.0860	(0.10)
R_lh	0.0569	0.0237	0.0175	0.0110	0.0084	0.0069	(0.01)
	0.1374	0.0746	0.0629	0.0430	0.0366	0.0324	(0.05)
	0.2003	0.1219	0.1065	0.0762	0.0677	0.0601	(0.10)
R_gamma2	0.0088	0.0024	0.0016	0.0008	0.0004	0.0004	(0.01)
	0.0562	0.0224	0.0178	0.0098	0.0078	0.0068	(0.05)
	0.1094	0.0534	0.0436	0.0274	0.0230	0.0202	(0.10)
<i>Chi-squared</i>							Nominal
(df)	(1)	(2)	(4)	(6)	(7)	(10)	level
robust	0.0411	0.0167	0.0054	0.0025	0.0019	0.0011	(0.01)
	0.1220	0.0746	0.0418	0.0279	0.0240	0.0184	(0.05)
	0.1956	0.1409	0.0936	0.0724	0.0668	0.0549	(0.10)
Z6	0.0887	0.0483	0.0235	0.0153	0.0136	0.0095	(0.01)
	0.1941	0.1410	0.0989	0.0798	0.0737	0.0608	(0.05)
	0.2680	0.2106	0.1682	0.1446	0.1375	0.1208	(0.10)
R_lh	0.0428	0.0133	0.0108	0.0097	0.0090	0.0073	(0.01)
	0.1064	0.0515	0.0365	0.0349	0.0325	0.0282	(0.05)
	0.1550	0.0912	0.0658	0.0635	0.0619	0.0552	(0.10)
R_gamma2	0.0428	0.0121	0.0029	0.0016	0.0011	0.0011	(0.01)
	0.1064	0.0504	0.0230	0.0158	0.0135	0.0103	(0.05)
	0.1550	0.0903	0.0506	0.0396	0.0350	0.0282	(0.10)

(Continued) Type-I Error rates ($n = 30$)

<i>Exponential</i>							Nominal
(λ)	(0.1)	(0.5)	(1)	(5)	(20)	(50)	level
robust	0.0169	0.0168	0.0174	0.0171	0.0179	0.0173	(0.01)
	0.0761	0.0760	0.0762	0.0752	0.0767	0.0749	(0.05)
	0.1405	0.1394	0.1398	0.1405	0.1416	0.1397	(0.10)
Z6	0.0481	0.0475	0.0478	0.0475	0.0472	0.0478	(0.01)
	0.1405	0.1404	0.1413	0.1405	0.1417	0.1415	(0.05)
	0.2146	0.2126	0.2112	0.2138	0.2124	0.2113	(0.10)
R_lh	0.0128	0.0126	0.0122	0.0138	0.0126	0.0129	(0.01)
	0.0513	0.0527	0.0520	0.0511	0.0526	0.0511	(0.05)
	0.0912	0.0923	0.0917	0.0918	0.0909	0.0917	(0.10)
R_gamma2	0.0118	0.0114	0.0112	0.0127	0.0114	0.0118	(0.01)
	0.0503	0.0514	0.0509	0.0500	0.0515	0.0500	(0.05)
	0.0902	0.0913	0.0905	0.0908	0.0896	0.0908	(0.10)
<i>Gamma</i>							Nominal
(shape, rate)	(0.5, 10)	(1, 10)	(5, 10)	(10, 10)	(50, 10)	(100, 10)	level
robust	0.0414	0.0171	0.0010	0.0006	0.0003	0.0004	(0.01)
	0.1232	0.0764	0.0180	0.0115	0.0083	0.0079	(0.05)
	0.1961	0.1423	0.0544	0.0411	0.0312	0.0289	(0.10)
Z6	0.0877	0.0478	0.0094	0.0061	0.0038	0.0035	(0.01)
	0.1924	0.1430	0.0610	0.0472	0.0351	0.0342	(0.05)
	0.2677	0.2137	0.1200	0.1020	0.0822	0.0789	(0.10)
R_lh	0.0428	0.0138	0.0066	0.0052	0.0036	0.0033	(0.01)
	0.1074	0.0516	0.0284	0.0227	0.0176	0.0167	(0.05)
	0.1577	0.0911	0.0561	0.0482	0.0378	0.0372	(0.10)
R_gamma2	0.0428	0.0123	0.0006	0.0005	0.0003	0.0004	(0.01)
	0.1074	0.0505	0.0099	0.0075	0.0057	0.0048	(0.05)
	0.1577	0.0900	0.0287	0.0231	0.0178	0.0176	(0.10)
<i>Weibull</i>							Nominal
(shape, scale)	(0.5, 1)	(1, 1)	(5, 1)	(20, 1)	(50, 1)	(100, 1)	level
robust	0.1546	0.0169	0.0003	0.0013	0.0017	0.0024	(0.01)
	0.2882	0.0752	0.0065	0.0190	0.0246	0.0270	(0.05)
	0.3795	0.1407	0.0260	0.0565	0.0680	0.0723	(0.10)
Z6	0.2734	0.0476	0.0029	0.0096	0.0136	0.0147	(0.01)
	0.3877	0.1421	0.0300	0.0628	0.0758	0.0791	(0.05)
	0.4499	0.2152	0.0733	0.1218	0.1397	0.1447	(0.10)
R_lh	0.3517	0.0124	0.0037	0.0123	0.0180	0.0214	(0.01)
	0.4856	0.0520	0.0193	0.0508	0.0631	0.0701	(0.05)
	0.5501	0.0908	0.0395	0.0885	0.1068	0.1150	(0.10)
R_gamma2	0.3517	0.0113	0.0001	0.0011	0.0019	0.0023	(0.01)
	0.4856	0.0508	0.0038	0.0148	0.0198	0.0228	(0.05)
	0.5501	0.0897	0.0116	0.0366	0.0464	0.0516	(0.10)

(Continued) Type-I Error rates ($n = 30$)

<i>Log-normal</i>							Nominal
(μ, τ)	(1, 0.01)	(1, 0.2)	(1, 0.4)	(1, 0.6)	(1, 0.8)	(1, 1)	level
robust	0.0003	0.0006	0.0036	0.0198	0.0606	0.1232	(0.01)
	0.0076	0.0118	0.0372	0.0934	0.1715	0.2634	(0.05)
	0.0272	0.0414	0.0908	0.1679	0.2606	0.3584	(0.10)
Z6	0.0032	0.0062	0.0214	0.0630	0.1323	0.2261	(0.01)
	0.0325	0.0465	0.0974	0.1751	0.2626	0.3594	(0.05)
	0.0750	0.1022	0.1684	0.2524	0.3405	0.4344	(0.10)
R_lh	0.0038	0.0050	0.0123	0.0338	0.1262	0.3186	(0.01)
	0.0182	0.0225	0.0471	0.1204	0.2946	0.5024	(0.05)
	0.0390	0.0464	0.0911	0.2020	0.3939	0.5930	(0.10)
R_gamma2	0.0003	0.0007	0.0037	0.0284	0.1257	0.3186	(0.01)
	0.0048	0.0090	0.0316	0.1160	0.2943	0.5024	(0.05)
	0.0167	0.0258	0.0738	0.1983	0.3938	0.5930	(0.10)
<i>Pareto</i>							Nominal
$(shape, scale)$	(2.5, 1)	(5, 1)	(10, 1)	(20, 1)	(50, 1)	(100, 1)	level
robust	0.3308	0.0805	0.0371	0.0254	0.0202	0.0184	(0.01)
	0.5106	0.1976	0.1226	0.0948	0.0828	0.0790	(0.05)
	0.6002	0.2921	0.2044	0.1668	0.1520	0.1462	(0.10)
Z6	0.5019	0.1581	0.0870	0.0653	0.0536	0.0508	(0.01)
	0.6165	0.2934	0.2027	0.1706	0.1519	0.1484	(0.05)
	0.6722	0.3677	0.2812	0.2437	0.2240	0.2206	(0.10)
R_lh	0.6822	0.1167	0.0403	0.0221	0.0166	0.0147	(0.01)
	0.7999	0.2576	0.1175	0.0797	0.0604	0.0557	(0.05)
	0.8426	0.3466	0.1865	0.1281	0.1047	0.0982	(0.10)
R_gamma2	0.6822	0.1166	0.0398	0.0212	0.0155	0.0134	(0.01)
	0.7999	0.2576	0.1172	0.0790	0.0595	0.0546	(0.05)
	0.8426	0.3466	0.1862	0.1275	0.1039	0.0972	(0.10)
<i>Beta</i>							Nominal
(a, b)	(0.5, 1)	(1, 1)	(2, 1)	(1, 0.5)	(2, 2)	(1, 2)	level
robust	0.0002	0.0000	0.0002	0.0003	0.0001	0.0001	(0.01)
	0.0033	0.0005	0.0043	0.0037	0.0017	0.0047	(0.05)
	0.0135	0.0044	0.0195	0.0132	0.0098	0.0188	(0.10)
Z6	0.0011	0.0001	0.0018	0.0012	0.0006	0.0017	(0.01)
	0.0133	0.0059	0.0210	0.0132	0.0128	0.0208	(0.05)
	0.0397	0.0232	0.0556	0.0400	0.0399	0.0543	(0.10)
R_lh	0.0001	0.0003	0.0033	0.0033	0.0009	0.0012	(0.01)
	0.0005	0.0024	0.0158	0.0143	0.0067	0.0054	(0.05)
	0.0012	0.0074	0.0334	0.0286	0.0158	0.0115	(0.10)
R_gamma2	0.0000	0.0000	0.0001	0.0001	0.0000	0.0000	(0.01)
	0.0002	0.0001	0.0012	0.0005	0.0005	0.0006	(0.05)
	0.0007	0.0002	0.0038	0.0014	0.0017	0.0019	(0.10)

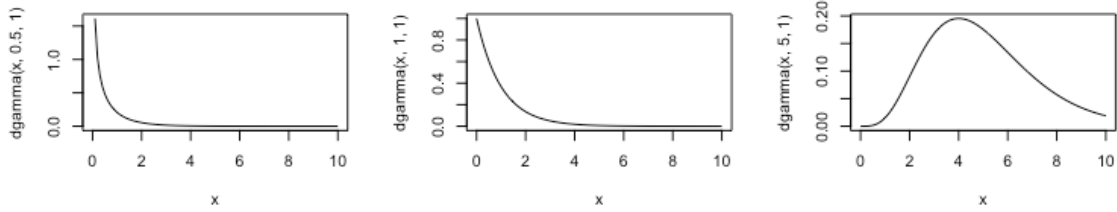
(Continued) Type-I Error rates ($n = 30$)

<i>Inv-gamma</i> (<i>shape, scale</i>)	(2.5, 1)	(5, 1)	(7.5, 1)	(10, 1)	(20, 1)	(50, 1)	Nominal level
robust	0.3209	0.0423	0.0146	0.0068	0.0018	0.0006	(0.01)
	0.4992	0.1479	0.0776	0.0523	0.0228	0.0120	(0.05)
	0.5911	0.2388	0.1535	0.1181	0.0666	0.0409	(0.10)
Z6	0.4868	0.1067	0.0531	0.0325	0.0126	0.0062	(0.01)
	0.6080	0.2423	0.1621	0.1245	0.0742	0.0476	(0.05)
	0.6641	0.3245	0.2441	0.1992	0.1382	0.1020	(0.10)
R_lh	0.7269	0.0809	0.0254	0.0168	0.0083	0.0047	(0.01)
	0.8397	0.2317	0.1015	0.0649	0.0346	0.0220	(0.05)
	0.8751	0.3381	0.1816	0.1215	0.0668	0.0454	(0.10)
R_gamma2	0.7269	0.0778	0.0182	0.0079	0.0016	0.0006	(0.01)
	0.8397	0.2304	0.0945	0.0543	0.0196	0.0090	(0.05)
	0.8751	0.3375	0.1767	0.1110	0.0487	0.0260	(0.10)

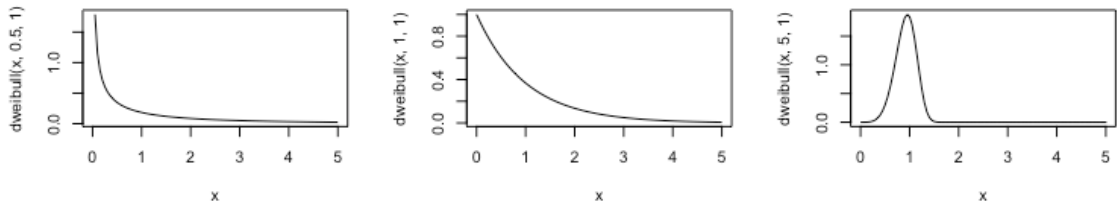
Table 19. Type-I Error rates from 10^5 simulations of samples of size $n = 30$ and significance levels $\alpha = 0.01, 0.05$, and 0.10 . Entries with bold font are more than 20% above the nominal level.

APPENDIX D: A SURVEY OF FAILED CASES

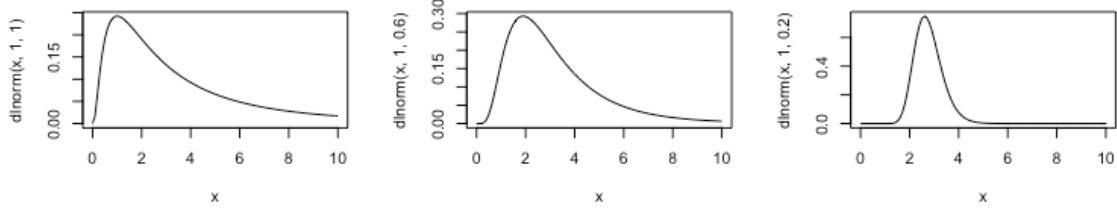
Gamma(0.5, 1); (1, 1); (5, 1) – Kurtosis: 12; 6; 1.2 - Skewness: 2.8; 2.0; 0.9



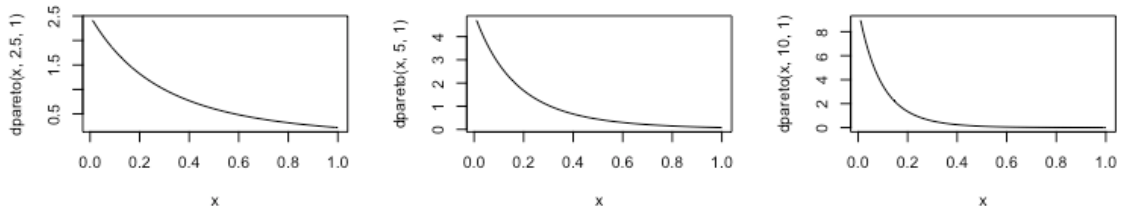
Weibull(0.5, 1); (1, 1); (5, 1) – Kurtosis: 91.4; 6.2; -0.1 – Skewness: 6.7; 2.0; -0.3



Log-normal(1, 1); (1, 0.6); (1, 0.2) – Kurtosis: 94.8; 6.3; -2.3 – Skewness: 2.3; 0.3; 0.0



Pareto(2.5, 1); (5, 1); (10, 1) – Kurtosis: NA; 70.8; 14.8 – Skewness: NA, 4.6, 2.8



Inv-gamma(2.5, 1); (5, 1); (7.5, 1) – Kurtosis: NA; 42.0; 10.1 – Skewness: -5.7, 3.5, 2.1

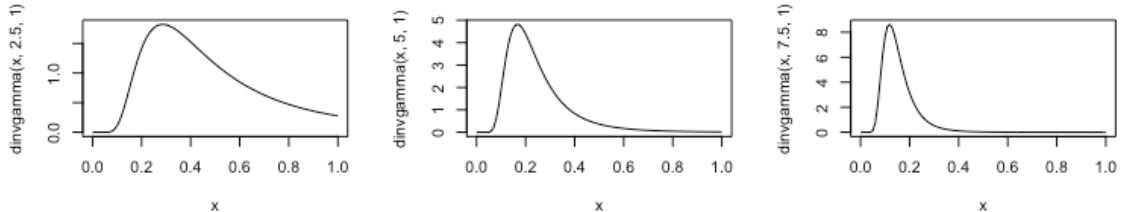


Figure 20. Each row shows the density of a particular distribution with three different parameter values. The excess kurtosis and skewness for each distribution is given. The distributions on the left cause all of the test statistics to have greatly inflated type-I error rates, while the distributions towards the right result in type-I error rate closer to the nominal level.

APPENDIX E: EXPANDED POWER COMPARISONS

Power with $\delta = 4$ ($n = 10$)							
<i>Normal</i> (μ, σ)	(0, 0.01)	(0, 0.5)	(0, 1)	(0, 5)	(0, 10)	(0, 100)	Alpha level
robust	0.0983	0.0983	0.1004	0.0983	0.0977	0.0993	(0.01)
	0.5056	0.5062	0.5067	0.5083	0.5052	0.5076	(0.05)
	0.7784	0.7772	0.7780	0.7771	0.7776	0.7746	(0.10)
Z6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	(0.01)
	0.2406	0.2404	0.2431	0.2382	0.2417	0.2404	(0.05)
	0.7406	0.7390	0.7430	0.7395	0.7413	0.7435	(0.10)
R_lh	0.2780	0.2659	0.2679	0.2754	0.2770	0.2803	(0.01)
	0.5960	0.5929	0.5915	0.5906	0.5953	0.5930	(0.05)
	0.7879	0.7857	0.7871	0.7875	0.7868	0.7877	(0.10)
R_gamma2	0.0368	0.0367	0.0363	0.0362	0.0367	0.0363	(0.01)
	0.3769	0.3793	0.3796	0.3762	0.3801	0.3761	(0.05)
	0.6538	0.6538	0.6558	0.6559	0.6573	0.6547	(0.10)
<i>t</i> (df)	(4)	(6)	(7)	(12)	(20)	(30)	Alpha level
robust	0.2079	0.1531	0.1398	0.1207	0.1083	0.1070	(0.01)
	0.5852	0.5447	0.5365	0.5165	0.5128	0.5085	(0.05)
	0.7610	0.7537	0.7589	0.7606	0.7650	0.7667	(0.10)
Z6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	(0.01)
	0.4148	0.3343	0.3136	0.2788	0.2610	0.2557	(0.05)
	0.7658	0.7467	0.7427	0.7368	0.7391	0.7389	(0.10)
R_lh	0.5831	0.5180	0.5033	0.4734	0.4598	0.4498	(0.01)
	0.7909	0.7752	0.7708	0.7659	0.7683	0.7656	(0.05)
	0.8621	0.8612	0.8627	0.8705	0.8746	0.8772	(0.10)
R_gamma2	0.1156	0.0725	0.0643	0.0510	0.0446	0.0413	(0.01)
	0.5369	0.4717	0.4507	0.4173	0.4052	0.3933	(0.05)
	0.7312	0.6954	0.6886	0.6747	0.6677	0.6646	(0.10)

(Continued) Power with $\delta = 4$ ($n = 10$)

<i>Chi-squared</i> (<i>df</i>)	(1)	(2)	(4)	(6)	(7)	(10)	Alpha level
robust	0.2973	0.2458	0.1946	0.1643	0.1560	0.1392	(0.01)
	0.5672	0.5571	0.5503	0.5478	0.5458	0.5398	(0.05)
	0.7055	0.7190	0.7344	0.7456	0.7484	0.7573	(0.10)
Z6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	(0.01)
	0.4200	0.3634	0.3262	0.3077	0.2974	0.2863	(0.05)
	0.7111	0.7114	0.7195	0.7244	0.7317	0.7338	(0.10)
R_lh	0.2044	0.1613	0.1872	0.2054	0.2120	0.2224	(0.01)
	0.4915	0.4704	0.4948	0.5119	0.5194	0.5333	(0.05)
	0.6210	0.6417	0.6780	0.7026	0.7106	0.7252	(0.10)
R_gamma2	0.2006	0.1254	0.0798	0.0638	0.0604	0.0520	(0.01)
	0.4899	0.4537	0.4348	0.4222	0.4175	0.4100	(0.05)
	0.6202	0.6332	0.6450	0.6524	0.6551	0.6576	(0.10)
<i>Exponential</i> (λ)	(0.1)	(0.5)	(1)	(5)	(20)	(50)	Alpha level
robust	0.2475	0.2442	0.2439	0.2438	0.2458	0.2452	(0.01)
	0.5540	0.5587	0.5552	0.5558	0.5545	0.5561	(0.05)
	0.7161	0.7202	0.7160	0.7192	0.7190	0.7170	(0.10)
Z6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	(0.01)
	0.3632	0.3666	0.3627	0.3629	0.3637	0.3634	(0.05)
	0.7118	0.7101	0.7118	0.7080	0.7125	0.7079	(0.10)
R_lh	0.1637	0.1631	0.1623	0.1656	0.1638	0.1626	(0.01)
	0.4739	0.4739	0.4736	0.4704	0.4752	0.4770	(0.05)
	0.6437	0.6428	0.6429	0.6428	0.6425	0.6408	(0.10)
R_gamma2	0.1270	0.1264	0.1254	0.1283	0.1274	0.1267	(0.01)
	0.4574	0.4576	0.4575	0.4541	0.4586	0.4604	(0.05)
	0.6347	0.6336	0.6341	0.6335	0.6336	0.6320	(0.10)
<i>Gamma</i> (<i>shape, rate</i>)	(0.5, 10)	(1, 10)	(5, 10)	(10, 10)	(50, 10)	(100, 10)	Alpha level
robust	0.2945	0.2466	0.1365	0.1184	0.1055	0.0992	(0.01)
	0.5641	0.5529	0.5391	0.5258	0.5144	0.5077	(0.05)
	0.7042	0.7207	0.7588	0.7685	0.7718	0.7771	(0.10)
Z6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	(0.01)
	0.4211	0.3633	0.2866	0.2659	0.2480	0.2447	(0.05)
	0.7101	0.7136	0.7323	0.7376	0.7407	0.7424	(0.10)
R_lh	0.2038	0.1609	0.2227	0.2302	0.2350	0.2532	(0.01)
	0.4893	0.4759	0.5294	0.5459	0.5620	0.5661	(0.05)
	0.6199	0.6399	0.7262	0.7475	0.7642	0.7694	(0.10)
R_gamma2	0.1999	0.1245	0.0538	0.0434	0.0382	0.0371	(0.01)
	0.4878	0.4594	0.4072	0.3967	0.3847	0.3836	(0.05)
	0.6190	0.6312	0.6561	0.6632	0.6626	0.6628	(0.10)

(Continued) Power with $\delta = 4$ ($n = 10$)

<i>Weibull</i>								Alpha
(shape, scale)	(0.5, 1)	(1, 1)	(5, 1)	(20, 1)	(50, 1)	(100, 1)		level
robust	0.4562	0.2475	0.0955	0.1393	0.1500	0.1572		(0.01)
	0.6627	0.5538	0.5062	0.5367	0.5433	0.5494		(0.05)
	0.7546	0.7167	0.7817	0.7540	0.7521	0.7509		(0.10)
Z6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000		(0.01)
	0.6186	0.3628	0.2358	0.2943	0.3139	0.3190		(0.05)
	0.7859	0.7122	0.7408	0.7290	0.7330	0.7354		(0.10)
R_lh	0.4907	0.1612	0.2726	0.4079	0.4812	0.5135		(0.01)
	0.7141	0.4746	0.6390	0.7287	0.7518	0.7656		(0.05)
	0.7902	0.6420	0.8236	0.8454	0.8514	0.8514		(0.10)
R_gamma2	0.4905	0.1246	0.0331	0.0600	0.0703	0.0737		(0.01)
	0.7140	0.4580	0.3418	0.4343	0.4569	0.4643		(0.05)
	0.7902	0.6333	0.5985	0.6637	0.6817	0.6829		(0.10)
<i>Log-normal</i>								Alpha
(μ, τ)	(1, 0.01)	(1, 0.2)	(1, 0.4)	(1, 0.6)	(1, 0.8)	(1, 1)		level
robust	0.1000	0.1200	0.1795	0.2666	0.3504	0.4282		(0.01)
	0.5086	0.5258	0.5614	0.5911	0.6270	0.6749		(0.05)
	0.7762	0.7697	0.7484	0.7427	0.7561	0.7772		(0.10)
Z6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000		(0.01)
	0.2424	0.2728	0.3388	0.4134	0.4940	0.5794		(0.05)
	0.7402	0.7384	0.7336	0.7381	0.7599	0.7892		(0.10)
R_lh	0.4323	0.2275	0.2234	0.2407	0.3300	0.4656		(0.01)
	0.7482	0.5508	0.5576	0.6136	0.6921	0.7651		(0.05)
	0.8662	0.7482	0.7461	0.7716	0.8111	0.8477		(0.10)
R_gamma2	0.0381	0.0470	0.0874	0.1737	0.3089	0.4611		(0.01)
	0.3867	0.4184	0.4965	0.5966	0.6890	0.7646		(0.05)
	0.6611	0.6810	0.7197	0.7652	0.8099	0.8476		(0.10)
<i>Pareto</i>								Alpha
(shape, scale)	(2.5, 1)	(5, 1)	(10, 1)	(20, 1)	(50, 1)	(100, 1)		level
robust	0.6100	0.3720	0.3030	0.2720	0.2549	0.2506		(0.01)
	0.7959	0.6395	0.5941	0.5718	0.5623	0.5586		(0.05)
	0.8597	0.7604	0.7327	0.7244	0.7209	0.7166		(0.10)
Z6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000		(0.01)
	0.7652	0.5148	0.4308	0.3960	0.3747	0.3710		(0.05)
	0.8842	0.7669	0.7366	0.7246	0.7152	0.7113		(0.10)
R_lh	0.6963	0.3106	0.2174	0.1866	0.1705	0.1656		(0.01)
	0.8801	0.6483	0.5526	0.5132	0.4879	0.4790		(0.05)
	0.9208	0.7737	0.7061	0.6722	0.6542	0.6478		(0.10)
R_gamma2	0.6960	0.2997	0.1939	0.1571	0.1369	0.1302		(0.01)
	0.8801	0.6461	0.5454	0.5018	0.4736	0.4641		(0.05)
	0.9208	0.7729	0.7025	0.6671	0.6464	0.6398		(0.10)

(Continued) Power with $\delta = 4$ ($n = 10$)

<i>Beta</i> (<i>a, b</i>)	(0.5, 1)	(1, 1)	(2, 1)	(1, 0.5)	(2, 2)	(1, 2)	Alpha level
robust	0.0772	0.0479	0.0881	0.0752	0.0631	0.0876	(0.01)
	0.4647	0.5348	0.4927	0.4627	0.5062	0.4944	(0.05)
	0.8558	0.9185	0.8098	0.8569	0.8518	0.8078	(0.10)
Z6	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	(0.01)
	0.1346	0.1272	0.1798	0.1331	0.1726	0.1790	(0.05)
	0.7190	0.7732	0.7263	0.7187	0.7622	0.7265	(0.10)
R_lh	0.0563	0.2060	0.3204	0.3238	0.2280	0.1335	(0.01)
	0.2002	0.5326	0.7080	0.7220	0.5892	0.3564	(0.05)
	0.3311	0.6830	0.8682	0.8654	0.7832	0.5360	(0.10)
R_gamma2	0.0160	0.0078	0.0258	0.0266	0.0151	0.0181	(0.01)
	0.1319	0.1408	0.2449	0.1669	0.2270	0.2049	(0.05)
	0.2669	0.3195	0.4470	0.3070	0.4687	0.4047	(0.10)
<i>Inv-gamma</i> (<i>shape, scale</i>)	(2.5, 1)	(5, 1)	(7.5, 1)	(10, 1)	(20, 1)	(50, 1)	Alpha level
robust	0.6035	0.3255	0.2470	0.2075	0.1509	0.1196	(0.01)
	0.8009	0.6309	0.5940	0.5761	0.5504	0.5260	(0.05)
	0.8625	0.7629	0.7504	0.7508	0.7594	0.7672	(0.10)
Z6	0.0001	0.0000	0.0000	0.0000	0.0000	0.0000	(0.01)
	0.7607	0.4831	0.4071	0.3697	0.3088	0.2706	(0.05)
	0.8817	0.7684	0.7462	0.7433	0.7358	0.7397	(0.10)
R_lh	0.7308	0.3003	0.2409	0.2286	0.2221	0.2286	(0.01)
	0.9058	0.6903	0.6158	0.5839	0.5502	0.5460	(0.05)
	0.9391	0.8214	0.7812	0.7632	0.7452	0.7439	(0.10)
R_gamma2	0.7304	0.2557	0.1513	0.1110	0.0646	0.0468	(0.01)
	0.9058	0.6837	0.5936	0.5454	0.4686	0.4199	(0.05)
	0.9391	0.8195	0.7745	0.7504	0.7111	0.6817	(0.10)

Table 20. Power study for the left-tailed hypothesis test of $\sigma^2 = \delta\sigma_0^2$, with $\delta = 4$, from 10^5 simulations of samples of size $n = 10$ and significance levels $\alpha = 0.01, 0.05$, and 0.10 .

Power with $\delta = 4$ ($n = 20$)

<i>Normal</i>							Alpha level
(μ, σ)	(0, 0.01)	(0, 0.5)	(0, 1)	(0, 5)	(0, 10)	(0, 100)	
robust	0.4774	0.4819	0.4805	0.4802	0.4812	0.4778	(0.01)
	0.9202	0.9197	0.9214	0.9194	0.9205	0.9207	(0.05)
	0.9867	0.9874	0.9874	0.9876	0.9878	0.9867	(0.10)
Z6	0.6809	0.6829	0.6815	0.6806	0.6788	0.6814	(0.01)
	0.9843	0.9836	0.9842	0.9841	0.9833	0.9835	(0.05)
	0.9981	0.9979	0.9981	0.9982	0.9977	0.9978	(0.10)
R_lh	0.7433	0.7432	0.7436	0.7432	0.7450	0.7450	(0.01)
	0.9600	0.9600	0.9613	0.9603	0.9598	0.9603	(0.05)
	0.9916	0.9913	0.9912	0.9909	0.9916	0.9911	(0.10)
R_gamma2	0.4840	0.4878	0.4880	0.4837	0.4854	0.4841	(0.01)
	0.9030	0.9028	0.9035	0.9028	0.9032	0.9038	(0.05)
	0.9751	0.9750	0.9749	0.9744	0.9758	0.9749	(0.10)
<i>t</i>							Alpha level
(df)	(4)	(6)	(7)	(12)	(20)	(30)	
robust	0.5023	0.4756	0.4736	0.4712	0.4693	0.4701	(0.01)
	0.7946	0.8214	0.8340	0.8684	0.8886	0.8996	(0.05)
	0.8851	0.9200	0.9318	0.9590	0.9725	0.9786	(0.10)
Z6	0.6984	0.6785	0.6765	0.6783	0.6778	0.6798	(0.01)
	0.8858	0.9191	0.9283	0.9555	0.9698	0.9748	(0.05)
	0.9299	0.9592	0.9668	0.9852	0.9923	0.9949	(0.10)
R_lh	0.8470	0.8711	0.8772	0.8978	0.9074	0.9111	(0.01)
	0.9214	0.9500	0.9562	0.9760	0.9830	0.9859	(0.05)
	0.9454	0.9684	0.9760	0.9891	0.9942	0.9960	(0.10)
R_gamma2	0.6206	0.5620	0.5484	0.5238	0.5088	0.5029	(0.01)
	0.8625	0.8713	0.8751	0.8911	0.9002	0.9029	(0.05)
	0.9191	0.9378	0.9451	0.9614	0.9699	0.9741	(0.10)
<i>Chi-squared</i>							Alpha level
(df)	(1)	(2)	(4)	(6)	(7)	(10)	
robust	0.4113	0.4369	0.4677	0.4787	0.4801	0.4840	(0.01)
	0.6716	0.7264	0.7824	0.8101	0.8214	0.8428	(0.05)
	0.7928	0.8456	0.8934	0.9182	0.9228	0.9400	(0.10)
Z6	0.5273	0.5379	0.5738	0.5942	0.6040	0.6228	(0.01)
	0.7870	0.8356	0.8846	0.9060	0.9167	0.9339	(0.05)
	0.8606	0.9066	0.9437	0.9595	0.9650	0.9745	(0.10)
R_lh	0.5290	0.5174	0.5346	0.5542	0.5667	0.5841	(0.01)
	0.7543	0.8009	0.8469	0.8717	0.8795	0.8969	(0.05)
	0.8372	0.8881	0.9317	0.9479	0.9538	0.9632	(0.10)
R_gamma2	0.5289	0.5157	0.5129	0.5113	0.5123	0.5065	(0.01)
	0.7543	0.8004	0.8416	0.8621	0.8674	0.8810	(0.05)
	0.8372	0.8880	0.9301	0.9446	0.9499	0.9582	(0.10)

(Continued) Power with $\delta = 4$ ($n = 20$)

<i>Exponential</i>							Alpha
(λ)	(0.1)	(0.5)	(1)	(5)	(20)	(50)	level
robust	0.4385	0.4377	0.4388	0.4372	0.4367	0.4387	(0.01)
	0.7250	0.7274	0.7244	0.7230	0.7238	0.7256	(0.05)
	0.8455	0.8445	0.8424	0.8462	0.8435	0.8449	(0.10)
Z6	0.5377	0.5358	0.5396	0.5367	0.5347	0.5391	(0.01)
	0.8358	0.8358	0.8340	0.8329	0.8329	0.8348	(0.05)
	0.9066	0.9068	0.9064	0.9071	0.9061	0.9075	(0.10)
R_lh	0.5205	0.5216	0.5185	0.5189	0.5192	0.5212	(0.01)
	0.7989	0.8006	0.8018	0.7995	0.7994	0.8005	(0.05)
	0.8883	0.8890	0.8897	0.8903	0.8869	0.8896	(0.10)
R_gamma2	0.5186	0.5200	0.5165	0.5171	0.5177	0.5191	(0.01)
	0.7985	0.8003	0.8014	0.7989	0.7990	0.8000	(0.05)
	0.8881	0.8887	0.8895	0.8901	0.8868	0.8894	(0.10)
<i>Gamma</i>							Alpha
(shape, rate)	(0.5, 10)	(1, 10)	(5, 10)	(10, 10)	(50, 10)	(100, 10)	level
robust	0.4097	0.4370	0.4854	0.4874	0.4795	0.4848	(0.01)
	0.6717	0.7235	0.8436	0.8762	0.9100	0.9163	(0.05)
	0.7924	0.8446	0.9399	0.9603	0.9814	0.9843	(0.10)
Z6	0.5300	0.5400	0.6251	0.6483	0.6714	0.6779	(0.01)
	0.7861	0.8358	0.9319	0.9543	0.9775	0.9808	(0.05)
	0.8633	0.9071	0.9744	0.9866	0.9958	0.9970	(0.10)
R_lh	0.5263	0.5180	0.5829	0.6193	0.6717	0.6844	(0.01)
	0.7539	0.8031	0.8961	0.9170	0.9411	0.9457	(0.05)
	0.8374	0.8888	0.9642	0.9753	0.9847	0.9862	(0.10)
R_gamma2	0.5263	0.5162	0.5050	0.5006	0.4980	0.4978	(0.01)
	0.7539	0.8026	0.8812	0.8943	0.9105	0.9134	(0.05)
	0.8374	0.8887	0.9594	0.9692	0.9766	0.9784	(0.10)
<i>Weibull</i>							Alpha
(shape, scale)	(0.5, 1)	(1, 1)	(5, 1)	(20, 1)	(50, 1)	(100, 1)	level
robust	0.4802	0.4405	0.4804	0.4811	0.4821	0.4831	(0.01)
	0.6862	0.7248	0.9314	0.8387	0.8196	0.8133	(0.05)
	0.7807	0.8454	0.9921	0.9367	0.9203	0.9167	(0.10)
Z6	0.6509	0.5408	0.6680	0.6282	0.6214	0.6198	(0.01)
	0.7891	0.8366	0.9887	0.9307	0.9129	0.9064	(0.05)
	0.8371	0.9058	0.9991	0.9732	0.9621	0.9580	(0.10)
R_lh	0.7729	0.5210	0.8393	0.8714	0.8699	0.8695	(0.01)
	0.8744	0.8008	0.9808	0.9686	0.9598	0.9574	(0.05)
	0.9096	0.8878	0.9962	0.9849	0.9793	0.9766	(0.10)
R_gamma2	0.7729	0.5191	0.3960	0.5062	0.5386	0.5446	(0.01)
	0.8744	0.8003	0.8123	0.8495	0.8562	0.8582	(0.05)
	0.9096	0.8876	0.9242	0.9300	0.9320	0.9342	(0.10)

(Continued) Power with $\delta = 4$ ($n = 20$)

<i>Log-normal</i>							Alpha
<i>(μ, τ)</i>	(1, 0.01)	(1, 0.2)	(1, 0.4)	(1, 0.6)	(1, 0.8)	(1, 1)	level
robust	0.4794	0.4884	0.4856	0.4807	0.4922	0.5214	(0.01)
	0.9206	0.8731	0.7979	0.7412	0.7184	0.7208	(0.05)
	0.9874	0.9583	0.8977	0.8454	0.8146	0.8079	(0.10)
Z6	0.6802	0.6517	0.6116	0.5960	0.6151	0.6609	(0.01)
	0.9840	0.9528	0.8891	0.8400	0.8148	0.8138	(0.05)
	0.9982	0.9849	0.9436	0.8990	0.8714	0.8609	(0.10)
R_lh	0.7750	0.6242	0.6354	0.7151	0.7898	0.8463	(0.01)
	0.9619	0.9229	0.9133	0.9156	0.9251	0.9354	(0.05)
	0.9907	0.9774	0.9666	0.9588	0.9565	0.9575	(0.10)
R_gamma2	0.4989	0.5351	0.6178	0.7136	0.7897	0.8463	(0.01)
	0.9126	0.9112	0.9117	0.9155	0.9251	0.9354	(0.05)
	0.9799	0.9746	0.9662	0.9587	0.9565	0.9575	(0.10)
<i>Pareto</i>							Alpha
<i>(shape, scale)</i>	(2.5, 1)	(5, 1)	(10, 1)	(20, 1)	(50, 1)	(100, 1)	level
robust	0.6642	0.4958	0.4594	0.4452	0.4415	0.4393	(0.01)
	0.8184	0.7157	0.7109	0.7174	0.7219	0.7248	(0.05)
	0.8839	0.8104	0.8191	0.8288	0.8382	0.8408	(0.10)
Z6	0.8117	0.6250	0.5757	0.5562	0.5437	0.5425	(0.01)
	0.8868	0.8150	0.8139	0.8223	0.8311	0.8329	(0.05)
	0.9122	0.8689	0.8779	0.8914	0.8977	0.9035	(0.10)
R_lh	0.9311	0.7361	0.6282	0.5709	0.5406	0.5280	(0.01)
	0.9655	0.8886	0.8452	0.8247	0.8118	0.8065	(0.05)
	0.9746	0.9307	0.9092	0.8991	0.8942	0.8909	(0.10)
R_gamma2	0.9311	0.7361	0.6278	0.5700	0.5392	0.5262	(0.01)
	0.9655	0.8886	0.8452	0.8246	0.8115	0.8061	(0.05)
	0.9746	0.9307	0.9092	0.8990	0.8941	0.8908	(0.10)
<i>Beta</i>							Alpha
<i>(a, b)</i>	(0.5, 1)	(1, 1)	(2, 1)	(1, 0.5)	(2, 2)	(1, 2)	level
robust	0.5434	0.7049	0.5075	0.5454	0.5804	0.5087	(0.01)
	0.9984	0.9997	0.9842	0.9985	0.9942	0.9837	(0.05)
	1.0000	1.0000	0.9998	1.0000	0.9999	0.9998	(0.10)
Z6	0.4111	0.5823	0.5599	0.4131	0.6585	0.5579	(0.01)
	0.9996	1.0000	0.9979	0.9994	0.9996	0.9978	(0.05)
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(0.10)
R_lh	0.1110	0.6438	0.9318	0.9441	0.7672	0.3132	(0.01)
	0.3483	0.7946	0.9926	0.9737	0.9394	0.6532	(0.05)
	0.5127	0.8767	0.9974	0.9809	0.9792	0.8067	(0.10)
R_gamma2	0.0827	0.0939	0.2133	0.1006	0.2245	0.1827	(0.01)
	0.3253	0.4171	0.5865	0.3338	0.6652	0.5733	(0.05)
	0.4948	0.6260	0.7578	0.4979	0.8423	0.7585	(0.10)

(Continued) Power with $\delta = 4$ ($n = 20$)

<i>Inv-gamma</i> (<i>shape, scale</i>)	(2.5, 1)	(5, 1)	(7.5, 1)	(10, 1)	(20, 1)	(50, 1)	Alpha level
robust	0.6681	0.5165	0.5000	0.4911	0.4923	0.4870	(0.01)
	0.8189	0.7452	0.7616	0.7785	0.8268	0.8760	(0.05)
	0.8810	0.8364	0.8596	0.8795	0.9237	0.9588	(0.10)
Z6	0.8081	0.6372	0.6183	0.6172	0.6380	0.6575	(0.01)
	0.8888	0.8348	0.8553	0.8728	0.9163	0.9531	(0.05)
	0.9135	0.8889	0.9102	0.9284	0.9617	0.9846	(0.10)
R_lh	0.9483	0.7905	0.7151	0.6746	0.6208	0.6183	(0.01)
	0.9757	0.9321	0.9223	0.9170	0.9139	0.9220	(0.05)
	0.9824	0.9632	0.9628	0.9640	0.9708	0.9771	(0.10)
R_gamma2	0.9483	0.7904	0.7133	0.6691	0.5903	0.5372	(0.01)
	0.9757	0.9321	0.9222	0.9167	0.9112	0.9122	(0.05)
	0.9824	0.9632	0.9628	0.9640	0.9703	0.9751	(0.10)

Table 21. Power study for the left-tailed hypothesis test of $\sigma^2 = \delta\sigma_0^2$, with $\delta = 4$, from 10^5 simulations of samples of size $n = 20$ and significance levels $\alpha = 0.01, 0.05$, and 0.10 .

Power with $\delta = 4$ ($n = 30$)

<i>Normal</i>							Alpha
(μ, σ)	(0, 0.01)	(0, 0.5)	(0, 1)	(0, 5)	(0, 10)	(0, 100)	level
robust	0.8426	0.8441	0.8436	0.8445	0.8423	0.8454	(0.01)
	0.9943	0.9942	0.9945	0.9948	0.9947	0.9945	(0.05)
	0.9997	0.9997	0.9997	0.9997	0.9997	0.9997	(0.10)
Z6	0.9584	0.9580	0.9589	0.9592	0.9589	0.9592	(0.01)
	0.9986	0.9984	0.9987	0.9986	0.9985	0.9986	(0.05)
	0.9999	0.9999	0.9998	0.9998	0.9998	0.9999	(0.10)
R_lh	0.9575	0.9576	0.9585	0.9588	0.9593	0.9590	(0.01)
	0.9981	0.9985	0.9982	0.9984	0.9984	0.9982	(0.05)
	0.9998	0.9998	0.9999	0.9998	0.9998	0.9999	(0.10)
R_gamma2	0.8823	0.8824	0.8831	0.8839	0.8830	0.8837	(0.01)
	0.9933	0.9932	0.9934	0.9931	0.9933	0.9933	(0.05)
	0.9990	0.9992	0.9991	0.9991	0.9991	0.9992	(0.10)
<i>t</i>							Alpha
(df)	(4)	(6)	(7)	(12)	(20)	(30)	level
robust	0.6702	0.7078	0.7200	0.7669	0.7979	0.8115	(0.01)
	0.8697	0.9181	0.9344	0.9686	0.9814	0.9879	(0.05)
	0.9274	0.9638	0.9733	0.9921	0.9969	0.9985	(0.10)
Z6	0.8367	0.8747	0.8878	0.9198	0.9390	0.9484	(0.01)
	0.9128	0.9525	0.9632	0.9860	0.9934	0.9956	(0.05)
	0.9359	0.9705	0.9790	0.9940	0.9980	0.9990	(0.10)
R_lh	0.9056	0.9361	0.9458	0.9709	0.9819	0.9871	(0.01)
	0.9571	0.9798	0.9856	0.9950	0.9979	0.9986	(0.05)
	0.9712	0.9888	0.9928	0.9982	0.9995	0.9998	(0.10)
R_gamma2	0.8399	0.8479	0.8538	0.8687	0.8823	0.8904	(0.01)
	0.9404	0.9646	0.9714	0.9857	0.9908	0.9927	(0.05)
	0.9626	0.9830	0.9881	0.9960	0.9982	0.9988	(0.10)
<i>Chi-squared</i>							Alpha
(df)	(1)	(2)	(4)	(6)	(7)	(10)	level
robust	0.5015	0.5768	0.6503	0.6932	0.7065	0.7366	(0.01)
	0.7481	0.8240	0.8891	0.9178	0.9288	0.9462	(0.05)
	0.8506	0.9106	0.9531	0.9704	0.9749	0.9839	(0.10)
Z6	0.6645	0.7214	0.7927	0.8271	0.8430	0.8670	(0.01)
	0.8117	0.8750	0.9270	0.9499	0.9565	0.9699	(0.05)
	0.8650	0.9192	0.9604	0.9742	0.9791	0.9871	(0.10)
R_lh	0.7332	0.7826	0.8278	0.8504	0.8588	0.8751	(0.01)
	0.8831	0.9350	0.9664	0.9777	0.9801	0.9869	(0.05)
	0.9289	0.9688	0.9890	0.9940	0.9949	0.9975	(0.10)
R_gamma2	0.7332	0.7825	0.8255	0.8439	0.8502	0.8618	(0.01)
	0.8831	0.9350	0.9662	0.9771	0.9796	0.9859	(0.05)
	0.9289	0.9688	0.9889	0.9939	0.9947	0.9972	(0.10)

(Continued) Power with $\delta = 4$ ($n = 30$)

<i>Exponential</i>							Alpha
(λ)	(0.1)	(0.5)	(1)	(5)	(20)	(50)	level
robust	0.5738	0.5770	0.5762	0.5751	0.5749	0.5754	(0.01)
	0.8254	0.8242	0.8224	0.8233	0.8243	0.8232	(0.05)
	0.9096	0.9098	0.9096	0.9117	0.9114	0.9106	(0.10)
Z6	0.7224	0.7201	0.7249	0.7220	0.7244	0.7227	(0.01)
	0.8743	0.8732	0.8756	0.8740	0.8744	0.8749	(0.05)
	0.9194	0.9188	0.9188	0.9199	0.9183	0.9190	(0.10)
R_lh	0.7805	0.7809	0.7835	0.7814	0.7822	0.7824	(0.01)
	0.9348	0.9343	0.9356	0.9348	0.9358	0.9344	(0.05)
	0.9687	0.9689	0.9699	0.9692	0.9690	0.9692	(0.10)
R_gamma2	0.7804	0.7809	0.7834	0.7814	0.7821	0.7823	(0.01)
	0.9348	0.9342	0.9356	0.9348	0.9358	0.9344	(0.05)
	0.9687	0.9689	0.9699	0.9691	0.9690	0.9692	(0.10)
<i>Gamma</i>							Alpha
(shape, rate)	(0.5, 10)	(1, 10)	(5, 10)	(10, 10)	(50, 10)	(100, 10)	level
robust	0.5026	0.5719	0.7354	0.7809	0.8293	0.8389	(0.01)
	0.7467	0.8227	0.9448	0.9691	0.9892	0.9917	(0.05)
	0.8497	0.9092	0.9837	0.9927	0.9986	0.9992	(0.10)
Z6	0.6657	0.7234	0.8708	0.9061	0.9454	0.9524	(0.01)
	0.8142	0.8751	0.9708	0.9856	0.9964	0.9974	(0.05)
	0.8646	0.9194	0.9866	0.9947	0.9991	0.9996	(0.10)
R_lh	0.7345	0.7822	0.8775	0.9029	0.9292	0.9357	(0.01)
	0.8854	0.9354	0.9867	0.9920	0.9966	0.9970	(0.05)
	0.9288	0.9682	0.9971	0.9988	0.9996	0.9997	(0.10)
R_gamma2	0.7345	0.7822	0.8644	0.8809	0.8929	0.8952	(0.01)
	0.8854	0.9354	0.9857	0.9905	0.9949	0.9951	(0.05)
	0.9288	0.9682	0.9969	0.9986	0.9994	0.9995	(0.10)
<i>Weibull</i>							Alpha
(shape, scale)	(0.5, 1)	(1, 1)	(5, 1)	(20, 1)	(50, 1)	(100, 1)	level
robust	0.5074	0.5782	0.8629	0.7270	0.7072	0.6956	(0.01)
	0.6991	0.8250	0.9977	0.9420	0.9227	0.9176	(0.05)
	0.7892	0.9100	1.0000	0.9838	0.9720	0.9678	(0.10)
Z6	0.6911	0.7206	0.9584	0.8666	0.8436	0.8355	(0.01)
	0.7747	0.8734	0.9995	0.9694	0.9537	0.9491	(0.05)
	0.8122	0.9183	1.0000	0.9862	0.9767	0.9724	(0.10)
R_lh	0.8856	0.7799	0.9871	0.9799	0.9718	0.9695	(0.01)
	0.9398	0.9347	0.9996	0.9967	0.9930	0.9913	(0.05)
	0.9572	0.9696	1.0000	0.9986	0.9966	0.9959	(0.10)
R_gamma2	0.8856	0.7798	0.7618	0.8194	0.8297	0.8340	(0.01)
	0.9398	0.9346	0.9590	0.9632	0.9630	0.9642	(0.05)
	0.9572	0.9696	0.9881	0.9851	0.9849	0.9848	(0.10)

(Continued) Power with $\delta = 4$ ($n = 30$)

<i>Log-normal</i>							Alpha
<i>(μ, τ)</i>	(1, 0.01)	(1, 0.2)	(1, 0.4)	(1, 0.6)	(1, 0.8)	(1, 1)	level
robust	0.8438	0.7777	0.6742	0.6038	0.5697	0.5668	(0.01)
	0.9944	0.9668	0.8925	0.8160	0.7706	0.7489	(0.05)
	0.9996	0.9916	0.9517	0.8959	0.8512	0.8260	(0.10)
Z6	0.9591	0.9061	0.8136	0.7446	0.7221	0.7269	(0.01)
	0.9988	0.9839	0.9317	0.8672	0.8259	0.8113	(0.05)
	0.9999	0.9932	0.9568	0.9045	0.8638	0.8452	(0.10)
R_lh	0.9518	0.9098	0.9093	0.9199	0.9332	0.9457	(0.01)
	0.9980	0.9935	0.9886	0.9823	0.9787	0.9770	(0.05)
	0.9998	0.9990	0.9965	0.9926	0.9878	0.9843	(0.10)
R_gamma2	0.8975	0.8994	0.9086	0.9199	0.9332	0.9457	(0.01)
	0.9958	0.9931	0.9885	0.9823	0.9787	0.9770	(0.05)
	0.9996	0.9989	0.9965	0.9926	0.9878	0.9843	(0.10)
<i>Pareto</i>							Alpha
<i>(shape, scale)</i>	(2.5, 1)	(5, 1)	(10, 1)	(20, 1)	(50, 1)	(100, 1)	level
robust	0.6863	0.5644	0.5588	0.5636	0.5683	0.5709	(0.01)
	0.8257	0.7617	0.7777	0.7974	0.8140	0.8175	(0.05)
	0.8824	0.8438	0.8669	0.8872	0.9009	0.9044	(0.10)
Z6	0.8383	0.7195	0.7072	0.7142	0.7198	0.7195	(0.01)
	0.8791	0.8203	0.8368	0.8534	0.8640	0.8692	(0.05)
	0.8983	0.8580	0.8801	0.8974	0.9106	0.9139	(0.10)
R_lh	0.9716	0.8934	0.8406	0.8110	0.7950	0.7867	(0.01)
	0.9838	0.9605	0.9485	0.9423	0.9376	0.9362	(0.05)
	0.9879	0.9760	0.9729	0.9717	0.9699	0.9701	(0.10)
R_gamma2	0.9716	0.8934	0.8406	0.8110	0.7950	0.7866	(0.01)
	0.9838	0.9605	0.9485	0.9423	0.9376	0.9362	(0.05)
	0.9879	0.9760	0.9729	0.9717	0.9699	0.9701	(0.10)
<i>Beta</i>							Alpha
<i>(a, b)</i>	(0.5, 1)	(1, 1)	(2, 1)	(1, 0.5)	(2, 2)	(1, 2)	level
robust	0.9939	0.9989	0.9562	0.9941	0.9843	0.9564	(0.01)
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(0.05)
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(0.10)
Z6	0.8960	0.9905	0.9429	0.8964	0.9893	0.9426	(0.01)
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(0.05)
	1.0000	1.0000	1.0000	1.0000	1.0000	1.0000	(0.10)
R_lh	0.1996	0.7657	0.9971	0.9893	0.9361	0.5216	(0.01)
	0.5093	0.9067	0.9997	0.9935	0.9923	0.8444	(0.05)
	0.6770	0.9539	0.9999	0.9960	0.9984	0.9317	(0.10)
R_gamma2	0.1900	0.2652	0.4620	0.1959	0.5584	0.4484	(0.01)
	0.5029	0.6584	0.8027	0.4901	0.8949	0.8129	(0.05)
	0.6725	0.8188	0.9050	0.6485	0.9636	0.9174	(0.10)

(Continued) Power with $\delta = 4$ ($n = 30$)

<i>Inv-gamma</i> (<i>shape, scale</i>)	(2.5, 1)	(5, 1)	(7.5, 1)	(10, 1)	(20, 1)	(50, 1)	Alpha level
robust	0.6967	0.6064	0.6319	0.6555	0.7151	0.7808	(0.01)
	0.8274	0.8009	0.8351	0.8664	0.9240	0.9673	(0.05)
	0.8822	0.8723	0.9080	0.9309	0.9708	0.9917	(0.10)
Z6	0.8403	0.7516	0.7723	0.7934	0.8515	0.9086	(0.01)
	0.8808	0.8488	0.8819	0.9066	0.9531	0.9840	(0.05)
	0.8992	0.8834	0.9160	0.9367	0.9754	0.9935	(0.10)
R_lh	0.9810	0.9390	0.9234	0.9146	0.9053	0.9086	(0.01)
	0.9887	0.9807	0.9829	0.9866	0.9894	0.9934	(0.05)
	0.9911	0.9892	0.9920	0.9944	0.9976	0.9990	(0.10)
R_gamma2	0.9810	0.9390	0.9234	0.9145	0.9040	0.9004	(0.01)
	0.9887	0.9807	0.9829	0.9866	0.9893	0.9932	(0.05)
	0.9911	0.9892	0.9920	0.9944	0.9976	0.9990	(0.10)

Table 22. Power study for the left-tailed hypothesis test of $\sigma^2 = \delta\sigma_0^2$, with $\delta = 4$, from 10^5 simulations of samples of size $n = 30$ and significance levels $\alpha = 0.01, 0.05$, and 0.10 .

APPENDIX F: R CODE

The source code is available online at github.com/tgrimes/Thesis.

```
#####
#
# Setting up the simulation: the code here reproduces the results given in
# appendix A, B, C and E, and is provided as an example. The various parameters can
# be adjusted to set up different simulations.
#
#####

#Get current working directory. Any source files should also be located here.
dir <- getwd()

#Required packages:
packages <- c("boot", "distr", "ggplot2", "gridExtra", "MASS",
              "moments", "polynom", "pracma", "actuar")

#Download and install any missing packages.
if(length(setdiff(packages, rownames(installed.packages())) > 0) {
  install.packages(setdiff(packages, rownames(installed.packages())))
}
#Load packages.
supply(packages, require, character.only = TRUE)

#Simulation parameters.
# n = 30   #Sample size.
m = 10^5   #Number of simulations.
# significance.level = NULL
# ht.list <- NULL
# delta.set <- c(1, 4)
tail <- "left"

results <- NULL
timestart <- NULL
timeend <- NULL

#-----
#Code for verifying test statistics. (Refer to appendix A.)
#-----
results <- verify.normal()
results <- c(results, verify.chisq())
```

```

results <- c(results, verify.exp())
results <- c(results, verify.gamma())
results <- c(results, verify.weibull())
results <- c(results, verify.lognormal())

filename <- "all_tests"
dir.save <- paste(dir, "/results/appendixA/", filename, sep = "")
dir.create(dir.save)
save.image(file = paste(dir.save, "/env", sep = ""))

#-----
#Code for simulation with all 11 test statistics. (Refer to appendix B.)
#-----
ht.list <- c("ht.chisq", "ht.robust", "ht.z6", "ht.saddle", "ht.saddle.gamma2",
            "ht.saddle.normal", "ht.saddle.chisq", "ht.saddle.exp", "ht.saddle.gamma",
            "ht.saddle.weibull", "ht.saddle.lognormal")
delta.set <- c(1, 2, 3, 4)

for(n in c(10)) {
  for(significance.level in c(0.05)) {
    timestart <- format(Sys.time(), "%Y_%m_%d_%H.%M.%S")
    results <- run.simulation(n, m, significance.level, ht.list, delta.set, tail)
    timeend <- format(Sys.time(), "%Y_%m_%d_%H.%M.%S")

    filename <- paste("n", n, "_m", m, "_signf", significance.level, sep = "")
    dir.save <- paste(dir, "/results/appendixB/", filename, sep = "")
    dir.create(dir.save)
    save.image(file = paste(dir.save, "/env", sep = ""))

    #Print out tables of results
    for(delta in c(1)) {
      for(coverage.list in results) {
        tab <- get.table(coverage.list, delta = delta)
        print(paste(tab[[1]], "significance =", significance.level))
        print((tab[[2]])[,])
      }
    }

    #Print out graphs of results:
    # for(coverage.list in results) {
    #   tests = NULL
    #   graph.overall(coverage.list, tests, significance.level)
    # }

```

```

    print(paste("n =", n, "- start:", timestart, "    end:", timeend))
  }
}

#-----
#Code for type-I error rates and power study. (Refer to appendix C and E.)
#-----
ht.list <- c("ht.robust", "ht.z6", "ht.saddle", "ht.saddle.gamma2")
delta.set <- c(1, 4)

for(n in c(10, 20, 30)) {
  for(significance.level in c(0.01, 0.05, 0.1)) {
    timestart <- format(Sys.time(), "%Y_%m_%d_%H.%M.%S")
    results <- run.simulation(n, m, significance.level, ht.list, delta.set, tail)
    timeend <- format(Sys.time(), "%Y_%m_%d_%H.%M.%S")

    filename <- paste("n", n, "_m", m, "_signf", significance.level, sep = "")
    dir.save <- paste(dir, "/results/appendixCandE/", filename, sep = "")
    dir.create(dir.save)
    save.image(file = paste(dir.save, "/env", sep = ""))

    #Print out tables of results
    for(delta in c(1)) {
      for(coverage.list in results) {
        tab <- get.table(coverage.list, delta = delta)
        print(paste(tab[[1]], "significance =", significance.level))
        print((tab[[2]][,])
      }
    }

    #Print out graphs of results:
    # for(coverage.list in results) {
    #   tests = NULL
    #   graph.overall(coverage.list, tests, significance.level)
    # }

    print(paste("n =", n, "- start:", timestart, "    end:", timeend))
  }
}

#####
#
#   Example for running a simulation for a particular population distribution.
#
#####

```



```

#Example: Maxwell-Boltzmann distribution
library("shotGroups") #Library containing rMaxwell().
rdist <- rMaxwell      #Function that generates random variables from the distribution.
variance.fn <- function(a) { a^2*(3*pi - 8)/pi } #Variance of distribution.
param <- list(a = c(0.5, 1, 1.5, 2, 4, 8, 10, 50)) #Parameters values to consider. If
                                                    # multiple parameters are provided,
                                                    # all combinations are used.

header = "Maxwell-Boltzmann (scale)"
results <- run.simulation(n = 10, rdist = rdist, param = param, variance.fn = variance.fn,
                        header = header, DEFAULT.DISTS = FALSE, DIST = NULL)

#####
#
# Retrieving stored results and printing graphs and tables.
#
#####

#The location of the stored environment should be specified by the variable "dir.load".
#The example here may need to be adjusted depending on the implementation.
dir.load <- paste(dir, "/results/appendixCandE/n10_m1e+05_signf0.05 /env", sep = "")
load(dir.load)

#Prints type-I error table. For other tables, specify desired delta value(s).
for(delta in c(1)) {
  for(coverage.list in results) {
    tab <- get.table(coverage.list, delta = delta)
    print(paste(tab[[1]], "significance =", significance.level)) #Optional: header for table.
    print((tab[[2]]),)
  }
}

#Generates graphs of power curves.
for(coverage.list in results) {
  tests = c("ht.saddle", "ht.saddle2") #Optional: specify which tests to show on graph.
  graph.overall(coverage.list, tests)
}

#####
#
# The simulation procedure.
#
#####

```

```

run.simulation <- function(n = 20, m = 1000, sig.level = 0.05, ht.list = NULL,
                           delta.set = c(1, 2, 3, 4), tail = "left", DEFAULT.DISTS = TRUE,
                           DIST = NULL, rdist = NULL, param = NULL,
                           variance.fn = NULL, header = NULL) {

  if(is.null(ht.list)) {
    #Default list of hypothesis tests to run.
    ht.list <- c("ht.robust", "ht.z6", "ht.saddle", "ht.saddle.gamma2")
  }

  results <- NULL

  if(!is.null(rdist)) {
    #If a distribution is provided, run the simulation on it.
    coverage <- simulate(rdist, param, variance.fn, delta.set, n, m, ht.list, sig.level,
                        tail, header)
    results <- c(results, list(coverage))
  }

  if(DEFAULT.DISTS == TRUE || any(strcmpi("norm", DIST))) {
    #Normal distribution.
    print("Normal:")
    param <- list(mu.set = c(0),
                  sigma.set = c(0.01, 0.5, 1, 5, 10, 100))
    header <- list("Normal (mu, sigma)")
    rdist <- function(n, mu, sigma) { rnorm(n, mu + 10*sigma, sigma) }
    variance.fn <- function(mu, sigma) { sigma^2 }
    coverage <- simulate(rdist, param, variance.fn, delta.set, n, m, ht.list, sig.level,
                        tail, header)
    results <- c(results, list(coverage))
  }

  if(DEFAULT.DISTS == TRUE || any(strcmpi("t", DIST))) {
    #t distribution.
    print("t:")
    param <- list(df.set = c(4, 6, 7, 12, 20, 30))
    header <- list("t (df)")
    rdist <- function(n, df) { rt(n, df) + 1000 }
    variance.fn <- function(df) { df/(df - 2) }
    coverage <- simulate(rdist, param, variance.fn, delta.set, n, m, ht.list, sig.level,
                        tail, header)
    results <- c(results, list(coverage))
  }
}

```

```

if(DEFAULT.DISTS == TRUE || any(strcmpi("chisq", DIST))) {
  #chi-sq distribution.
  print("Chi-sq:")
  param <- list(df.set = c(1, 2, 4, 6, 7, 10))
  header <- list("chi-sq (df)")
  rdist <- rchisq
  variance.fn <- function(df) { 2*df }
  coverage <- simulate(rdist, param, variance.fn, delta.set, n, m, ht.list, sig.level,
                      tail, header)
  results <- c(results, list(coverage))
}

```

```

#Exponential distribution.
if(DEFAULT.DISTS == TRUE || any(strcmpi("exp", DIST))) {
  #Exponential distribution.
  print("Exp:")
  param <- list(lambda.set = c(0.1, 0.5, 1, 5, 20, 50))
  header <- list("exp (lambda)")
  rdist <- rexp
  variance.fn <- function(lambda) { 1/lambda^2 }
  coverage <- simulate(rdist, param, variance.fn, delta.set, n, m, ht.list, sig.level,
                      tail, header)
  results <- c(results, list(coverage))
}

```

```

if(DEFAULT.DISTS == TRUE || any(strcmpi("gamma", DIST))) {
  #Gamma distribution.
  print("Gamma:")
  param <- list(alpha.set = c(0.5, 1, 5, 10, 50, 100),
                beta.set = c(10))
  header <- list("gamma (alpha, beta)")
  rdist <- rgamma
  variance.fn <- function(alpha, beta) { alpha/beta^2 }
  coverage <- simulate(rdist, param, variance.fn, delta.set, n, m, ht.list, sig.level,
                      tail, header)
  results <- c(results, list(coverage))
}

```

```

if(DEFAULT.DISTS == TRUE || any(strcmpi("weibull", DIST))) {
  #Weibull distribution.
  print("Weibull:")
  param <- list(shape.set = c(0.5, 1, 5, 20, 50, 100),
                scale.set = c(1))
  header <- list("weibull (shape, scale)")
}

```

```

rdist <- rweibull
variance.fn <- function(shape, scale) { scale^2 * (gamma(1 + 2/shape) - (gamma(1 +
1/shape)^2)) }
coverage <- simulate(rdist, param, variance.fn, delta.set, n, m, ht.list, sig.level,
                    tail, header)
results <- c(results, list(coverage))
}

if(DEFAULT.DISTS == TRUE || any(strcmpi("lnorm", DIST))) {
  #Lognormal distribution.
  print("Log-Normal:")
  param <- list(lmu.set = c(1),
               lsigma.set = c(0.01, 0.2, 0.4, 0.6, 0.8, 1))
  header <- list("Lognormal (lmu, lsigma)")
  rdist <- rlnorm
  variance.fn <- function(lmu, lsigma) { (exp(lsigma^2) - 1) * exp(2 * lmu + lsigma^2) }
  coverage <- simulate(rdist, param, variance.fn, delta.set, n, m, ht.list, sig.level,
                    tail, header)
  results <- c(results, list(coverage))
}

if(DEFAULT.DISTS == TRUE || any(strcmpi("pareto", DIST))) {
  #Pareto distribution.
  print("Pareto:")
  rdist <- rpareto
  variance.fn <- function(shape, scale) { scale^2*shape/((shape-1)^2*(shape-2)) }
  param <- list(shape.set = c(2.5, 5, 10, 20, 50, 100),
               scale.set = c(1))
  header <- list("Pareto (scale, shape)")
  coverage <- simulate(rdist, param, variance.fn, delta.set, n, m, ht.list, sig.level,
                    tail, header)
  results <- c(results, list(coverage))
}

if(DEFAULT.DISTS == TRUE || any(strcmpi("beta", DIST))) {
  #Beta distribution.
  print("Beta:")
  rdist <- rbeta
  variance.fn <- function(a, b) { (a*b)/((a + b)^2*(a + b + 1)) }
  param <- list(shape = c(0.5, 1, 2),
               scale = c(0.5, 1, 2))
  header <- list("Beta (a, b)")
  coverage <- simulate(rdist, param, variance.fn, delta.set, n, m, ht.list, sig.level,
                    tail, header)

```

```

    results <- c(results, list(coverage))
  }

  if(DEFAULT.DISTS == TRUE || any(strcmpi("inverse-gamma", DIST))) {
    #Inverse-gamma distribution.
    print("Inv-Gamma:")
    rdist <- rinvgamma
    variance.fn <- function(shape, scale) { scale^(-2)/((shape - 1)^2*(shape - 2)) }
    param <- list(shape = c(2.5, 5, 7.5, 10, 20, 50),
                  scale = c(1))
    header <- list("Inverse-gamma (shape, scale)")
    coverage <- simulate(rdist, param, variance.fn, delta.set, n, m, ht.list, sig.level,
                        tail, header)
    results <- c(results, list(coverage))
  }

  return(results)
}

simulate <- function(rdist, param, variance.fn, delta.set, n, m, ht.list,
                    sig.level = 0.05, tail = "left", header = NULL) {

  num.param <- length(param)
  coverage.list <- header
  if(num.param == 1) {
    for(param1 in param[[1]]) {
      coverage.data <- NULL
      for(delta in delta.set) {
        rdist.wpar <- function(n) { rdist(n, param1) }
        variance <- variance.fn(param1)
        coverage <- sim.dist(rdist.wpar, variance, delta, n, m, ht.list,
                             sig.level, tail)
        coverage.data <- rbind(coverage.data,
                              data.frame(Test = ht.list,
                                           Delta = rep(delta, length(ht.list)),
                                           Coverage = coverage,
                                           param1 = rep(param1, length(ht.list))))
      }
      coverage.list <- c(coverage.list, list(coverage.data))
    }
  } else if(num.param == 2) {
    for(param1 in param[[1]]) {

```

```

for(param2 in param[[2]]) {
  coverage.data <- NULL
  for(delta in delta.set) {
    rdist.wpar <- function(n) { rdist(n, param1, param2) }
    variance <- variance.fn(param1, param2)
    coverage <- sim.dist(rdist.wpar, variance, delta, n, m, ht.list,
                        sig.level, tail)
    coverage.data <- rbind(coverage.data,
                          data.frame(Test = ht.list,
                                      Delta = rep(delta, length(ht.list)),
                                      Coverage = coverage,
                                      param1 = rep(param1, length(ht.list)),
                                      param2 = rep(param2, length(ht.list))))
  }
  coverage.list <- c(coverage.list, list(coverage.data))
}
}
} else {
  return(NA)
}

return(coverage.list)
}

sim.dist <- function(rdist.wpar, variance, delta, n, m, ht.list,
                    sig.level = 0.05, tail = "left", nuisance = NULL) {

  x <- matrix(rdist.wpar(n*m), nrow = n, ncol = m)

  sigma.sq.0 <- delta * variance

  results <- NULL
  for(i in 1:length(ht.list)) {
    results <- cbind(results, apply(x, 2, get(ht.list[i]),
                                   sigma.sq.0 = sigma.sq.0,
                                   significance.level = sig.level,
                                   tail = tail,
                                   nuisance))
  }

  #Account for any samples that resulted in a NaN.
  print(apply(results, 2, function(x) sum(is.nan(x))))

```

```

coverage <- apply(results, 2, function(x) mean(x[!is.nan(x)]))
return(coverage)
}

#####
#
# Functions for each hypothesis test. As shown, these procedures can only
# conduct left tailed tests.
#
#####

#Chi-squared test of variance.
ht.chisq <- function(data, sigma.sq.0, significance.level = 0.05,
                    tail = "left", nuisance = NULL) {
  n <- length(data)
  s2 <- var(data)
  t <- s2/sigma.sq.0*(n - 1)

  if(tail == "left") {
    if(t < qchisq(significance.level, n - 1)) {
      return(TRUE)
    } else {
      return(FALSE)
    }
  } else {
    return(NA)
  }
}

#The robust chi-squared test.
ht.robust <- function(data, sigma.sq.0, significance.level = 0.05,
                    tail = "left", nuisance = NULL) {
  eta.hat <- kurtosis(data)
  d.hat <- 1/(1 + eta.hat/2)
  n <- length(data)
  s2 <- var(data)
  t <- s2*(n - 1)*d.hat/sigma.sq.0

  if(tail == "left") {
    if(t < qchisq(significance.level, ceiling((n - 1)*d.hat))) {
      return(TRUE)
    } else {

```

```

    return(FALSE)
  }
} else {
  return(NA)
}
}

#The Z6 test statistic.
ht.z6 <- function(data, sigma.sq.0, significance.level = 0.05,
  tail = "left", nuisance = NULL) {
  #Center data:
  data <- data - mean(data)

  k <- (all.cumulants(all.moments(data, 6)))[2:7]
  s2 <- var(data)
  s4 <- s2^2
  n <- length(data)
  if(k[4] + 2 * s4 < 0) {
    k[4] <- 0
  }
  b1.hat <- -sqrt(s4/(k[4] + 2*s4))
  b2.hat <- (k[6] + 12*k[4]*s2 + 4*k[3]^2 + 8*s2^3)/(k[4] + 2*s4)^(3/2)

  temp <- (k[4]*sigma.sq.0)/(n*s2) + (2*sigma.sq.0^2)/(n - 1)
  if(temp < 0) {
    #Test procedure failed.
    return(NaN)
  }

  z6 <- (s2 - sigma.sq.0)/(sqrt(temp))

  if(tail == "left") {
    z.alpha <- qnorm(significance.level)
    right <- z.alpha + sqrt(1/n)*(b1.hat + b2.hat*((z.alpha)^2 - 1)/6)
    if(is.nan(z6) || is.nan(right)) {
      #Test procedure failed.
      return(NaN)
    }
    if(z6 < right) {
      return(TRUE)
    } else {
      return(FALSE)
    }
  }
}

```



```

    } else {
      return(NA)
    }
  }

#The R_norm test statistic. Refer to section (2.4.1).
ht.saddle.normal <- function(obs, sigma.sq.0, significance.level = 0.05,
                             tail = "left", nuisance = NULL) {

  n <- length(obs)

  if(var(obs) > sigma.sq.0) {
    return(FALSE)
  }

  mu <- nuisance
  if(is.null(nuisance)) {
    #If mu is not provided, calculate its estimator.
    mu <- mean(obs)
  }

  s.sq <- var(obs)

  #Exponential Family setup:
  t <- function(x) { -0.5*(x - mu)^2 + 0.5*mu^2 }
  t.bar <- 0.5*(mu^2 - s.sq)
  k <- function(theta) { 0.5*mu^2*theta - 0.5*log(theta) }
  kp <- function(theta) { 0.5*mu^2 - 0.5/theta }
  kpp <- function(theta) { 0.5/theta^2 }
  theta.hat <- 1/s.sq
  theta.0 <- 1/sigma.sq.0

  temp <- 2*n*((theta.hat - theta.0)*t.bar - k(theta.hat) + k(theta.0))
  if(temp < 0) {
    #Test procedure failed.
    return(NaN)
  }

  ts.R <- sign(theta.hat - theta.0)*sqrt(temp)
  #ts.U <- sqrt(n)*(theta.hat - theta.0)*sqrt(kpp(theta.hat))
  ts.U <- sqrt(n/2)*(1-theta.0/theta.hat)
  ts <- ts.R + 1/ts.R*log(ts.U/ts.R)

  if(is.nan(ts)) {

```

```

#Test procedure failed.
return(NaN)
}

if(tail == "left") {
  if(ts >= qnorm(1 - significance.level)) {
    return(TRUE)
  } else {
    return(FALSE)
  }
} else {
  return(NA)
}
}

#The R_chisq test statistic. Refer to section (2.4.2).
ht.saddle.chisq <- function(obs, sigma.sq.0, significance.level = 0.05,
                             tail = "left", nuisance = NULL) {
  n <- length(obs)

  if(var(obs) > sigma.sq.0) {
    return(FALSE)
  }

  #Exponential Family setup.
  t <- function(x) { 0.5*log(x) }
  t.bar <- mean(t(obs))
  k <- function(theta) { theta*log(2)/2 + lgamma(theta/2) }
  kp <- function(theta) { log(2)/2 + digamma(theta/2)/2 }
  kpp <- function(theta) { trigamma(theta/2)/4 }
  theta.hat <- 2*lgamma(2*t.bar - log(2))
  theta.0 <- sigma.sq.0/2

  temp <- 2*n*((theta.hat - theta.0)*t.bar - k(theta.hat) + k(theta.0))
  if(temp < 0) {
    #Test procedure failed.
    return(NaN)
  }

  ts.R <- sign(theta.hat - theta.0)*sqrt(temp)
  ts.U <- sqrt(n)*(theta.hat - theta.0)*sqrt(kpp(theta.hat))
  ts <- ts.R + 1/ts.R*log(ts.U/ts.R)

```

```

if(is.nan(ts)) {
  #Test procedure failed.
  return(NaN)
}

if(tail == "left") {
  if(ts <= qnorm(significance.level)) {
    return(TRUE)
  } else {
    return(FALSE)
  }
} else {
  return(NA)
}
}

#The R_exp test statistic. Refer to section (2.4.3).
ht.saddle.exp <- function(obs, sigma.sq.0, significance.level = 0.05,
                           tail = "left", nuisance = NULL) {
  n <- length(obs)

  if(var(obs) > sigma.sq.0) {
    return(FALSE)
  }

  #Exponential Family setup.
  t <- function(x) { -x }
  t.bar <- mean(t(obs))
  k <- function(theta) { -log(theta) }
  kp <- function(theta) { -1/theta }
  kpp <- function(theta) { 1/theta^2 }
  theta.hat <- -1/t.bar
  theta.0 <- sqrt(1/sigma.sq.0)

  temp <- 2*n*(((theta.hat - theta.0)*t.bar - k(theta.hat) + k(theta.0))
  if(temp < 0) {
    #Test procedure failed.
    return(NaN)
  }

  ts.R <- sign(theta.hat - theta.0)*sqrt(temp)

```

```
ts.U <- sqrt(n)*(theta.hat - theta.0)*sqrt(kpp(theta.hat))
ts <- ts.R + 1/ts.R*log(ts.U/ts.R)
```

```
if(is.nan(ts)) {
  #Test procedure failed.
  return(NaN)
}
```

```
if(tail == "left") {
  if(ts >= qnorm(1 - significance.level)) {
    return(TRUE)
  } else {
    return(FALSE)
  }
} else {
  return(NA)
}
}
```

```
#The R_gamma test statistic. Refer to section (2.4.4).
ht.saddle.gamma <- function(obs, sigma.sq.0, significance.level = 0.05,
                             tail = "left", nuisance = NULL) {
  n <- length(obs)
```

```
  alpha.hat <- nuisance
  if(is.null(nuisance)) {
    #Approximate maximum likelihood estimator.
    log.xbar <- log(mean(obs))
    logx.bar <- mean(log(obs))
    s <- log.xbar - logx.bar
    alpha.hat <- (3 - s + sqrt((s-3)^2 + 24*s))/(12*s)
  }
```

```
#Exponential Family setup.
t <- function(x) { -x }
t.bar <- mean(t(obs))
k <- function(theta) { -alpha.hat*log(theta) }
kp <- function(theta) { -alpha.hat/theta }
kpp <- function(theta) { alpha.hat/theta^2 }
theta.hat <- -alpha.hat/t.bar
theta.0 <- sqrt(alpha.hat/sigma.sq.0)
```

```

temp <- 2*n*((theta.hat - theta.0)*t.bar - k(theta.hat) + k(theta.0))
if(temp < 0) {
  #Test procedure failed.
  return(NaN)
}

```

```

ts.R <- sign(theta.hat - theta.0)*sqrt(temp)
ts.U <- sqrt(n)*(theta.hat - theta.0)*sqrt(kpp(theta.hat))
ts <- ts.R + 1/ts.R*log(ts.U/ts.R)

```

```

if(is.nan(ts)) {
  #Test procedure failed.
  return(NaN)
}

```

```

if(tail == "left") {
  if(ts >= qnorm(1 - significance.level)) {
    return(TRUE)
  } else {
    return(FALSE)
  }
} else {
  return(NA)
}
}

```

```

#The adjusted gamma test. Refer to section (2.4.5).
ht.saddle.gamma2 <- function(obs, sigma.sq.0, significance.level = 0.05,
                             tail = "left", nuisance = NULL) {

```

```

  n <- length(obs)

  alpha.hat <- nuisance
  if(is.null(nuisance)) {
    #Approximate maximum likelihood estimator.
    log.xbar <- log(mean(obs))
    logx.bar <- mean(log(obs))
    s <- log.xbar - logx.bar
    alpha.hat <- (3 - s + sqrt((s-3)^2 + 24*s))/(12*s)
  }

```

```

  obs <- obs/alpha.hat

```

```
sigma.sq.0 <- sigma.sq.0/alpha.hat

alpha.hat <- 1

#Exponential Family setup.
t <- function(x) { -x }
t.bar <- mean(t(obs))
k <- function(theta) { -alpha.hat*log(theta) }
kp <- function(theta) { -alpha.hat/theta }
kpp <- function(theta) { alpha.hat/theta^2 }
theta.hat <- -alpha.hat/t.bar
theta.0 <- sqrt(alpha.hat/sigma.sq.0)

temp <- 2*n*((theta.hat - theta.0)*t.bar - k(theta.hat) + k(theta.0))
if(temp < 0) {
  #Test procedure failed.
  return(NaN)
}

ts.R <- sign(theta.hat - theta.0)*sqrt(temp)
ts.U <- sqrt(n)*(theta.hat - theta.0)*sqrt(kpp(theta.hat))
ts <- ts.R + 1/ts.R*log(ts.U/ts.R)

if(is.nan(ts)) {
  #Test procedure failed.
  return(NaN)
}

if(tail == "left") {
  if(ts >= qt(1 - significance.level/2, n)) {
    return(TRUE)
  } else {
    return(FALSE)
  }
} else {
  return(NA)
}
}
```

```

n <- length(obs)

if(var(obs) > sigma.sq.0) {
  return(FALSE)
}

shape <- nuisance
if(is.null(nuisance)) {
  shape <- tryCatch({
    shape <- (fitdistr(obs, "weibull"))$estimate[[1]]
  }, error = function(e) {
    #print("fitdist failed, returning NULL.")
    return(NULL)
  })
  if(is.null(shape)) {
    #Failed to fit distribution.
    return(NaN)
  }
}

#Exponential Family setup.
t <- function(x) { -(x^shape) }
t.bar <- mean(t(obs))
k <- function(theta) { -log(theta) }
kp <- function(theta) { -1/theta }
kpp <- function(theta) { 1/theta^2 }
theta.hat <- -1/t.bar
theta.0 <- ((gamma(1 + 2/shape) - (gamma(1 + 1/shape))^2)/sigma.sq.0)^(shape/2)

temp <- 2*n*((theta.hat - theta.0)*t.bar - k(theta.hat) + k(theta.0))
if(is.nan(temp)) {
  #Test procedure failed.
  return(NaN)
}
if(temp < 0) {
  #Test procedure failed.
  return(NaN)
}

ts.R <- sign(theta.hat - theta.0)*sqrt(temp)
ts.U <- sqrt(n)*(theta.hat - theta.0)*sqrt(kpp(theta.hat))
ts <- ts.R + 1/ts.R*log(ts.U/ts.R)

```

```

if(is.nan(ts)) {
  #Test procedure failed.
  return(NaN)
}

if(tail == "left") {
  if(ts >= qnorm(1 - significance.level)) {
    return(TRUE)
  } else {
    return(FALSE)
  }
} else {
  return(NA)
}
}

#The R_Inorm test statistic. Refer to section (2.4.7).
ht.saddle.lognormal <- function(obs, sigma.sq.0, significance.level = 0.05,
                                tail = "left", nuisance = NULL) {

  n <- length(obs)

  if(var(obs) > sigma.sq.0) {
    return(FALSE)
  }

  lsigma.sq <- nuisance
  if(is.null(nuisance)) {
    lsigma.sq <- tryCatch({
      lsigma.sq <- ((fitdistr(obs, "lognormal"))$estimate[[2]])^2
    }, error = function(e) {
      print("fitdist failed, returning NULL.")
      return(NULL)
    })
    if(is.null(lsigma.sq)) {
      #Failed to fit distribution.
      return(NaN)
    }
  }

  #Exponential Family setup.
  t <- function(x) { log(x)/lsigma.sq }
  t.bar <- mean(t(obs))
  k <- function(theta) { theta^2/(2 * lsigma.sq) }

```



```

kp <- function(theta) { theta/lsigma.sq }
kpp <- function(theta) { 1/lsigma.sq }
theta.hat <- lsigma.sq * t.bar
theta.0 <- (log(sigma.sq.0/(exp(2 * lsigma.sq) - exp(lsigma.sq))))/2

temp <- 2*n*((theta.hat - theta.0)*t.bar - k(theta.hat) + k(theta.0))
if(temp < 0) {
  #Test procedure failed.
  return(NaN)
}

ts.R <- sign(theta.hat - theta.0)*sqrt(temp)
ts.U <- sqrt(n)*(theta.hat - theta.0)*sqrt(kpp(theta.hat))
ts <- ts.R + 1/ts.R*log(ts.U/ts.R)

if(is.nan(ts)) {
  #Test procedure failed.
  return(NaN)
}

if(tail == "left") {
  if(ts <= qnorm(significance.level)) {
    return(TRUE)
  } else {
    return(FALSE)
  }
} else {
  return(NA)
}
}

#The R_lh test statistic. Refer to section (2.4.8).
ht.saddle <- function(obs, sigma.sq.0, significance.level = 0.05,
  tail = "left", nuisance = NULL) {

  #Tests corresponding to the distributions considered. These must be the names
  #of the functions that conduct the test.
  tests <- c("ht.saddle.normal", "ht.saddle.gamma2")

  #Fit normal.
  fits <- tryCatch({
    (fitdistr(obs, "normal"))$loglik
  }, error = function(e) {

```



```

    i <- i + 1
  }

  layout <- NULL
  for(divisor in c(3, 4, 5, 6, 7, 2)) {
    if((n %% divisor) == 0) {
      for(k in 1:(n/divisor)) {
        layout <- rbind(layout, ((divisor*k - (divisor - 1)):(divisor*k)))
      }
      layout <- rbind(layout, rep(n + 1, divisor))
      get.plot(plots, layout = layout)
    }
  }
  return(NULL)
}
}
}

get.coverage.plot <- function(coverage.data, parameter.vals, tests = NULL,
                             significance.level) {

  main <- paste("(", parameter.vals[1], sep = "")
  i <- 2
  while(i <= length(parameter.vals)) {
    main <- paste(main, ", ", parameter.vals[i])
    i <- i + 1
  }
  main <- paste(main, ")", sep = "")

  if(!is.null(tests)) {
    coverage.data <- subset(coverage.data, Test %in% tests)
  }

  plot <- ggplot(data = coverage.data, aes(x = Delta, y = Coverage, linetype = Test)) +
    geom_line() +
    geom_point(size=2, shape=21, fill="white", alpha = 0.5) +
    geom_hline(yintercept = significance.level) +
    labs(title = main, y = "Power") +
    scale_y_continuous(breaks = c(0, 0.2, 0.4, 0.6, 0.8, 1)) +
    theme(axis.title.x = element_text(size = 8),
          axis.title.y = element_text(size = 8),
          plot.title = element_text(size = 10),
          legend.key.size = unit(1, "cm"),
          legend.title = element_text(size = 10, face = "bold"))

```

```
}
```

```
get.plot <- function(plots, distribution, parameter.names, parameter.vals,
                     layout = rbind(1:3, 4:6, 7:9, rep(10,3))) {
```

```
  g <- ggplotGrob(plots[[1]] + theme(legend.position="bottom"))$grobs
  legend <- g[[which(sapply(g, function(x) x$name) == "guide-box")]]
  lheight <- sum(legend$height)
  gl <- lapply(plots, function(x) x + theme(legend.position="none"))
  grid.arrange(grobs = c(gl, list(legend)), layout_matrix = layout,
               heights = grid::unit.c(rep((unit(1, "npc") - lheight) *
                                           (1/(nrow(layout)-1)),nrow(layout)-1),
                                       lheight))
}
```

```
}
```

```
table.single.dist <- function(data, params) {
```

```
  Tests <- unique(data[, 1])
  Deltas <- unique(data[, 2])
```

```
  coverage.table <- data.frame(Test = Tests)
  n <- length(Tests)
  d <- length(Deltas)
```

```
  i <- 1
  for(i in 1:d) {
    coverage.table <- cbind(coverage.table, data[((i - 1)*n + 1):(i*n), 3])
  }
```

```
  names <- NULL
  for(i in 1:d) {
    names <- c(names, paste("Delta =", Deltas[i]))
  }
```

```
  colnames(coverage.table) <- c("Tests", names)
  return(coverage.table)
}
```

```
table.overall <- function(data.list, delta = 1) {
  #First item in list is description of distribution. Take this and remove it.
```

```

description <- data.list[[1]]
if(delta == 1) {
  description <- paste("Population: ", description, ": Type-I-Error rate.", sep = "")
} else {
  description <- paste("Population: ", description, ": Power, delta = ", delta, ".", sep = "")
}
data.list[[1]] <- NULL

tests <- unique((data.list[[1]])[, 1])
deltas <- unique((data.list[[1]])[, 2])

n <- length(tests)    #Number of tests
d <- length(deltas)   #Number of delta levels.
num.dataframes <- length(data.list)

coverage.table <- data.frame(test = tests)
i <- match(delta, deltas)
for(data in data.list) {
  coverage.table <- cbind(coverage.table, data[((i - 1)*n + 1):(i*n), 3])
}

col.names <- NULL
for(i in 1:num.dataframes) {
  name <- paste("(", (data.list[[i]][1, 4], sep = "")
  if(ncol(data.list[[i]]) > 4) {
    for(k in 5:ncol(data.list[[i]])) {
      name <- paste(name, ", ", (data.list[[i]][1, k], sep = "")
    }
  }
  name <- paste(name, ")", sep = "")
  col.names <- c(col.names, name)
}

coverage.table <- cbind(coverage.table,
  apply(coverage.table[, 2:(num.dataframes + 1)], 1, min),
  apply(coverage.table[, 2:(num.dataframes + 1)], 1, median),
  apply(coverage.table[, 2:(num.dataframes + 1)], 1, max))
colnames(coverage.table) <- c("Tests", col.names, "min", "median", "max")

return(list(desc = description, table = coverage.table))
}

```

```
get.table <- function(data, delta = 1) {  
  if(is.data.frame(data)) {  
    table.single.dist(data)  
  } else {  
    table.overall(data, delta)  
  }  
}
```

VITA

Tyler Luke Grimes was born . After graduating from High School, he entered the University of Central Florida in Orlando, Florida, where he studied computer science and mathematics. After receiving his Bachelor of Science in Mathematics in May of 2014, he joined the University of North Florida in Jacksonville, Florida, to pursue a Master of Science degree in statistics while working as a graduate teaching assistant. Tyler is expected to graduate in August of 2016.