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On Representations of the Jacobi Group and Differential Equations

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On Representations of the Jacobi Group and
Differential Equations

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CHAPTER 1

Introduction

In this paper we investigate a representation of the Jacobi group in terms of a simple differential equation. In [1], the authors show that in PDEs with nontrivial Lie symmetry algebras, the Lie symmetry naturally yield Fourier and Laplace transforms of fundamental solutions. They then derive explicit formulas for such transformations in terms of the coefficients of the PDE. Using these facts, and borrowing from an idea of [2] the authors of [5] provide a geometric construction of a unitary lowest weight representation \mathcal{H}^+ and a unitary highest weight representation \mathcal{H}^- of a double cover of the conformal group, then provide explicit orthonormal bases for the spaces consisting of weight vectors. This is an algebraic construction, but very similar. Using this template, and an approach called Langlands decomposition (program), [3] then classified global $\widetilde{SL}(2, R)$ representations of the Schrodinger equation with singular potential.

We now follow this path in order to study global $\widetilde{SL}(2, R)$ representations of a general differential equation of the form $u_{xx} + \alpha u_x + \beta u_t = 0$. We will begin with a review of linear algebra and lie algebras, then a review of group theory and Lie group theory, with a specific emphasis on matrix Lie groups. We then introduce the Jacobi group and our differential equation, and define our Lie group of interest, which is the semi-direct product of the metaplectic group and the Heisenberg group. We are then able to induce a faithful representation, and as a result we can classify the solution space of the homogenous equation.

CHAPTER 2

Vector Spaces & Lie Algebras

One of the main strengths of representation theory of Lie groups is the fact that many complicated problems at the level of the Lie group can be reduced to the study of a similar problem at the level of the Lie algebra, which is a much simpler mathematical object. In short, a Lie algebra is a vector space with a binary operation, called the Lie bracket, that is bilinear, anticommutative, and satisfies the Jacobi identity. The latter is just a measure of the non-associativity of this operation. Therefore, begin with the formal definition of vector spaces, Lie algebras, and related topics from linear algebra. This chapter can be considered a review for the expert reader and as such, it can be omitted. We include this chapter for self-containment.

1. Linear Algebra Review

Roughly, a vector space is a set of vectors endowed with two operations a vector addition and a scalar multiplication that satisfy certain conditions. Scalars come from a numeric field, for us the fields will be either the real numbers \mathbb{R} or the complex numbers \mathbb{C} . Formally, the definitions of a field and a vector space are given below.

DEFINITION 1.1 (Field). *A field \mathbb{F} is a set on which two operations $+$ and \cdot (called addition and multiplication, respectively) are defined so that, for each pair of elements x, y in F , there are unique elements $x + y$ and $x \cdot y$ in F for which the following conditions hold for all elements a, b, c in F .*

- (1) $a + b = b + a$ and $a \cdot b = b \cdot a$ (commutativity of addition and multiplication)
- (2) $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ (associativity of addition and multiplication)
- (3) There exist distinct elements 0 and 1 in \mathbb{F} such that

$$0 + a = a \qquad \text{and} \qquad 1 \cdot a = a$$

(existence of identity elements for addition and multiplication)

- (4) For each element a in \mathbb{F} and each nonzero element b in \mathbb{F} , there exist elements c and d in F such that

$$a + c = 0 \qquad \text{and} \qquad b \cdot d = 1$$

(existence of inverses for addition and multiplication)

- (5) $a \cdot (b + c) = a \cdot b + a \cdot c$ (distributivity of multiplication over addition)

DEFINITION 1.2 (Vector Space). A vector space (or linear space) V over a field \mathbb{F} consists of a set on which two operations (called addition and scalar multiplication, respectively) are defined so that for each pair of elements x, y , in V there is a unique element $x + y$ in V , and for each element a in F and each element x in V there is a unique element ax in V , such that the following conditions hold.

- (1) For all x, y in V , $x + y = y + x$ (commutativity of addition).
- (2) For all x, y, z in V , $(x + y) + z = x + (y + z)$ (associativity of addition)
- (3) There exists an element in V denoted by 0 such that $x + 0 = x$ for each x in V .
- (4) For each element x in V there exists an element y in V such that $x + y = 0$.
- (5) For each element x in V , $1x = x$.
- (6) For each pair of elements a, b in \mathbb{F} and each element x in V , $(ab)x = a(bx)$.
- (7) For each element a in \mathbb{F} and each pair of elements x, y in V , $a(x + y) = ax + ay$.
- (8) For each pair of elements a, b in \mathbb{F} and each element x in V , $(a+b)x = ax + bx$.

As noted above, the elements of the field F are called *scalars* and the elements of the vector space V are called *vectors*. One of the main concepts in linear algebra is the concept of a linear combination. Through them we can write any vector in terms of a special (often finite) set of vectors, that will be denominated a basis of the vector space.

DEFINITION 1.3 (Linear Combination). Let V be a vector space and S a nonempty subset of V . A vector $v \in V$ is called a linear combination of vectors of S if there exist a finite number of vectors u_1, u_2, \dots, u_n in S and scalars a_1, a_2, \dots, a_n in \mathbb{F} such that

$v = a_1u_1 + a_2u_2 + \dots + a_nu_n$. In this case we also say that v is a linear combination of u_1, u_2, \dots, u_n and call a_1, a_2, \dots, a_n the coefficients of the linear combination.

Since vectors can be written as linear combinations of other vectors, when given one such representation, it is natural to ask if this representation is as optimal as possible. In a sense, we would like to avoid redundancy. For example,

$$v = a_1u_1 + a_2u_2 = (a_1 - a_3)u_1 + (a_1 - a_3)u_2 + a_3(u_1 + u_2).$$

However, the latter expression is not as optimal as the first. To address this issue, we introduce the concept of linear dependence and linear independence. We would like to say that the vector $u_1 + u_2$ depends (linearly) on the vectors u_1 and u_2 . Therefore, it is not necessary in the representation of v . In the following definitions we formalize this concept.

DEFINITION 1.4 (Linearly Dependent). *A subset S of a vector space V is called linearly dependent if there exist a finite number of distinct vectors u_1, u_2, \dots, u_n in S and scalars a_1, a_2, \dots, a_n , not all zero, such that*

$$a_1u_1 + a_2u_2 + \dots + a_nu_n = 0.$$

In this case we also say that the vectors of S are linearly dependent. For any vectors u_1, u_2, \dots, u_n , we have $a_1u_1 + a_2u_2 + \dots + a_nu_n = 0$ if $a_1 = a_2 = \dots = a_n = 0$. We call this the trivial representation of 0 as a linear combination of u_1, u_2, \dots, u_n . Thus, for a set to be linearly dependent, there must exist a nontrivial representation of 0 as a linear combination of vectors in the set.

DEFINITION 1.5 (Linearly Independent). *A subset S of a vector space that is not linearly dependent is called linearly independent.*

As before, we say that the vectors of S are linearly independent. It is useful to note the the fact that a set is linearly independent if and only if the only representation of 0 as linear combinations of its vectors is the trivial representation.

DEFINITION 1.6 (Span). *Let $S \subset V$ be a subset of a vector space over the field \mathbb{F} . Then, the span of S is defined as*

$$\text{span}_{\mathbb{F}}(S) = \{a_1u_1 + a_2u_2 + \dots + a_nu_n \mid a_i \in \mathbb{F} \text{ and } u_i \in S\}.$$

The sub-index \mathbb{F} will be used only when the field needs to be specified or it is unclear from the context. The following theorem is important, but the proof follows trivially from the definition of linear dependence. Hence, the theorem is presented without a proof.

THEOREM 1.1. *Let S be a linearly independent subset of a vector space V , and let v be a vector in V that is not in S . Then, $S \cup \{v\}$ is linearly dependent if and only if $v \in \text{span}(S)$.*

The contrapositive of the implication is how this theorem is often used. That is, if a vector $v \notin \text{span}(S)$ is attached to S , the resulting set $S \cup \{v\}$ is still linearly independent. The idea is that if we know that the number of generators of a vector space is finite, this process must end at some point. Hopefully, this process would yield an optimal generating set for the vector space. An optimal generating set, will be called a basis.

DEFINITION 1.7 (Basis). *A basis β for a vector space V is a linearly independent subset of V that generates V i.e. $\text{span}(\beta) = V$. If β is a basis for V , we also say that the vectors of β form a basis for V .*

If the cardinality of a basis β of the vector space V is finite, the vector space V is said to be *finite dimensional*. It is *infinite dimensional* otherwise.

One of the main advantages of having a basis for a vector space is that often the calculations can be reduced to calculating the desired properties only for the basis elements and extending the property by linearity. If the vector space is finite dimensional, another advantage of fixing a basis is that the representation of any vector as a linear combination of elements of the basis is unique. This allows to identify the vector with a coordinate vector in \mathbb{F}^n .

THEOREM 1.2. *Let V be a finite dimensional vector space and $\beta = \{u_1, u_2, \dots, u_n\}$ be a subset of V . Then β is a basis for V if and only if each $v \in V$ can be uniquely expressed as a linear combination of vectors of β , that is, can be expressed in the form*

$$v = a_1u_1 + a_2u_2 + \dots + a_nu_n$$

for unique scalars a_1, a_2, \dots, a_n .

PROOF. We show the implication first. By the definition of a basis, it is clear that $v \in V$ implies that there exists constants $a_1, \dots, a_n \in \mathbb{F}$ such that

$$v = a_1u_1 + a_2u_2 + \dots + a_nu_n.$$

Suppose that there exists constants $b_1, \dots, b_n \in \mathbb{F}$ such that

$$v = a_1u_1 + a_2u_2 + \dots + a_nu_n = b_1u_1 + b_2u_2 + \dots + b_nu_n.$$

Then, subtracting $b_1u_1 + b_2u_2 + \dots + b_nu_n$ on both sides, by the distributive property of the scalar multiplication, we obtain that

$$(a_1 - b_1)u_1 + (a_2 - b_2)u_2 + \dots + (a_n - b_n)u_n = 0.$$

However, the linear independence of the set β implies that

$$a_1 - b_1 = \dots = a_n - b_n = 0.$$

This implies the uniqueness of the expression

$$v = a_1u_1 + a_2u_2 + \dots + a_nu_n.$$

For the converse, it is clear that the hypothesis implies $\text{span}(\beta) = V$. Now, note that

$$0 = 0u_1 + \dots + 0u_n.$$

Since, this representation of the zero vector is unique, any other null linear combination must have zero coefficients. Hence, β is linearly independent, thusly a basis. \square

Suppose that the vector space V has a basis β and consider a proper subset $\gamma \subset \beta$. The set $\text{span}(\gamma)$ is a vector space in itself. Moreover, $\text{span}(\gamma) \subset V$. A subset of a vector space that is a vector space in its own right, is called a subspace.

DEFINITION 1.8 (Subspace). *Let V be a vector space over a field \mathbb{F} . A subset $W \subset V$ is called a subspace of V if and only if W is a vector space over \mathbb{F} with respect to the same operations as V*

Instead of verifying that a prospective subspace satisfies the eight axioms that define a vector space, it is convenient to note that by virtue of being a subset of a vector space, many of the properties of the respective operations are preserved. The

following theorem gives us a way to simplify the process of proving that a given set forms a subspace. The proof is elementary, hence it is omitted.

THEOREM 1.3. *A subset $W \subset V$ is a subspace of V if and only if $0 \in W$ and $\alpha u + v \in W$ whenever $\alpha \in \mathbb{F}$ and $u, v \in W$.*

In the category of vector spaces, the morphisms are linear transformations or linear maps. The following definition introduces the concept.

DEFINITION 1.9 (Linear Transformation/Map). *Let V and W be vector spaces (over \mathbb{F}). We call a function $T : V \rightarrow W$ a linear transformation from V to W if, for all $x, y \in V$ and $c \in \mathbb{F}$, we have*

- (1) $T(x + y) = T(x) + T(y)$ and
- (2) $T(cx) = cT(x)$.

It is common to simply call T linear. It is important to also have the following properties to prove linearity.

- (1) If T is linear, then $T(0) = 0$.
- (2) T is linear if and only if $T(cx + y) = cT(x) + T(y)$ for all $x, y \in V$ and $c \in \mathbb{F}$.
- (3) If T is linear, then $T(x - y) = T(x) - T(y)$ for all $x, y \in V$.
- (4) T is linear if and only if, for $x_1, x_2, \dots, x_n \in V$ and $a_1, a_2, \dots, a_n \in \mathbb{F}$, we have

$$T\left(\sum_{i=1}^n a_i x_i\right) = \sum_{i=1}^n a_i T(x_i).$$

A generalization of linearity to functions on two variables is the idea of bilinearity. We shall use this concept below, so we record the definition here.

DEFINITION 1.10 (Bilinear Map). *Let U, V, W be vector spaces over a field \mathbb{F} . A map $H : U \times V \rightarrow W$ of ordered pairs of vectors to \mathbb{F} is called a bilinear form on V if*

- (1) $H(ax_1 + x_2, y) = aH(x_1, y) + H(x_2, y)$ for all $x_1, x_2 \in U$ $y \in V$ and $a \in \mathbb{F}$.
- (2) $H(x, ay_1 + y_2) = aH(x, y_1) + H(x, y_2)$ for all $x \in U$, $y_1, y_2 \in V$ and $a \in \mathbb{F}$.

EXAMPLE 1.1 (\mathbb{R}^n).

Consider, for simplicity, the real coordinate space in three dimensions \mathbb{R}^3 . This can be written as $\mathbb{R}^3 = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ or $(a \ b \ c)$, where $a, b, c \in \mathbb{R}$. Note that \mathbb{R} is in fact a field, and \mathbb{R}^3 is a vector space. Consider four vectors in \mathbb{R}^3 , namely

$$S = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\}.$$

It is simple to verify that $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 1 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, and therefore $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ is linearly dependent in the set. The subset

$$(1.1) \quad S' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}$$

is linearly independent. Since the linear combination $a \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + c \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ for any $a, b, c \in \mathbb{R}$, S' is a basis for \mathbb{R}^3 . In fact, this specific basis is referred to as the *standard basis*, and the basis vectors are written as $\{e_1, e_2, e_3\}$, where e_i is a vector with a 1 in the i -th row and 0's in all others. Notice, however, that the subset $S'' = \left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix} \right\}$ is also a basis for \mathbb{R}^3 , as $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = -1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} - 1 \begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$. In fact, there are infinitely many basis for this vector space. However, notice how $|S'| = |S''| = 3$. This is always the case, any two bases of a finite dimensional vector space always have the same cardinality. This is called the dimension of the vector space.

EXAMPLE 1.2 ($M_n(\mathbb{F})$).

Let $M_n(\mathbb{F})$ denote the set of all $n \times n$ matrices with entries in \mathbb{F} . Then, $M_n(\mathbb{F})$ is a vector space with the coordinate-wise addition and coordinate-wise scalar multiplication.

DEFINITION 1.11 (Dimension). *Let V be a vector space. The dimension of the vector space, $\dim V$, is defined as the cardinality of any of its bases.*

EXAMPLE 1.3 (Homogeneous Differential Equations).

In this example we illustrate the fact that solution spaces of linear differential equations form subspaces. In this case, a finite dimensional subspace of the space of

two times differentiable functions. Consider the set

$$\mathcal{S} = \left\{ f \in C^2(\mathbb{R}) \mid \frac{d^2 f}{dx^2} + f = 0 \right\}.$$

We show that \mathcal{S} is a subspace of the vector space $C^2(\mathbb{R})$ of two times continuously differentiable functions.

As it is clear that $0 \in \mathcal{S}$, it suffices to show that $\alpha f + g \in \mathcal{S}$ whenever $\alpha \in \mathbb{R}$ and $f, g \in \mathcal{S}$. So, note that

$$\frac{d^2}{dx^2}(\alpha f + g) + \alpha f + g = \alpha \left(\frac{d^2 f}{dx^2} + f \right) + \frac{d^2 g}{dx^2} + g = 0.$$

Hence, \mathcal{S} is a subspace.

This fact will be useful for us, as our goal is to realize the space of solutions of a general linear differential equation as a representation of a Lie group/algebra. Representations are, roughly, vector spaces on which a linear action of these algebraic structures can be defined.

2. Lie Algebras

In this section, we introduce the definition and results in the representation theory of Lie algebras to be used later in this thesis. This section is standard, so may be omitted by the expert reader.

DEFINITION 2.1 (Lie Algebra). *A finite dimensional real or complex Lie algebra is a finite-dimensional real or complex vector space \mathfrak{g} , together with a map $[\cdot, \cdot]$ from $\mathfrak{g} \times \mathfrak{g}$ into \mathfrak{g} , with the following properties:*

- (1) $[\cdot, \cdot]$ is bilinear.
- (2) $[\cdot, \cdot]$ is skew symmetric: $[X, Y] = -[Y, X]$ for all $X, Y \in \mathfrak{g}$.
- (3) The Jacobi Identity holds:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0$$

for all $X, Y, Z \in \mathfrak{g}$

Two elements X and Y of a Lie algebra \mathfrak{g} *commute* if $[X, Y] = 0$. A Lie algebra \mathfrak{g} is *commutative* if $[X, Y] = 0$ for all $X, Y \in \mathfrak{g}$. The map $[\cdot, \cdot]$ is referred to as the *bracket operation* on \mathfrak{g} .

EXAMPLE 2.1 (General linear Lie algebra, $\mathfrak{gl}(2; \mathbb{F})$).

Let \mathbb{F} be \mathbb{R} or \mathbb{C} . Then,

$$\mathfrak{gl}(2; \mathbb{F}) = M_2(\mathbb{F})$$

is a Lie algebra with the bracket operation defined by

$$[X, Y] = XY - YX$$

for all $X, Y \in \mathfrak{gl}(2; \mathbb{F})$.

As in the case of vector spaces and subspaces, a subset of a Lie algebra that forms a Lie algebra with respect to the same operations is called a Lie subalgebra. The formal definition is given below.

DEFINITION 2.2 (Lie Subalgebra). *A subalgebra of a real or complex Lie algebra \mathfrak{g} is a vector subspace \mathfrak{h} of \mathfrak{g} such that $[H_1, H_2] \in \mathfrak{h}$ for all H_1 and $H_2 \in \mathfrak{h}$. If \mathfrak{g} is a complex Lie algebra and \mathfrak{h} is a real subspace of \mathfrak{g} which is closed under brackets, then \mathfrak{h} is said to be a real subalgebra of \mathfrak{g} .*

EXAMPLE 2.2 (Special linear Lie algebra, $\mathfrak{sl}(2; \mathbb{F})$).

Let \mathbb{F} be \mathbb{R} or \mathbb{C} . Then,

$$\mathfrak{sl}(2; \mathbb{F}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in \mathbb{F} \right\}$$

is a Lie subalgebra of $\mathfrak{gl}(2; \mathbb{F})$.

A special example of a subalgebra (in fact an ideal) of a Lie algebra is its center. This is the subalgebra of elements that commute with all other elements of the Lie algebra.

DEFINITION 2.3 (Center). *The center of a Lie algebra \mathfrak{g} is the set of all $X \in \mathfrak{g}$ for which $[X, Y] = 0$ for all $Y \in \mathfrak{g}$.*

The morphisms in this category are called homomorphisms. They are essentially linear transformations that play nicely with the bracket operation.

DEFINITION 2.4 (Lie Algebra Homomorphism). *If \mathfrak{g} and \mathfrak{h} are Lie algebras, then a linear map $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$ is called a Lie algebra homomorphism if $\phi([X, Y]) = [\phi(X), \phi(Y)]$ for all $X, Y \in \mathfrak{g}$.*

If, in addition, ϕ is one-to-one and onto, then ϕ is called a Lie algebra isomorphism. A Lie algebra isomorphism of a Lie algebra with itself is called a Lie algebra automorphism.

DEFINITION 2.5 (Adjoint Map / Adjoint Representation). *If \mathfrak{g} is a Lie algebra and X is an element of \mathfrak{g} , define a linear map $\text{ad}_X : \mathfrak{g} \rightarrow \mathfrak{g}$ by*

$$\text{ad}_X(Y) = [X, Y]$$

The map $X \mapsto \text{ad}_X$ is the adjoint map or adjoint representation

PROPOSITION 1. *If \mathfrak{g} is a Lie algebra, then*

$$\text{ad}_{[X, Y]} = \text{ad}_X \text{ad}_Y - \text{ad}_Y \text{ad}_X = [\text{ad}_X, \text{ad}_Y]$$

that is, $\text{ad} : \mathfrak{g} \rightarrow \text{End}(\mathfrak{g})$ is a Lie algebra homomorphism

PROOF. This follows from the Jacobi identity. Let $Z \in \mathfrak{g}$ be arbitrary. Then,

$$\begin{aligned} \text{ad}_{[X, Y]}(Z) &= [[X, Y], Z] \\ &= [X, [Y, Z]] - [Y, [X, Z]] \\ &= \text{ad}_X \circ \text{ad}_Y(Z) - \text{ad}_Y \circ \text{ad}_X(Z). \end{aligned}$$

□

In this thesis, for decomposition purposes, we use direct and semidirect sums of Lie algebras. We start with the definition of a direct sum.

DEFINITION 2.6 (Direct Sum). *If \mathfrak{g}_1 and \mathfrak{g}_2 are Lie algebras, the direct sum of \mathfrak{g}_1 and \mathfrak{g}_2 is the vector space sum of \mathfrak{g}_1 and \mathfrak{g}_2 , with bracket given by*

$$[(X_1, X_2), (Y_1, Y_2)] = ([X_1, Y_1], [X_2, Y_2])$$

If \mathfrak{g} is a Lie algebra and \mathfrak{g}_1 and \mathfrak{g}_2 are subalgebras, we say that \mathfrak{g} decomposes as the Lie algebra direct sum of \mathfrak{g}_1 and \mathfrak{g}_2 if \mathfrak{g} is the direct sum of \mathfrak{g}_1 and \mathfrak{g}_2 as vector spaces and $[X_1, X_2] = 0$ for all $X_1 \in \mathfrak{g}_1$ and $X_2 \in \mathfrak{g}_2$.

3. Examples of Classical Algebras

In order to understand the subject of Lie algebras, we should learn the general theory, but also investigate many examples. Many of the most important examples make up the classical algebras; the general and special linear Lie algebras, the unitary and orthogonal Lie algebras, and the symplectic Lie algebra.

EXAMPLE 3.1 (General linear Lie algebra, $\mathfrak{gl}(n; \mathbb{F})$).

Let \mathbb{F} be \mathbb{R} or \mathbb{C} . Then,

$$\mathfrak{gl}(n; \mathbb{F}) = M_n(\mathbb{F})$$

is a Lie algebra with the bracket operation defined by

$$[X, Y] = XY - YX$$

for all $X, Y \in \mathfrak{gl}(n; \mathbb{F})$.

The remaining examples that we will investigate are Lie subalgebras of $\mathfrak{gl}(n; \mathbb{F})$

EXAMPLE 3.2 (Special linear Lie algebra, $\mathfrak{sl}(n; \mathbb{F})$).

Let \mathbb{F} be \mathbb{R} . Then, $\mathfrak{sl}(n; \mathbb{F})$ is a Lie algebra of real matrices with trace zero.

For any n , $\mathfrak{sl}(n; \mathbb{F})$ is a Lie subalgebra of $\mathfrak{gl}(n; \mathbb{F})$

EXAMPLE 3.3 (Orthogonal Lie algebra, $\mathfrak{o}(n, k)$).

If J is the $(n+k) \times (n+k)$ diagonal matrix with the first n diagonal entries equal to 1 and the last k diagonal entries equal to minus one:

$$J = \begin{pmatrix} I_n & 0 \\ 0 & -I_k \end{pmatrix}$$

Then the Lie algebra $\mathfrak{o}(n, k)$ consists precisely of those matrices $X \in M_{n+k}(\mathbb{R})$ such that

$$JX^{tr}J = -X.$$

EXAMPLE 3.4 (Special Unitary Lie algebra, $\mathfrak{su}(n; \mathbb{F})$).

Let \mathbb{F} be \mathbb{R} or \mathbb{C} . The lie algebra $su(n)$ consists of the $n \times n$ skew-Hermitian matrices with trace zero.

4. Representations of Lie Algebras

The main idea of Lie theory is to study different representations of Lie groups and Lie algebras. There is an intrinsic connection between the representations of these corresponding objects. In this section we introduce the definition of a representation of a Lie algebra, a few results, and we construct an irreducible representation of $\mathfrak{sl}(2; \mathbb{R})$ which serves as the prototype for the theory of representations of Lie algebras.

DEFINITION 4.1 (Representation of a Lie algebra). *Let V be a vector space over the field \mathbb{R} or \mathbb{C} . A representation of a Lie algebra \mathfrak{g} is a pair (V, φ) where φ is a homomorphism $\varphi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ such that*

$$\varphi([A, B]) = [\varphi(A), \varphi(B)] = \varphi(A)\varphi(B) - \varphi(B)\varphi(A)$$

DEFINITION 4.2 (\mathfrak{g} -module). *Let \mathfrak{g} be a Lie algebra over a field k and V be a vector space. A \mathfrak{g} -module is a pair (V, \cdot) where $\cdot : V \rightarrow V$ is a map satisfying the following conditions:*

For all $x, y \in \mathfrak{g}; v, w \in V; \lambda, \mu \in k$

- (1) $(\lambda x + \mu y) \cdot v = \lambda(x \cdot v) + \mu(y \cdot v)$
- (2) $x \cdot (\lambda v + \mu w) = \lambda(x \cdot v) + \mu(x \cdot w)$
- (3) $[x, y] \cdot v = x(y \cdot v) - y(x \cdot v)$

In essence, there is no difference between representations of \mathfrak{g} and \mathfrak{g} -modules. The choice to use the language of representations or the language of modules is one of convenience.

LEMMA 1. *Given a representation $\varphi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$, there is a \mathfrak{g} -module (V, \cdot) defined in the natural way, by defining for all $x \in \mathfrak{g}, v \in V$*

$$x \cdot v = \varphi(x)(v)$$

PROOF. Assume (V, φ) is a representation. We verify that (V, \cdot) is a \mathfrak{g} -module

- (1) $(\lambda x + \mu y) \cdot v = \varphi(\lambda x + \mu y)(v) = (\lambda\varphi(x) + \mu\varphi(y))(v)$
 $= \lambda\varphi(x)(v) + \mu\varphi(y)(v) = \lambda(x \cdot v) + \mu(y \cdot v)$
- (2) $x \cdot (\lambda v + \mu w) = \varphi(x)(\lambda v + \mu w) = \lambda\varphi(x)(v) + \mu\varphi(x)(w) = \lambda(x \cdot v) + \mu(x \cdot w)$
- (3) $[x, y] \cdot v = \varphi([x, y])(v) = [\varphi(x), \varphi(y)](v) = \varphi(x)(\varphi(y)(v)) - \varphi(y)(\varphi(x)(v)) =$
 $x \cdot (y \cdot v) - y \cdot (x \cdot v).$

Conversely, if (V, \cdot) is a \mathfrak{g} -module, we define $\varphi : \mathfrak{g} \rightarrow \mathfrak{gl}(V)$ by the linear map $\phi(x) \cdot v \mapsto x \cdot v$, so

$$AB - BA = \varphi(AB - BA) = \varphi([A, B]) = [A, B]$$

shows the representation is valid. □

For this reason, unless necessary for clarity, we may refer to the representation V , with the representation homomorphism φ and the \mathfrak{g} -module operation \cdot implied.

DEFINITION 4.3 (Invariant Subspace). *Let V be a \mathfrak{g} -module, a submodule, or subrepresentation of V is a subspace $W \subset V$ which is invariant under the action of \mathfrak{g} i.e., for all $x \in \mathfrak{g}$ and $w \in W$, $x \cdot w \in W$.*

DEFINITION 4.4 (Irreducible Representation). *A representation V of a Lie algebra \mathfrak{g} is said to be irreducible, or simple, if it is non-zero and has no submodules other than $\{0\}$ and V itself.*

DEFINITION 4.5 (Completely Reducible Representation). *A completely reducible representation or semi-simple representation is a representation that can be written as the direct sum of simple, irreducible representations.*

At this point, we should examine

$$\mathfrak{sl}(2; \mathbb{F}) = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \mid a, b, c \in \mathbb{F} \right\}$$

a little closer. First, if we notice that $\mathfrak{sl}(2; \mathbb{F})$ is spanned by the three elements

$$E = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad H = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

as

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix} = aH + bE + cF.$$

These three basis elements generate $\mathfrak{sl}(2; \mathbb{F})$. Furthermore, they satisfy the following bracket operations

$$\begin{aligned} [E, F] &= EF - FE = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = H, \\ [E, H] &= EH - HE = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = -2E, \\ [F, H] &= FH - HF = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = -2E, \end{aligned}$$

and $[X, Y] = -[Y, X]$ for all X, Y in $\mathfrak{sl}(2; \mathbb{F})$ implies that $\mathfrak{sl}(2; \mathbb{F})$ is closed under the bracket operation.

DEFINITION 4.6 (Weight Spaces and Weight Vectors of a Representation). *Let \mathfrak{g} be a Lie algebra over a field k . Let V be a finite dimensional representation of $\mathfrak{sl}(2; \mathbb{F})$ and consider V as a \mathfrak{g} -module, as described above. Let $\lambda \in k$. We can define a vector subspace of V by*

$$V_\lambda = \{v \in V \mid h \cdot v = \lambda v\}$$

If λ is not an eigenvalue of $\varphi(H)$ then $V_\lambda = \{0\}$. Whenever $V_\lambda \neq \{0\}$ we call λ a weight of H in V and V_λ its associated weight space.

LEMMA 2. *Let $\{H, E, F\}$ be the standard basis for $\mathfrak{sl}(2)$. Suppose that v is a weight vector of H with weight λ . Then, $E.v$ (resp. $F.v$) is a weight vector of H with weight $\lambda + 2$ (resp. $\lambda - 2$).*

PROOF. $[H, E] = 2E$, so for any $v \in V$ we have $H(Ev) = (EH + 2E)v = \lambda(Ev) + 2(Ev) = (2 + \lambda)Ev$. Therefore either $Ev = 0$ or Ev is an eigenvector of H . Similarly, $[H, F] = -2F$, so for any $v \in V$ we have $(H(Fv)) = (FH - 2F)v = \lambda(Fv) - 2(Fv) = (\lambda - 2)Fv$.

□

From this basic fact, we can completely characterize the finite dimensional irreducible representations of $\mathfrak{sl}(2; \mathbb{F})$. Let $V_\mu := \{v \in V \mid H.v = \mu v\}$ denote the weight space relative to H with weight $\mu \in \mathbb{Z}$.

THEOREM 4.1. *Let V be a finite dimensional irreducible representation of $\mathfrak{sl}(2; \mathbb{R})$. Then,*

$$V = \bigoplus_{j=0}^m V_{-m+2j},$$

where $m = \dim V - 1$.

EXAMPLE 4.1 (V_m and the Classification of irreducible $\mathfrak{sl}(2)$ modules).

If we consider the vector space $\mathbb{R}[X, Y]$, the ring of polynomials in two variables, with coefficients in \mathbb{R} , consisting of all possible sums of the form $\sum_{i=0}^m a_i X^i Y^{m-i}$, where $a_i \in \mathbb{R}$. X and Y are base vectors, which give the vector space of all the linear combinations of X and Y : $\mathbb{R}X + \mathbb{R}Y \subseteq \mathbb{R}[X, Y]$. We begin by constructing a family of irreducible representations of $\mathfrak{sl}(2; \mathbb{F})$. For each integer $m \geq 1$, let V_m be the subspace of homogeneous polynomials in X and Y of degree m . For $m = 0$, the vector space V_0 consists of the constant polynomial, thus $\dim(V_0) = 1$, and for $m \geq 1$ the space has the following basis:

$$\mathfrak{B} = \{X^m, X^{m-1}Y, \dots, XY^{m-1}, Y^m\}.$$

This is a vector space of dimension $m + 1$. In order to realize V_m as an $\mathfrak{sl}(2, \mathbb{R})$ -module we specify a Lie algebra homomorphism $\varphi : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \mathfrak{gl}(V_m)$. In this way, we realize all the irreducible representations of $\mathfrak{sl}(2, \mathbb{R})$, since for each degree, we will find the representation for it. By specifying where φ sends each of the basis elements, we realize the impact of φ on $\mathfrak{sl}(2, \mathbb{R})$. Define:

$$(4.1) \quad \varphi(E) = X \frac{\partial}{\partial Y}$$

$$(4.2) \quad \varphi(F) = Y \frac{\partial}{\partial X}$$

$$(4.3) \quad \varphi(H) = X \frac{\partial}{\partial X} - Y \frac{\partial}{\partial Y}$$

Consider the homomorphism on an arbitrary coordinate of the polynomial, $X^i Y^j$. Then we have the following relations:

$$(4.4) \quad \varphi(E)(X^i Y^j) = j X^{i+1} Y^{j-1}$$

$$(4.5) \quad \varphi(F)(X^i Y^j) = i X^{i-1} Y^{j+1}$$

$$(4.6) \quad \varphi(H)(X^i Y^j) = (a - b) X^i Y^j$$

Notice that the overall degree of the polynomial is therefore preserved.

LEMMA 3. *The preceding relations make V_m an $\mathfrak{sl}(2, \mathbb{R})$ representation*

We omit the details of the proof, but the reader can easily check that

$$[\varphi(E), \varphi(F)] = \varphi([E, F]) = \varphi(H),$$

$$[\varphi(H), \varphi(E)] = \varphi([H, E]) = 2\varphi(E),$$

$$[\varphi(H), \varphi(F)] = -2\varphi(F).$$

We can also realize the action of E, F , and H on V_m by matrices with respect to the basis $X^m, X^{m-1}Y, \dots, XY^{m-1}, Y^m$.

In order to underscore the flexibility of Lie algebras, we will visualize V_m in terms of $(m+1)$ -dimensional matrices with respect to the basis $\mathfrak{B} = X^m, X^{m-1}Y, \dots, XY^{m-1}, Y^m$

$$\varphi_m(E) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & m \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

$$\varphi_m(F) = \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ m & 0 & \dots & 0 & 0 \\ 0 & m-1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\varphi_m(H) = \begin{pmatrix} m & 0 & \dots & 0 & 0 \\ 0 & m-2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & -m+2 & 0 \\ 0 & 0 & \dots & 0 & -m \end{pmatrix}.$$

CHAPTER 3

Lie Groups

Lie groups are essentially groups that can be endowed with a manifold structure that makes the group multiplication and inversion into smooth operations. As such, there is a heavy reliance on the topological properties of the manifold. For example, each Lie group has a Lie algebra associated to it. This Lie algebra is uniquely defined as the tangent space to the manifold at the identity as its base point. However, given a Lie algebra, it is not immediately clear how to choose a corresponding Lie group. One way to go about this is by looking at the one-parameter group generated by each basis element of the Lie algebra. However, this procedure would only generate the connected component of the identity even if the group is disconnected.

In this section, we look at the basic concepts of Lie theory that will be needed to tackle our main problem. The definitions here come from [4] and

1. Review of Group Theory and Manifolds

We start our study of Lie groups with the algebraic structure that underlies the definition of a Lie group, namely, groups.

DEFINITION 1.1 (Group). A **binary operation** on a set G is a function $G \times G \rightarrow G$ that assigns to each pair $(a, b) \in G \times G$ a unique element $a \cdot b$, or ab in G , called the product of a and b . A **group** (G, \cdot) is a set G together with a product $(a, b) \mapsto a \cdot b$ that satisfies the following axioms:

(1) The product is associative. That is,

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

(2) There exists an element $e \in G$, called the **identity element**, such that for any element $a \in G$

$$e \cdot a = a \cdot e = a$$

- (3) For each element $a \in G$, there exists an **inverse element** in G , denoted by a^{-1} , such that

$$a \cdot a^{-1} = a^{-1} \cdot a = e$$

A group G with the property that $a \cdot b = b \cdot a$ for all $a, b \in G$ is called **Abelian**. We write ab instead of $a \cdot b$, unless the operation needs to be specified to avoid confusion.

DEFINITION 1.2 (Subgroup). A **subgroup** H of a group G is a subset H of G such that when the group operation of G is restricted to H , H is a group in its own right.

EXAMPLE 1.1 (General Linear Group $\text{GL}(2; \mathbb{F})$).

For any field \mathbb{F} , the set

$$\text{GL}(2; \mathbb{F}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc \neq 0 \in \mathbb{F} \right\}$$

with the usual matrix multiplication is a group. Note that the condition of the nonzero determinant is needed to ensure that inverses exist.

EXAMPLE 1.2 (Special Linear Group $\text{SL}(2; \mathbb{F})$).

For any field \mathbb{F} , the set

$$\text{SL}(2; \mathbb{F}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid ad - bc = 1 \in \mathbb{F} \right\}$$

is a subgroup of the general linear group $\text{GL}(2; \mathbb{F})$.

DEFINITION 1.3 (Cosets). Let G be a group and H a subgroup of G . We define a **left coset** of H with **representative** $g \in G$ to be the set

$$gH = \{gh : h \in H\}.$$

Right cosets can be defined similarly by

$$Hg = \{hg : h \in H\}$$

DEFINITION 1.4 (Normal Subgroup). *A subgroup H of a group G is **normal** in G if $gH = Hg$ for all $g \in G$. That is, a normal subgroup of a group G is one in which the right and left cosets are precisely the same.*

DEFINITION 1.5 (Direct Products). *Let G be a group and H and K subgroups satisfying:*

- (1) H and K are normal in G
- (2) $H \cap K = e$
- (3) $HK = G$

Then we say G is isomorphic to the direct product of H and K , or $G \cong H \times K$.

DEFINITION 1.6 (Semidirect Products). *Suppose now we relax the first condition. Let G be a group and H and K subgroups satisfying:*

- (1) H is normal in G
- (2) $H \cap K = e$
- (3) $HK = G$

Then we say G is isomorphic to the semidirect product of H and K , or $G \cong H \rtimes K$.

DEFINITION 1.7 (Manifold). *A manifold is an object M that looks locally like a piece of \mathbb{R}^n . More precisely, an n -dimensional manifold is a second-countable, Hausdorff topological space with the property that each $m \in M$ has a neighborhood that is homeomorphic to an open subset of \mathbb{R}^n .*

A two-dimensional torus, for example looks a locally but not globally like \mathbb{R}^2 and is, thus, a two-dimensional manifold.

DEFINITION 1.8 (Smooth Manifold). *A smooth manifold is a manifold M together with a collection of local coordinates covering M such that the change-of-coordinates map between two overlapping coordinate systems is smooth.*

DEFINITION 1.9 (Lie Group). *A Lie group is a smooth manifold G which is also a group and such that the product*

$$G \times G \rightarrow G$$

and the inverse map $G \rightarrow G$ are smooth.

2. Matrix Lie Groups

EXAMPLE 2.1 (General Linear Groups over \mathbb{R} and \mathbb{C}).

The general linear group over the real numbers, denoted $GL(n; \mathbb{R})$, is the group of all $n \times n$ invertible matrices with real entries. The *general linear group* over the complex numbers, denoted $GL(n; \mathbb{C})$, is the group of all $n \times n$ invertible matrices with complex entries.

Let $M_n(\mathbb{C})$ denote the space of all $n \times n$ matrices with complex entries. Let A_m be a sequence of complex matrices in $M_n(\mathbb{C})$. We say that A_m *converges* to a matrix A if each entry of A_m converges (as $m \rightarrow \infty$) to the corresponding entry of A (i.e. if $(A_m)_{kl}$ converges to A_{kl} for all $1 \leq k, l \leq n$).

DEFINITION 2.1 (Matrix Lie Group). A matrix Lie group is a subgroup G of $GL(n; \mathbb{C})$ with the following property: if A_m is any sequence of matrices in G , and A_m converges to some matrix A then either $A \in G$, or A is not invertible.

This is equivalent to saying that a matrix Lie group is a *closed subgroup* of $GL(n; \mathbb{C})$. For our specific analysis, we have a stronger property that if A_m is any sequence of matrices in G and A_m converges to some matrix A , then $A \in G$ (i.e., that G is closed in $M_n(\mathbb{C})$).

EXAMPLE 2.2 (The Special Linear group $SL(n; \mathbb{C})$).

The special linear group $SL(n; \mathbb{C})$ is the set of $n \times n$ matrices with determinant 1.

EXAMPLE 2.3 (The Unitary group $U(n)$).

An $n \times n$ complex matrix A is said to be unitary if the column vectors of A are orthonormal, that is, if

$$(2.1) \quad \sum_{l=1}^n \bar{A}_{lj} A_{lk} = \delta_{jk}.$$

We may rewrite the above equation as $\sum_{l=1}^n (A^*)_{jl} A_{lk} = \delta_{jk}$, where δ_{jk} is the Kronecker delta, equal to 1 if $j = k$ and equal to zero if $j \neq k$. Here A^* is the adjoint of

A , defined by $(A^*)_{jk} = \bar{A}_{kj}$. Since this implies $A^*A = I$, we have that A is unitary if and only if $A^* = A^{-1}$, which is to say that $U(n)$ is a subset of $GL(n)$, or invertible.

EXAMPLE 2.4 (The Special Unitary group $SU(n; \mathbb{C})$).

The special unitary group $SU(n)$ is the subgroup of $U(n)$ consisting of unitary matrices with determinant 1.

Note that $SU(n; \mathbb{C})$ is a subgroup of both $SL(n; \mathbb{C})$ and $U(n)$.

EXAMPLE 2.5 (The Orthogonal group $O(n)$).

The orthogonal group $O(n)$ is the group of distance preserving transformations of a Euclidean space, consisting of the set of $n \times n$ matrices whose inverse equal their transpose, that is

$$O(n) = \{Q \in GL(n) | Q^T = Q^{-1}\}$$

The subset of $O(n)$ with determinant 1 is also a matrix Lie group, $O(n)$, called the special orthogonal group.

DEFINITION 2.2 (The generalized orthogonal group $O(n; k)$). *Let n and k be positive integers, and consider \mathbb{R}^{n+k} . Define a symmetric bilinear form $[\cdot, \cdot]_{n,k}$ on \mathbb{R}^{n+k} by the formula*

$$[x, y]_{n,k} = x_1y_1 + \cdots + x_ny_n - x_{n+1}y_{n+1} - \cdots - x_{n+k}y_{n+k}$$

The set of $(n+k) \times (n+k)$ real matrices A which preserve this form (i.e., such that $[Ax, Ay]_{n,k} = [x, y]_{n,k}$ for all $x, y \in \mathbb{R}^{n+k}$ is the generalized orthogonal group $O(n; k)$. It is a subgroup of $GL(n+k; \mathbb{R})$ and a matrix Lie group.

If A is an $(n+k) \times (n+k)$, let $A^{(i)}$ denote the i^{th} column vector of A , that is

$$A^{(i)} = \begin{pmatrix} A_{1,i} \\ \vdots \\ A_{n+k,i} \end{pmatrix}.$$

Then, A is in $O(n; k)$ if and only if the following conditions are satisfied:

$$\begin{aligned} [A^{(l)}, A^{(j)}]_{n,k} &= 0 & l \neq j, \\ [A^{(l)}, A^{(l)}]_{n,k} &= 1 & 1 \leq l \leq n, \\ [A^{(l)}, A^{(l)}]_{n,k} &= -1 & n+1 \leq l \leq n+k, \end{aligned}$$

Let g denote the $(n+k) \times (n+k)$ diagonal matrix with ones in the first n diagonal entries and minus ones in the last k diagonal entries. Then, A is in $O(n; k)$ if and only if $A^{tr}gA = g$. Taking the determinant of this equation gives $(\det A)^2 \det g = \det g$, or $(\det A)^2 = 1$. Thus, for any A in $O(n; k)$, $\det A = \pm 1$. Of particular interest in physics is the *Lorentz group* $O(3; 1)$.

The group \mathbb{C}^* of nonzero complex numbers under multiplication is isomorphic to $GL(1, \mathbb{C})$, thus we will regard \mathbb{C}^* as a matrix Lie group.

DEFINITION 2.3 (Euclidean Group $E(n)$). *The Euclidean Group $E(n)$ is the group of all one-to-one, onto, distance-preserving maps of \mathbb{R}^n to itself, that is, maps $f : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that*

$$d(f(x), f(y)) = d(x, y) = |x - y|$$

for all $x, y \in \mathbb{R}^n$.

DEFINITION 2.4 (Translation). *For $x \in \mathbb{R}^n$, define the translation by x , denoted T_x , by*

$$T_x(y) = x + y$$

The set of translations is also a subgroup of $E(n)$

A very important group in physics is the Poincaré group. We spend some time studying this important group.

DEFINITION 2.5 (Poincaré Group). *We define the Poincaré Group $P(n; 1)$ to be the group of all transformations of \mathbb{R}^{n+1} of the form*

$$T = T_x A$$

with $x \in \mathbb{R}^{n+1}$ and $A \in O(n; 1)$.

This is the group of affine transformations of \mathbb{R}^{n+1} which preserve the Lorentz "distance"

$$d_L(x, y) = (x_1 - y_1)^2 + \cdots + (x_n - y_n)^2 - (x_{n+1} - y_{n+1})^2$$

An affine transformation is one of the form $x \rightarrow Ax + b$, where A is a linear transformation and b is constant. In the case of the Poincaré group, the transformation belongs to the generalized orthogonal group for the bilinear form with signature $(n, 1)$. In the following proposition, we study certain properties of the most general real orthogonal group.

PROPOSITION 2. *Let $[\cdot, \cdot]_{n,k}$ be the symmetric bilinear form on \mathbb{R}^{n+k} preserved by the generalized orthogonal group $O(n; k)$. Let g be the $(n+k) \times (n+k)$ diagonal matrix with first n diagonal entries equal to one and last k diagonal entries equal to minus one:*

$$g = \begin{pmatrix} I_n & 0 \\ 0 & -I_k \end{pmatrix}.$$

Then,

(1) For all $x, y \in \mathbb{R}^{n+k}$,

$$[x, y]_{n,k} = \langle x, gy \rangle$$

(2) An $(n+k) \times (n+k)$ real matrix A is in $O(n; k)$ if and only if $A^{\text{tr}} g A = g$

(3) $O(n; k)$ and $SO(n; k) := O(n; k) \cap SL(n+k; \mathbb{R})$ are subgroups of $GL(n+k; \mathbb{R})$ and are matrix Lie groups.

Proof:

(1) Remember the definition of $[x, y]_{n,k}$ provided in the definition of the generalized

orthogonal group. Let $x, y \in \mathbb{R}^{n+k}$. Then

$$\begin{aligned} \langle x, gy \rangle &= x_1gy_1 + x_2gy_2 + \cdots + x_ngy_n + x_{n+1}gy_{n+1} + \cdots + x_{n+k}gy_{n+k} \\ &= x_1y_1 + x_2y_2 + \cdots + x_ny_n - x_{n+1}y_{n+1} - \cdots - x_{n+k}y_{n+k} = [x, y]_{n,k} \end{aligned}$$

(2) (\Rightarrow) Let $A \in O(n; k)$. Let $A^{(i)}$ denote the i^{th} column vector of A . Then we know

$$\begin{aligned} [A^{(l)}, A^{(j)}]_{n,k} &= 0 & l \neq j, \\ [A^{(l)}, A^{(l)}]_{n,k} &= 1 & l \leq l \leq n, \\ [A^{(l)}, A^{(l)}]_{n,k} &= -1 & n+l \leq l \leq n+k \end{aligned}$$

By (1), this means that

$$\begin{aligned} \langle A^{(l)}, gA^{(j)} \rangle &= 0 & l \neq j, \\ \langle A^{(l)}, gA^{(l)} \rangle &= 1 & l \leq l \leq n, \\ \langle A^{(l)}, gA^{(l)} \rangle &= -1 & n+l \leq l \leq n+k \end{aligned}$$

Notice that this implies

$$A^T gA = \begin{pmatrix} (A^{(1)})^T gA^{(1)} & (A^{(1)})^T gA^{(2)} & \cdots & (A^{(1)})^T gA^{(l)} \\ (A^{(2)})^T gA^{(1)} & (A^{(2)})^T gA^{(2)} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ (A^{(l)})^T gA^{(1)} & (A^{(l)})^T gA^{(2)} & \cdots & (A^{(l)})^T gA^{(l)} \end{pmatrix}.$$

Since $(A^{(j)})^T gA^{(k)} = \langle A^{(j)}, gA^{(k)} \rangle = [A^{(j)}, A^{(k)}]_{n,k}$, we have that

$$\begin{aligned}
A^T g A &= \begin{pmatrix} [A^{(1)}, A^{(1)}]_{n,k} & [A^{(1)}, A^{(2)}]_{n,k} & \dots & [A^{(1)}, A^{(l)}]_{n,k} \\ [A^{(2)}, A^{(1)}]_{n,k} & [A^{(2)}, A^{(2)}]_{n,k} & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ [A^{(l)}, A^{(1)}]_{n,k} & [A^{(l)}, A^{(2)}]_{n,k} & \dots & [A^{(l)}, A^{(l)}]_{n,k} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & -1 \end{pmatrix} = g.
\end{aligned}$$

(\Leftarrow) The converse follows at once from the previous calculation.

(3) By (2), we have that $A \in O(n, k)$, and

$$A^T g A = g \Rightarrow (A^T)^{-1} A^T g A = g A = (A^T)^{-1} g$$

We can show that A preserves the form $[Ax, Ay]_{n,k} = [x, y]_{n,k}$ for all $x, y \in \mathbb{R}^{n+k}$.

$$[Ax, Ay]_{n,k} = \langle Ax, gAy \rangle = \langle x, A^T g Ay \rangle = \langle x, gy \rangle = [x, y]_{n,k}$$

PROPOSITION 3. *Every element T of $P(n; 1)$ can be written uniquely as a linear transformation followed by a translation, that is, in the form*

$$T = T_x A$$

with $x \in \mathbb{R}^{n+1}$ and $A \in O(n; 1)$.

We briefly investigate three important topological properties of matrix Lie groups, each of which is satisfied by some groups but not others.

DEFINITION 2.6 (Compactness). *A matrix Lie group $G \subset GL(n; \mathbb{C})$ is said to be compact if it is compact in the usual topological sense as a subset of $M_n(\mathbb{C}) \cong \mathbb{R}^{2n^2}$*

Explicitly, this means that G is compact if and only if whenever $A_m \in G$ and $A_m \rightarrow A$, then A is in G , and there exists a constant C such that for all $A \in G$, we have $|A_{jk}| \leq C$ for all $1 \leq j, k \leq n$.

DEFINITION 2.7 (Connectedness). *A matrix Lie group G is said to be connected if for all A and B in G , there exists a continuous path $A(t)$, $a \leq t \leq b$, lying in G with $A(a) = A$ and $A(b) = B$. For any matrix Lie group G , the identity component of G , denoted G_0 , is the set of $A \in G$ for which there exists a continuous path $A(t)$, $a \leq t \leq b$, lying in G with $A(a) = I$ and $A(b) = A$.*

This property is normally called path connected in topology, but it can be shown that a matrix Lie group is connected if and only if it is path connected.

To show that a matrix Lie group G is connected, it suffices to show that each $A \in G$ can be connected to the identity by a continuous path lying in G .

PROPOSITION 4. *If G is a matrix Lie group, the identity component G_0 of G is a normal subgroup of G .*

PROOF. If A and B are any two elements of G_0 , then there are continuous paths $A(t)$ and $B(t)$ connecting I to A and to B in G . Then the path $A(t)B(t)$ is a continuous path connecting I to AB in G , and $(A(t))^{-1}$ is a continuous path connecting I to A^{-1} in G . Thus, both AB and A^{-1} belong to G_0 , showing that G_0 is a subgroup of G . Now suppose A is in G_0 and B is any element of G . Then there is a continuous path $A(t)$ connecting I to A in G , and the path $BA(t)B^{-1}$ connects I to BAB^{-1} in G . Thus, $BAB^{-1} \in G_0$. \square

DEFINITION 2.8 (Simple Connectedness). *A matrix Lie group G is said to be simply connected if it is connected and, in addition, every loop in G can be shrunk continuously to a point in G .*

More precisely, assume that G is connected. Then G is simply connected if for every continuous path $A(t)$, $0 \leq t \leq 1$, lying in G and with $A(0) = A(1)$, there exists a continuous function $A(s, t)$, $0 \leq s, t \leq 1$, taking values in G and having the following properties:

- (1) $A(s, 0) = A(s, 1)$ for all s ,
- (2) $A(0, t) = A(t)$, and
- (3) $A(1, t) = A(1, 0)$ for all t .

One should think of $A(t)$ as a loop and $A(s, t)$ as a family of loops, parameterized by the variable s which shrinks $A(t)$ to a point. Condition (1) says that for each value of the parameter s , we have a loop; Condition (2) says that when $s = 0$ the loop is the specified loop $A(t)$; and Condition (3) says that when $s = 1$ our loop is a point. The condition of simple connectedness is important because for simply connected groups, there is a particularly close relationship between the group and the Lie algebra.

If a matrix Lie group G is not simply connected, the degree to which it fails to be simply connected is encoded into the fundamental group of G .

3. One Parameter Subgroups

DEFINITION 3.1 (One-Parameter Subgroup). *A function $A : \mathbb{R} \rightarrow \text{GL}(n; \mathbb{C})$ is called a one-parameter subgroup of $\text{GL}(n; \mathbb{C})$ if*

- (1) *A is continuous*
- (2) *$A(0) = I$*
- (3) *$A(t + s) = A(t)A(s)$ for all $t, s \in \mathbb{R}$*

The types of one-parameter subgroups that we are interested in are those generated by elements of the Lie algebra. To do this, we need to define the matrix exponential.

DEFINITION 3.2 (Exponential of a Matrix). *If X is an $n \times n$ matrix, we define the exponential of X , denoted e^X , or \exp^X , by the usual power series*

$$e^X = \sum_{m=0}^{\infty} \frac{X^m}{m!}$$

Something must be said about the convergence of the series. It is, however, a standard argument using the diagonalizable and nilpotent parts of the matrix. Hence, we omit the argument and assume the convergence.

With the exponential in hand, we can define a one parameter subgroup in the following way. Assume $X \in \mathfrak{g} \subset \mathfrak{gl}(n; \mathbb{F})$ and define

$$A(s) = e^{sX}.$$

Now we show that this is a one parameter subgroup. It is clear by the definition that A is continuous and that $A(0) = I$. So, it just a matter of observing that

$$A(t)A(s) = e^{tX}e^{sX} = e^{(t+s)X} = A(t+s).$$

In general the law of exponents for numbers does not hold for matrices, unless the matrices commute, which is clear in this case.

The following proposition gives a type of converse to our previous observation.

PROPOSITION 5. *If $A(\cdot)$ is a one-parameter subgroup of $\mathrm{GL}(n; \mathbb{C})$, there exists a unique $n \times n$ complex matrix X such that*

$$A(t) = e^{tX}.$$

EXAMPLE 3.1 ($\mathfrak{sl}(2; \mathbb{R})$ and $\mathrm{SL}(2; \mathbb{R})$).

Consider the basis $\{H := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, E := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, F := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\}$. Then, the one parameter subgroups generated by them are subgroups of $\mathrm{SL}(2; \mathbb{R})$. They are explicitly given by

$$\begin{aligned} A &:= \{\exp sH \mid s \in \mathbb{R}\} = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \mid a \in \mathbb{R} \right\} \\ N^+ &:= \{\exp sE \mid s \in \mathbb{R}\} = \left\{ \begin{pmatrix} 1 & s \\ 0 & 1 \end{pmatrix} \mid s \in \mathbb{R} \right\} \\ N^- &:= \{\exp sF \mid s \in \mathbb{R}\} = \left\{ \begin{pmatrix} 1 & 0 \\ s & 1 \end{pmatrix} \mid s \in \mathbb{R} \right\}. \end{aligned}$$

DEFINITION 3.3 (Lie Algebra of a Lie Group). *Let G be a matrix Lie group. The Lie algebra of G , denoted \mathfrak{g} , is the set of all matrices X such that e^{tX} is in G for all $t \in \mathbb{R}$.*

Equivalently, X is in \mathfrak{g} if and only if the entire one-parameter subgroup generated by X lies in G .

PROPOSITION 6. *Let G be a matrix Lie group, and X an element of its Lie algebra. Then e^X is an element of the identity component G_0 of G .*

PROOF. By definition of the Lie algebra, e^{tX} lies in G for all real t . However, as t varies from 0 to 1, e^{tX} is a continuous path connecting the identity to e^X . \square

3.1. More Examples of Lie algebras of Lie groups.

EXAMPLE 3.2 (The Lie algebra of $\text{SO}(n; k)$ and $\text{O}(n; k)$).

If J is defined by

$$J = \left(\begin{array}{c|c} I_n & 0 \\ \hline 0 & -I_k \end{array} \right)$$

where J is an $n + k \times n + k$ matrix, then the Lie algebra of $\text{O}(n; k)$ consists precisely of those real matrices X such that

$$JX^T = -XJ.$$

The Lie algebra of $\text{SO}(n; k)$ is the same as that of $\text{O}(n; k)$ and it is denoted by $\mathfrak{o}(n, k)$.

EXAMPLE 3.3 (The Heisenberg group and its Lie algebra).

The Heisenberg group is used to describe one-dimensional quantum mechanical systems, and is used as a model for the Heisenberg-Weyl commutation relations. The Heisenberg group H is the group of 3×3 upper triangular matrices of the form

$$H = \left\{ \left(\begin{array}{ccc} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{array} \right) \mid a, b, c \in \mathbb{R} \right\}$$

The Lie algebra of the Heisenberg group H is the space of all matrices of the form

$$X = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}$$

with $a, b, c \in \mathbb{R}$

and the exponential map

$$\exp \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} = \sum_{k=0}^{\infty} \frac{1}{k!} \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}^k = \begin{pmatrix} 1 & a & b + \frac{1}{2}a \cdot c \\ 0 & 1 & c \\ 0 & 0 & 1 \end{pmatrix}.$$

EXAMPLE 3.4 (The Lie algebra of $\text{U}(n)$ and $\text{SU}(n)$).

Recall that A is unitary if and only if $A^* = A^{-1}$, and that $U(n)$ is the group of unitary $n \times n$ matrices. Similarly $SU(n)$ is the subgroup of $U(n)$ with determinant 1. The Lie algebra of $U(n)$ consists of all complex matrices satisfying $X^* = -X$ and the Lie algebra of $SU(n)$ consists of all complex matrices satisfying $X^* = -X$ and $\text{Tr}(X) = 0$. These Lie algebras are denoted $\mathfrak{u}(n)$ and $\mathfrak{su}(n)$, respectively.

We specialize to the case of $\mathfrak{su}(2)$. By the previous argument,

$$\mathfrak{su}(2) = \left\{ \begin{pmatrix} ix & -\bar{b} \\ b & -ix \end{pmatrix} \mid x \in \mathbb{R}, b \in \mathbb{C} \right\}.$$

As a real Lie algebra, $\mathfrak{su}(2)$ has a basis $\{iH, X^+, X^-\}$, where H is defined as we did for $\mathfrak{sl}(2; \mathbb{R})$, $X^+ = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$, and $X^- = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

4. Lie Group Representations

The morphisms in the category of groups are group homomorphisms. For Lie groups, we require that the homomorphisms be continuous with respect to the respective manifold structures of the groups.

DEFINITION 4.1 (Lie Group Homomorphism). *Let G and H be matrix Lie groups. A map Φ from G to H is called a Lie group homomorphism if*

- (1) Φ is a group homomorphism and
- (2) Φ is continuous

If in addition, Φ is one-to-one and onto and the inverse map Φ^{-1} is continuous, then Φ is called a Lie group isomorphism.

The condition that Φ be continuous should be regarded as a technicality, in that it is very difficult to give an example of a group homomorphism between two matrix Lie groups which is not continuous. Note that the inverse of a Lie group isomorphism is continuous (by definition) and a group homomorphism (by elementary group theory), and thus a Lie group isomorphism. If G and H are matrix Lie groups and there exists a Lie group isomorphism from G to H , then G and H are said to be isomorphic, and we write $G \cong H$.

A special case of group homomorphisms is when the codomain of the homomorphism is the group of invertible linear operators on a vector space V . These homomorphisms are called representations of the group G . In a sense, they can be thought of the way that elements of the abstract group can be represented as invertible linear transformations, i.e. matrices.

DEFINITION 4.2 (Finite-Dimensional Complex or Real Representation). *Let G be a matrix Lie group. A finite-dimensional complex representation of G is a Lie group homomorphism*

$$\Pi : G \rightarrow \mathrm{GL}(V),$$

where V is a finite-dimensional complex vector space with $\dim(V) \geq 1$.

Similar to the argument for Lie algebras, Lie groups have a natural equivalency to modules. We can state this as follows:

LEMMA 4. *Given a representation $\Pi : G \rightarrow \mathrm{GL}(V)$, there is a G -module (V, \cdot) defined in the natural way, by defining for all $x \in G, v \in V$*

$$x \cdot v = \Pi(x)(v)$$

PROPOSITION 7. *Let G and H be matrix Lie groups, with Lie algebras \mathfrak{g} and \mathfrak{h} , respectively. Suppose that $\Phi : G \rightarrow H$ is a Lie group homomorphism. Then there exists a unique real-linear map $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$ such that*

$$\Phi(e^X) = e^{\phi(X)}$$

for all $X \in \mathfrak{g}$. The map ϕ has the following additional properties:

- (1) $\phi(AXA^{-1}) = \Phi(A)\phi(X)\Phi(A)^{-1}$, for all $X \in \mathfrak{g}, A \in G$
- (2) $\phi([X, Y]) = [\phi(X), \phi(Y)]$, for all $X, Y \in \mathfrak{g}$.
- (3) $\phi(X) = \frac{d}{dt}\Phi(e^{tX})|_{t=0}$, for all $X \in \mathfrak{g}$

We illustrate the previous proposition by differentiating the action of $SU(2)$ on $V_m(\mathbb{C}^2)$ to recover the action of $\mathfrak{sl}(2; \mathbb{C})$ calculated in Example 4.1.

EXAMPLE 4.1. (V_m as a representation of $SU(2)$)

Let $V_m(\mathbb{C}^2)$ be as in Example 4.1, with the exception that the domain of the polynomials is \mathbb{C}^2 . From the definition

$$\mathrm{SU}(2) = \left\{ \begin{pmatrix} a & -\bar{b} \\ b & \bar{a} \end{pmatrix} \mid a, b \in \mathbb{C} \right\}.$$

Given $(x, y)^T \in \mathbb{C}^2$, $g \in \mathrm{SU}(2)$ acts naturally on $(x, y)^T$ by left multiplication, i.e. $g(x, y)^T$. In detail, we compute the product

$$g^{-1}(x, y)^T = \begin{pmatrix} \bar{a}x + \bar{b} \\ ay - bx \end{pmatrix}.$$

Now, let $P(x, y) \in V_m(\mathbb{C}^2)$ we define the action of $g \in \mathrm{SU}(2)$ on $P(x, y)$ by

$$\Pi(g)(P(x, y)) := g.P(x, y) = P(g^{-1}(x, y)) = P(\bar{a}x + \bar{b}, ay - bx).$$

We show that $\Pi : \mathrm{SU}(2) \rightarrow \mathrm{GL}(V_m(\mathbb{C}^2))$ is a group representation, alternatively, we show that the previous equation defines a group action.

By differentiating the actions we recover the actions of $\mathfrak{su}(2)$:

$$\pi(H)(P) = -xP_y + yP_x$$

$$\pi(X^+)(P) = -yP_x$$

$$\pi(X^-)(P) = -xP_y.$$

You can see this explicitly by considering the action on $x^j y^k$.

That is,

$$\pi(H)(x^j y^k) = (-j + k)x^j y^k$$

$$\pi(X^+)(x^j y^k) = -jx^{j-1}y^{k+1}$$

$$\pi(X^-)(x^j y^k) = -kx^{j+1}y^{k-1}.$$

Note that this generates the eigenvectors of $\pi(H)$, with eigenvalues $-n, = n + 2, \dots, n - 2, n$, which are weights of the representation. The $\pi(X^+)$ are "raising operators" that increase this eigenvalue by 2, i.e., shift the weight by the one positive root. The $\pi(X^-)$ are "lowering operators" that decrease it by 2, i.e., shift it by the negative root.

EXAMPLE 4.2 (Left Multiplication and Left-Regular Representation).

Almost trivially the group G acts on itself by $\pi(g_1)(g_2) := g_1g_2$. This is the so called left-action of the group. Now consider the space of smooth complex valued functions on G , $C^\infty(G)$. We must note, that the theory is usually developed for square-integrable functions. However, the space of smooth functions is dense in this space and requires less care to exceptional points. Hence, we consider only smooth functions. Then the left regular action of G on $C^\infty(G)$ is defined by

$$l(g).f(h) = f(g^{-1}h).$$

PROPOSITION 8. *Suppose that G , H , and K are matrix Lie groups and $\Phi : H \rightarrow K$ and $\Psi : G \rightarrow H$ are Lie group homomorphisms. Let $\Lambda : G \rightarrow K$ be the composition of Φ and Ψ and let ϕ, ψ , and λ be the Lie algebra maps associated to Φ, Ψ , and Λ , respectively. Then we have*

$$\lambda = \phi \circ \psi.$$

PROOF. For any $X \in \mathfrak{g}$, $\Lambda(e^tX) = \Phi(\Psi(e^{tX})) = \Phi(e^{t\psi(X)}) = e^{t\phi(\psi(X))}$. \square

PROPOSITION 9. *If $\Phi : G \rightarrow H$ is a Lie group homomorphism and $\phi : \mathfrak{g} \rightarrow \mathfrak{h}$ is the associated Lie algebra homomorphism, then the kernel of Φ is a closed, normal subgroup of G and the Lie algebra of the kernel is given by*

$$\text{Lie}(\ker(\Phi)) = \ker(\phi)$$

5. Not All Lie Groups are Matrix Lie Groups

While many of the Lie groups of interest can be represented as matrix Lie groups, in this thesis our main object of study will be a semidirect product of the metaplectic group $\text{Mp}_2(\mathbb{R})$ and the Heisenberg group. Here we construct the metaplectic group, which is the unique double cover of $\text{SL}(2; \mathbb{R})$. The metaplectic group is the paramount example of a Lie group that cannot be realized as a closed subgroup of the general linear group, i.e. it is not a matrix Lie group. The representation we show is known as the Weyl representation. Start with

$$G_0 := \text{SL}(2; \mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{R}) : ad - bc = 1 \right\}.$$

G_0 acts naturally on the upper half-plane H

$$H := \{z \in \mathbb{C} \mid \text{Im } z > 0\}$$

by Möbius transformations. Namely,

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

We wish to consider the two-fold cover of G_0 . If we define $d : G_0 \times D \rightarrow \mathbb{C}$ by $d(g, z) := cz + d$ then there are exactly two smooth square roots of $d(g, z)$ for each $g \in G_0$ and $z \in D$. The double cover can be realized as:

$$\begin{aligned} \tilde{G}_0 &= \{(g, \epsilon) \mid g \in SL(2; \mathbb{R}) \text{ and smooth } \epsilon : D \rightarrow \mathbb{C} \\ &\text{such that } \epsilon(z)^2 = d(g, z) \text{ for } z \in D\} \end{aligned}$$

with the product defined by

$$(g_1, \epsilon_1(z))(g_2, \epsilon_2(z)) = (g_1 g_2, \epsilon_1(g_2 \cdot z) \epsilon_2(z)).$$

The mapping $(g, \epsilon) \mapsto g$ is a two-to-one map, reflecting the fact that the constructed group is a two-fold cover of $SL(2; \mathbb{R})$. We denote this group as $\widetilde{G}_0 := \text{Mp}_2(\mathbb{R})$

CHAPTER 4

Jacobi Group and the Differential Equation $u_{xx} + \alpha u_x + \beta u_t = 0$

In this chapter, we construct the machinery necessary to study the space of solutions to the equation

$$(0.1) \quad u_{xx} + \alpha u_x + \beta u_t = 0$$

as a global representation of the Jacobi group. In the previous chapters we introduced the special linear group $\mathrm{SL}(2; \mathbb{R})$ and the three-dimensional Heisenberg group H_3 . The Jacobi group is the semidirect product of the symplectic group $\mathrm{SL}(2; \mathbb{R})$ and the Heisenberg group (there is an n -dimensional analogue of this group, but we are only interested in the $n = 1$ case). For technical reasons we need to consider the metaplectic group $\mathrm{Mp}_2(\mathbb{R})$ instead of $\mathrm{SL}(2; \mathbb{R})$. Hence, we define the G as

$$G := \mathrm{Mp}_2(\mathbb{R}) \ltimes H_3$$

In [3], the structure of $\ker(\Omega + 3/8)$, for Casimir element Ω as a (\mathfrak{g}, K) -module is studied by using an isomorphic picture of $J'(q, r, s)$ denominated the compact picture and represented by $I''(q, r, s) \subset C^\infty(S^1 \times \mathbb{R})$. The explicit isomorphism between these spaces is easily calculated and it is defined by $f \leftrightarrow F$ if and only if,

$$(0.2) \quad f(t, x) = (1 + t^2)^{-r/2} e^{\frac{stx^2}{1+t^2} + \frac{\alpha(\alpha t - 2\beta x)}{4\beta}} F(\arctan t, x(1 + t^2)^{-1/2})$$

or,

$$(0.3) \quad F(\theta, y) = (\cos \theta)^{-r} e^{-sy^2 \tan \theta - \frac{\alpha(\alpha \tan \theta - 2y\beta \sec \theta)}{4\beta}} f(\tan \theta, y \sec \theta).$$

In particular, they obtain the following result, that we adapt to our representation.

PROPOSITION 10. *There exist a K -finite vector of weight m in $\ker(\Omega + 3/8) \subset I''(q, r, s)$,*

$$(0.4) \quad F_m(\theta, y) = e^{-im\theta/2} e^{-y^2/2} y^l {}_1F_1 \left(\frac{1 + 2l - m}{4}, l + \frac{1}{2}, y^2 \right)$$

iff $m \equiv 2l + q \pmod{4}$ for $l = 0$ or $l = 1$.

1. Langlands' Decomposition

Without appealing to the general theory, which reaches beyond the scope of this thesis, we introduce what is known as the Langlands' decomposition of a parabolic subgroup of the Jacobi group and its corresponding Lie algebra. Most of the results in this section are standard or well-known. As a result, many of the details are omitted. For a careful exposé on these constructions, the interested reader is referred to [3, 6, 7]. We consider this decomposition with respect to the minimal parabolic subalgebra of lower triangular matrices $\mathfrak{q} \subset \mathfrak{sl}(2, \mathbb{R})$. This subalgebra has the following Langlands decomposition $\mathfrak{m} \oplus \mathfrak{a} \oplus \bar{\mathfrak{n}}$, where

$$\begin{aligned}\mathfrak{m} &= \{\pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\} \\ \mathfrak{a} &= \left\{ \begin{pmatrix} t & 0 \\ 0 & -t \end{pmatrix} \mid t \in \mathbb{R} \right\} \\ \bar{\mathfrak{n}} &= \left\{ \begin{pmatrix} 0 & 0 \\ t & 0 \end{pmatrix} \mid t \in \mathbb{R} \right\} \\ \mathfrak{n} &= \left\{ \begin{pmatrix} 0 & t \\ 0 & 0 \end{pmatrix} \mid t \in \mathbb{R} \right\}.\end{aligned}$$

Let the exponential map be denoted by $\exp_{\tilde{G}_0} : \mathfrak{sl}(2, \mathbb{R}) \rightarrow \tilde{G}_0$. Then,

$$\begin{aligned}A &:= \exp_{\tilde{G}_0}(\mathfrak{a}) = \left\{ \left(\begin{pmatrix} t & 0 \\ 0 & t \end{pmatrix}, z \mapsto e^{-\frac{t}{2}} \right) \mid t \geq 0 \right\} \\ N &:= \exp_{\tilde{G}_0}(\mathfrak{n}) = \left\{ \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, z \mapsto 1 \right) \mid t \in \mathbb{R} \right\} \\ \bar{N} &:= \exp_{\tilde{G}_0}(\bar{\mathfrak{n}}) = \left\{ \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}, z \mapsto \sqrt{tz+1} \right) \mid t \in \mathbb{R} \right\}.\end{aligned}$$

Let $\mathfrak{k} := \left\{ \begin{pmatrix} 0 & -\theta \\ \theta & 0 \end{pmatrix} : \theta \in \mathbb{R} \right\}$ then

$$K := \exp_{\tilde{G}_0}(\mathfrak{k}) = \left\{ \left(\begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, z \mapsto \sqrt{\cos \theta - z \sin \theta} \right) \mid \theta \in \mathbb{R} \right\},$$

where $\sqrt{\cdot}$ denotes the principal square root in \mathbb{C} . Writing M for the centralizer of A in K then

$$M = \{m_j := \left(\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}^j, z \mapsto i^{-j} \right) \mid j = 0, 1, 2, 3\}.$$

Let $W \subset H_3$ be given by $W = \{(0, v, w) \mid v, w \in \mathbb{R}\} \cong \mathbb{R}^2$ and let $X := \{(x, 0, 0) \mid x \in \mathbb{R}\}$. Write \mathfrak{w} for the Lie algebra of W . Then $\bar{P} = M\bar{N} \times W$ is the analogue of a parabolic subgroup in G corresponding to $\bar{\mathfrak{p}} := \bar{\mathfrak{q}} \times \mathfrak{w}$. We notice that an element in $g = \left[\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \mapsto \epsilon(z) \right), (u, v, w) \right] \in G$ is in the image of the mapping $\bar{P} \times (N \times X) \rightarrow G$

given by $(\bar{p}, n) \mapsto \bar{p}n$, if $a \neq 0$. This induces a decomposition of such g into its \bar{P} and $N \times X$ components,

$$\begin{aligned} \left[\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, z \mapsto \epsilon(z) \right), (u, v, w) \right] &= \left[\left(\begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}, z \mapsto \epsilon(z + \frac{b}{a}) \right), (0, v, w + (u + \frac{bv}{a}v)) \right] \\ &\quad \cdot \left[\left(\begin{pmatrix} 1 & \frac{b}{a} \\ 0 & 1 \end{pmatrix}, z \mapsto 1 \right), (u + \frac{bv}{a}, 0, 0) \right]. \end{aligned}$$

On the open dense set where $a \neq 0$, let $\bar{p} : G \rightarrow \bar{P}$ and $n : G \rightarrow N \times X$ be the projections from the previous decomposition.

2. Characters and Induced Representations

The idea here will be to introduce a one-dimensional representation of the group G . From this representation we shall induce an infinite-dimensional representation. An isomorphic copy of this space will be used to realize the space of solutions to $u_{xx} + \alpha u_x + \beta u_t = 0$ as a global representation of the Lie group. We start with the definition of a character.

DEFINITION 2.1. *A character of a Lie group G is a one-dimensional representation $\xi : G \rightarrow \mathbb{C}$.*

It is well known that the character group on A is isomorphic to the additive group \mathbb{C} so any character on A can be indexed by a constant $r \in \mathbb{C}$ and defined by

$$\chi_r \left(\left(\begin{pmatrix} t & 0 \\ 0 & t^{-1} \end{pmatrix}, z \mapsto e^{-\frac{t}{2}} \right) \right) = t^r$$

for $t > 0$. A character on M is parameterized by $q \in \mathbb{Z}_4$ and defined by $\chi_q(m_j) = i^{jq}$. A character on W can be parameterized by $s \in \mathbb{C}$ and defined by,

$$\chi_s((0, v, w)) = e^{sw}.$$

Finally, any character on \bar{P} that is trivial on N is parameterized by a triplet (q, r, s) where $s, r \in \mathbb{C}$ and $q \in \mathbb{Z}_4$ and defined by

$$\chi_{q,r,s} \left(\left((-1)^j \begin{pmatrix} a & 0 \\ c & a^{-1} \end{pmatrix}, z \mapsto i^{-j} e^{-\frac{\alpha}{2}} \sqrt{acz + 1} \right), (0, v, w) \right) = i^{jq} |a|^r e^{sw}$$

With the desired character in hand, we briefly introduce the technique used to induce a representation from this character. For this discussion, consider G an arbitrary Lie group and H a closed subgroup of G . Then, suppose that $\rho : H \rightarrow \text{GL}(V)$

is a representation and that $C^\infty(G, V)$ is the space of smooth V -valued functions on G . Then,

$$\text{ind}_H^G(\rho) = \{f \in C^\infty(G, V) \mid f(xh) = \rho(h)^{-1}(f(x)) \forall (x, h) \in G \times H\}$$

is a representation of G with the group action given by the left-regular action [6]. Using H as the parabolic subgroup and $\rho = \chi_{q,r,s}$ the resulting induced representation will be denoted by $I(q, r, s)$ and defined by

$$I(q, r, s) := \{\phi : G \rightarrow \mathbb{C} \mid \phi \in C^\infty \text{ and } \phi(g\bar{p}) = \chi_{q,r,s}^{-1}(\bar{p})\phi(g) \text{ for } g \in G, \bar{p} \in \bar{P}\}$$

the G -action on $I(q, r, s)$ is the left-regular action of G .

3. The Non-Compact Picture

Since $H_3 = XW$ then $G = (N \times X)\bar{P}$ a.e. But $N \times X$ is isomorphic to \mathbb{R}^2 via $(t, x) \mapsto N_{t,x} := [(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}), z \mapsto 1], (x, 0, 0)]$. Since a section in the induced representation is determined by its restriction to $N \times X$, this restriction induces an injection of $I(q, r, s)$ into $C^\infty(\mathbb{R}^2)$ which is identified as

$$I'(q, r, s) = \{f \in C^\infty(\mathbb{R}^2) \mid f(t, x) = \phi(N_{t,x}) \text{ for some } \phi \in I(q, r, s)\}.$$

This space is endowed with the corresponding action so that the map $\phi \mapsto f$ where $f(t, x) = \phi(N_{t,x})$ becomes intertwining. Thus $I(q, r, s) \cong I'(q, r, s)$ as \widetilde{G}_0 -modules. As in the semisimple case, we will call this the non-compact picture.

The action of the group is calculated in [8] and given in the next proposition.

PROPOSITION 11. *Let $f \in I'(q, r, s)$, $(g, \epsilon) \in \widetilde{G}_0$, and $(u, v, w) \in H_3$. Then,*

$$(3.1a) \quad ((g, \epsilon).f)(t, x) = (a - ct)^{r-q/2} \epsilon(g^{-1}.(t+z)) e^{\frac{-scx^2}{a-ct}} f\left(\frac{dt-b}{a-ct}, \frac{x}{a-ct}\right)$$

$$(3.1b) \quad ((u, v, w).f)(t, x) = e^{-s(uv-2vx-tv^2+w)} f(t, x - u - tv).$$

Differentiating this action as in Proposition 7 in Chapter 3, we obtain the actions of the Lie algebra.

COROLLARY 1. *The action of $\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2, \mathbb{R})$ on $I'(q, r, s)$ is given by the differential operator*

$$(3.2) \quad (ct - a)x\partial_x + (ct^2 - 2at - b)\partial_t + (ra - csx^2 - rct).$$

An element $(u, v, w) \in \mathfrak{h}_3$ acts on $I'(q, r, s)_{\mu_1}$ by the differential operator

$$(3.3) \quad (tv - u)\partial_x + s(w - 2vx).$$

4. A Distinguished Isomorphic Representation

In this section we define a map between $I'(q, r, s)$ and $C^\infty(\mathbb{R}^2)$. We shall endow the image of this map with the appropriate G -action so that the action of the group and the map commute. These types of maps are called *intertwining maps*. These maps preserve the structure as representations of the group.

Define the following smooth map $\Psi : I'(q, r, s) \rightarrow C^\infty(\mathbb{R}^2)$ by

$$\Psi(f(t, x)) = e^{\frac{\alpha(\alpha t - 2\beta x)}{4\beta}} f(t, x).$$

Let $J'(q, r, s)$ be the image of $I'(q, r, s)$ under Ψ . Since Ψ is clearly injective, it suffices to equip $J'(q, r, s)$ with the appropriate G -action to obtain a G -representation isomorphic to $I(q, r, s)$. This action is calculated in the next proposition.

PROPOSITION 12. Let $f \in J'(q, r, s)$, $(g, \epsilon) \in \widetilde{G}_0$, and $(u, v, w) \in H_3(\mathbb{R})$. Then,

$$(4.1a) \quad ((g, \epsilon).f)(t, x) = (a - ct)^{r-q/2} \epsilon(g^{-1}.(t + z)) \\ \cdot e^{\frac{\alpha(\alpha t - 2\beta x)}{4\beta} - \frac{\alpha^2(dt-b) - 2\alpha\beta x + 4\beta s c x^2}{4\beta(a-ct)}} f\left(\frac{dt-b}{a-ct}, \frac{x}{a-ct}\right)$$

$$(4.1b) \quad ((u, v, w).f)(t, x) = e^{-s(uv - 2vx - tv^2 + w) - \alpha(tv+u)/2} f(t, x - u - tv).$$

PROOF. From the diagram,

$$\begin{array}{ccc} I'(q, r, s) & \xrightarrow{g} & I'(q, r, s) \\ \downarrow \Psi & & \downarrow \Psi \\ J'(q, r, s) & \xrightarrow{g} & J'(q, r, s) \end{array}$$

we see that the for the action of G on $J'(q, r, s)$ to be intertwining, we need

$$g.f = \Psi(g.\Psi^{-1}(f)),$$

for all $g \in G$, where the actions on right-hand-side are given by (3.1a) and (3.1b).

The result of these calculations are (4.1a) and (4.1b). \square

In the following, we give the actions of the Lie algebra \mathfrak{g} on $J'(q, r, s)$.

COROLLARY 2. *The action of $\begin{pmatrix} a & b \\ c & -a \end{pmatrix} \in \mathfrak{sl}(2; \mathbb{R})$ on $J'(q, r, s)$ is given by the differential operator*

$$(4.2) \quad (ct - a)x\partial_x + (ct^2 - 2at - b)\partial_t + (ra - csx^2 - rct) \\ + \frac{(2\alpha^2 t - 2\alpha\beta x)a + \alpha^2 b - (\alpha^2 t^2 - 2\alpha\beta tx)c}{4\beta}.$$

An element $(u, v, w) \in \mathfrak{h}_3$ acts on $J'(q, r, s)$ by the differential operator

$$(tv - u)\partial_x + s(w - 2vx) - \alpha(tv + u)/2.$$

PROOF. Follows from differentiating the actions (4.1a) and (4.1b). \square

From the corollary, we have that the actions of the standard \mathfrak{sl}_2 -triple, $\{H, E, F\}$, on $J'(q, r, s)$ are given by

$$(4.3) \quad H = -x\partial_x - 2t\partial_t + r + \frac{\alpha^2 t - \alpha\beta x}{2\beta} \\ E = -\partial_t + \frac{\alpha^2}{4\beta} \\ F = tx\partial_x + t^2\partial_t - sx^2 - rt - \frac{\alpha^2 t^2 - 2\alpha\beta tx}{4\beta}.$$

Let Ω be the Casimir element of $\mathfrak{sl}(2; \mathbb{R})$. It is well-known that with respect to the standard basis, Ω can be written as

$$(4.4) \quad \Omega = \frac{1}{2}H^2 - H + 2EF.$$

PROPOSITION 13. *The action of Ω on $J'(q, r, s)$ is given by*

$$\Omega = \frac{4r^2 - 4r(\alpha x - 2) - \alpha x(2 - \alpha x)}{8} - \frac{s\alpha^2 x^2}{2\beta} \\ - \frac{1}{2}x \left((2r - \alpha x + 1)\partial_x - x(4s\partial_t + \partial_x^2) \right).$$

In particular, Ω acts on $J'(q, -1/2, \beta/4)$ by,

$$\Omega + 3/8 = \frac{x^2}{2} (\partial_x^2 + \alpha\partial_x + \beta\partial_t).$$

PROOF. The result is obtained by a straightforward calculation using (4.4) and (4.3). \square

As a consequence of this result, the subspace of solutions to (0.1) in $J'(q, -1/2, \beta/4)$ is equal to $\ker(\Omega + 3/8) \subset J'(q, -1/2, \beta/4)$. In particular, the space is invariant under the action of \mathfrak{g} and by connectedness, under the action of G as well. The corresponding K -finite vector in $\ker(\Omega + 3/8) \subset J'(q, r, s)$ is given by

$$(4.5) \quad f_m(t, x) = x^l (1+t^2)^{-(r+l)/2} e^{\frac{(2st-1)x^2}{2(1+t^2)} + \frac{\alpha(\alpha t - 2\beta x)}{4\beta}} \left(\frac{1-it}{1+it} \right)^{m/4} \cdot {}_1F_1 \left(\frac{1+2l-m}{4}, l + \frac{1}{2}, \frac{x^2}{1+t^2} \right).$$

In particular, (4.5) is a solution to (0.1) for $m \in \mathbb{N}$ and $l = 0, 1$.

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