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The Subconstituent Algebra of a Hypercube

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THE SUBCONSTITUENT ALGEBRA OF A HYPERCUBE

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Contents

Abstract

We study the hypercube and the associated subconstituent algebra. Let Q_D denote the hypercube with dimension D and let X denote the vertex set of Q_D. Fix a vertex x in X. We denote by A the adjacency matrix of Q_D and by $A^* = A^*(x)$ the diagonal matrix with yy-entry equal to $D - 2i$, where i is the distance between x and y. The subconstitutent algebra $T = T(x)$ of Q_D with respect to x is generated by A and A^*. We show that

$$A^2 A^* - 2AA^*A + A^*A^2 = 4A^*$$

$$A^{*2}A - 2A^*AA^* + AA^{*2} = 4A.$$

Using these relations, we show that there exists a surjective $\mathbb{C}$-algebra homomorphism from the universal enveloping algebra of the Lie algebra $sl_2(\mathbb{C})$ to T.

Chapter 1: Introduction

This thesis is centered around the study of the subconstituent algebra of a distance-regular graph. The subconstituent algebra was first introduced by Terwilliger [10] in his study of Q-polynomial distance-regular graphs. Since then, the subconstituent algebra has been a significant tool in the theory of Q-polynomial distance-regular graphs. It has connections to various areas of mathematics, including association schemes [3], representation theory [4], coding theory [6], and the theory of double affine Hecke algebras [8], and is a powerful tool in the study of these areas. In this thesis, we consider the subconstituent algebra of the hypercube Q_D. The hypercube Q_D is a classical family of Q-polynomial distance-regular graphs, and it is bipartite with diameter D. The main results of this thesis can be summarized as follows. We denote by X the vertex set of the hypercube Q_D. Fix a vertex $x \in X$, and let $T = T(x)$ denote the subconstituent algebra of Q_D with respect to x. Additionally, we denote by $A^* = A^*(x)$ the diagonal matrix in $Mat_X(\mathbb{C})$ with yy-entry $D - 2i$, where i is the distance between x and y, and we refer to A^* as the dual adjacency matrix of Q_D. From the structure of Q_D, we obtain the following fundamental relations in the algebra T, which is our first main result; see Theorem 5.2.

$$A^2A^* - 2AA^*A + A^*A^2 = 4A^*$$
$$A^{*2}A - 2A^*AA^* + AA^{*2} = 4A.$$

These relations give rise to a surjective $\mathbb{C}$-algebra homomorphism ρ from the universal enveloping algebra of the Lie algebra $sl_2(\mathbb{C})$ and the subconstituent algebra T. Proving that ρ exists is the second main result of the thesis; see Theorem 7.4.

The paper is organized as follows. In section 2, we introduce the general concept of a graph, as well as its various properties. This leads to a discussion about a particular family of graphs, which have the property of being distance-regular. In section 3, we discuss the underlying algebra of distance-regular graphs, first by discussing the distance matrices $\{A_i\}_{i=0}^{D}$, of which we observe a fundamental property; $A_1 = A$ in particular generates an associative $\mathbb{C}$-algebra M. We then discuss the dual distances matrices, where $A_1^* = A^*$ generates an associative $\mathbb{C}$-algebra M^*. We finish section 3 with the definition of the the algebra T, generated by M and M^*. In section 4, we begin the discussion of a particular distance-regular graph, called the hypercube Q_D. After observing the properties of Q_D, we are led into the discussion of the fundamental relations in section 5. In Section 6, we introduce the raising and lowering maps, followed by the second main result in Section 7, which is a connection between the $sl_2(\mathbb{C})$ and T. This work is based on [8].

Chapter 2: Distance-Regular Graphs

In order to discuss the main results of the thesis, we need some basic ideas from graph theory. We discuss in this section the concept of a distance-regular graph and its various properties. For further details on distance-regular graphs, we refer the reader to [1].

DEFINITION 2.1 A *graph* is a pair $\Gamma = (X, R)$, where X is a non-empty set, and where $R = \{(a, b) : a, b \in X\}$. Each element $x \in X$ is called a *vertex* of Γ, and hence X is referred to as the *vertex set* of Γ. Additionally, R is called the *edge set*, and we say that $x, y \in X$ are *adjacent* if $(x, y) \in R$. If x is adjacent to y, then x is called a *neighbor* of y, and vice versa.

Let $\Gamma = (X, R)$ be a graph. If every vertex of Γ is adjacent to exactly k other vertices, then we say that Γ is *regular* with *valency* k. Let i be a natural number and let $x, y \in X$ be any vertices. By a *path of length* i, we mean a sequence of vertices $x_0x_1...x_i$ such that $x_0 = x$, $x_i = y$ and for $0 \leq j \leq i-1$, x_j, x_{j+1} are adjacent. If Γ contains a path from x to y for all $x, y \in X$, then we say that Γ is *connected*. We say that the graph Γ is *simple* if it contains no loops or multiple edges.

Throughout the paper, we will assume that Γ is simple and connected. For all $x, y \in X$ we refer to the length of the shortest path between x and y as the *distance* between x and y. This distance is denoted by $\partial(x, y)$. The *diameter* D of Γ is the largest distance that appears in Γ, that is, $D = \max\{\partial(x, y) \mid x, y \in X\}$. For $0 \leq i \leq D$, and for $x \in X$, we define the *ith subconstituent of* Γ *with respect to* x as $\Gamma_i(x) = \{z \in X \mid \partial(x, z) = i\}$. In other words, $\Gamma_i(x)$ denotes the set of vertices $z \in X$ that are at a distance i from x. Observe that $\Gamma_1(x)$ is the set of neighbors of x. If there exists a partition of the vertex set $X = A \sqcup B$, where neither A nor B contains an edge, then we say that Γ is *bipartite*. Here, the symbol $\sqcup$ denotes the disjoint union. We also note the Bipartite Characterization Theorem and refer the reader to [3] for further details. The following example is intended to illustrate the preceding definitions.

EXAMPLE 2.2. Consider the graph Γ below:

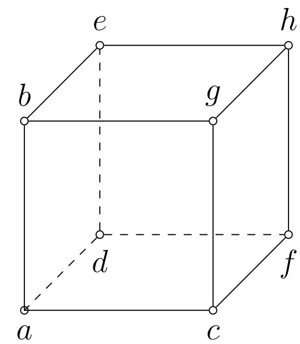


Figure 2(a)

In regards to the previous terminology of this section, there are a few things we can say about Γ. First, the vertex set is

$$X = \{a, b, c, d, e, f, g\},$$

and the edge set, which has pairs of adjacent vertices as its elements, is

$$R = \{(a, b), (a, c), (a, d), (b, e), (b, g), (c, g), (c, f), (d, e), (d, f), (h, f), (h, e), (h, g)\}.$$

We see that Γ is regular with valency $k = 3$, since any vertex is adjacent to exactly three other distinct vertices. The graph Γ is connected, and for any two vertices x, y, we have $\partial(x, y) \leq 3$. Thus, $D = 3$. We also see that, for example, $\Gamma_2(b) = \{d, c, h\}$. Let $A = \{a, e, f, g\}$ and let $B = \{b, c, d, h\}$. Notice that $A \sqcup B = X$, and that no two vertices that are both in A or both in B form an edge. This shows that Γ is bipartite.

A special family of graphs are the *distance-regular* graphs. However, it is more reasonable to have the following definition come first.

Definition 2.3. (Intersection number) Let $\Gamma = (X, R)$ denote a graph with diameter D. Let h, i, j denote integers with $0 \leq h, i, j \leq D$. For any two vertices x, y in X with $\partial(x, y) = h$, we define the number

$$p^h_{ij}(x, y) = |\Gamma_i(x) \cap \Gamma_j(y)|.$$

We refer to the numbers $p^h_{ij}(x, y)$ as the *intersection numbers* of Γ with respect to x and y.

Example 2.4. Recall the graph Γ of Example 2.2. Consider vertices a and e. We observe that $\partial(a, e) = 2$. Suppose that we are interested in determining how many vertices lie simultaneously at a distance 1 from a and 3 from e. By inspection of Γ, we find

$$p^2_{13}(a, e) = |\Gamma_1(a) \cap \Gamma_3(e)| = |\{b, d, c\} \cap \{c\}| = |\{c\}| = 1.$$

Definition 2.5. (Distance-Regular Graph) Let $\Gamma = (X, R)$ denote a connected graph with diameter D. We say that Γ is *distance-regular* whenever, for integers $0 \leq h, i, j \leq D$, and for any vertices x, y in X with $\partial(x, y) = h$, the numbers $p^h_{ij}(x, y)$ depend only on h, i, j and not on the chosen vertices x and y. We now write $p^h_{ij}(x, y) = p^h_{ij}$.

Let A be the square matrix indexed by the vertex set of Γ such that for a pair of vertices x, y,

$$(A)_{xy} = \begin{cases} 1 & \text{if } x \sim y \\ 0 & \text{otherwise.} \end{cases}$$

we refer to A as the *adjacency matrix* of Γ.

Lemma 2.6 Let Γ be a graph with diameter D, and let A be the adjacency matrix of Γ. For $0 \leq k \leq D$, and for x, y vertices in X,

$$(A^k)_{xy} = \text{ number of } k\text{-walks between } x \text{ and } y,$$

Proof. We show this property by induction. Since a 0-walk can only exist between vertices x, y if and only if x and y are the same vertex, and since $A^0 = I$, the case when $k = 0$ is trivial. Now when $k = 1$, we have

$$(A)_{xy} = \begin{cases} 1 = \text{ number of 1-walks between } x \text{ and } y, \\ \\ 0, \end{cases}$$

which shows that the result holds for $k = 1$ by the definition of A. Now suppose the statement holds up to $k - 1$. Then we have

$$(A^k)_{xy} = (A^{k-1} \cdot A)_{xy} = \sum_{w \in X} (A^{k-1})_{wx} (A)_{wy}.$$

Now we observe that, by the induction hypothesis, $(A^{k-1})_{wx}$ is equal to the number of $k-1$-walks from x to w and $(A)_{wy}$ is equal to the number of 1-walks from w to y. Thus the above summand is equal to the number of k-walks from x to y; the result follows. $\square$

Suppose that Γ is distance-regular. Then for $0 \leq h, i, j \leq D$, the intersection numbers have the property that

$$p^h_{ij} = p^h_{ji} \tag{1}$$

and

$$p^i_{1j} = 0, \qquad \text{if } |i - j| > 1 \tag{2}$$

To see this, consider the triangle below.

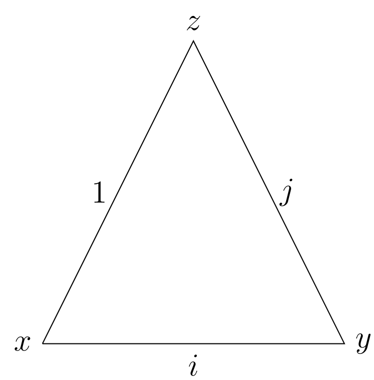


Figure 2(b)

By definition of the intersection numbers, the above triangle can be formed (that is, for x, y with $\partial(x, y) = i$, there is a vertex z such that $\partial(x, z) = 1$ and $\partial(z, y) = j$), if $p^i_{1j} \neq 0$. But by the triangle inequality, this means we must have

$$i + 1 \geq j$$

$$j + 1 \geq i$$

$$i + j \geq 1$$

Rearranging terms in inequalities (1) and (2) yields $-1 \leq i - j \leq 1$, which implies that $|i - j| \leq 1$. Then it follows that if $|i - j| > 1$, then $p^i_{1j} = 0$.

This property allows us to define the numbers a_i, b_i, c_i such that

$$\begin{aligned} c_i &= p^i_{1,i-1} \quad && 1 \le i \le D, \\ a_i &= p^i_{1,i} && 0 \le i \le D, \\ b_i &= p^i_{1,i+1} && 0 \le i \le D-1. \end{aligned} \tag{3}$$

For notational convenience, we denote $c_0 = 0$ and $b_D = 0$.

For the graph of Example 2.2, by definition the intersection numbers a_i, b_i, c_i are given as follows.

$$\begin{aligned} c_0 &= 0, \qquad & a_0 &= 0, \qquad & b_0 &= 3, \\ c_1 &= 1, & a_1 &= 0, & b_1 &= 2, \\ c_2 &= 2, & a_2 &= 0, & b_2 &= 1, \\ c_3 &= 3, & a_3 &= 0, & b_3 &= 0. \end{aligned} \tag{4}$$

Thus, we can conclude that the cube in Example 2.2 is distance-regular.

Observe that distance-regular graphs are also regular with valency $k = p^0_{11} = b_0$. The intersection numbers a_i, b_i, c_i can be expressed as follows.

$$\begin{aligned} c_i &= |\Gamma_{i-1}(x) \cap \Gamma_1(y)|, \\ a_i &= |\Gamma_i(x) \cap \Gamma_1(y)|, \\ b_i &= |\Gamma_{i+1}(x) \cap \Gamma_1(y)|, \end{aligned}$$

for $x, y \in X$ with $\partial(x, y) = i$. The distance diagram of Figure 2(c) illustrates this concept.

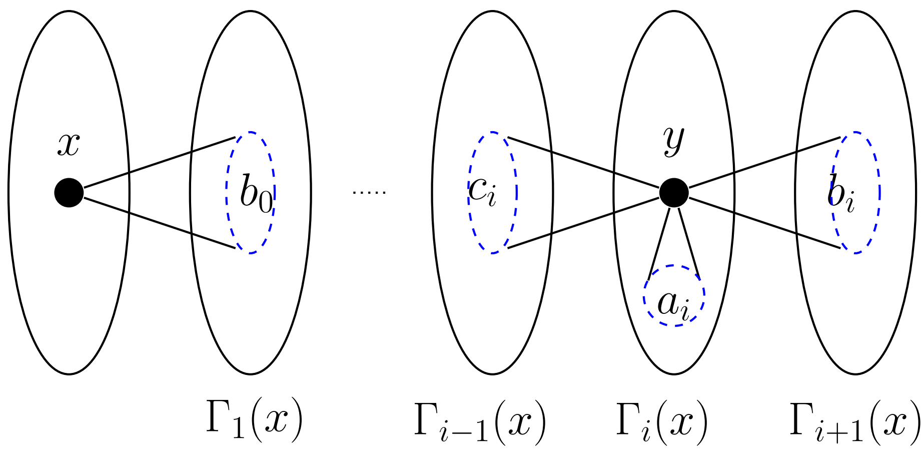


Figure 2(c)

Notice from Figure 2(c) that $|\Gamma_1(x)| = b_0 = k$. Additionally, observe that for any vertex y such that $\partial(x, y) = i$, $0 \le i \le D$, we have

$$c_i + a_i + b_i = |\Gamma_1(y)| = k.$$

LEMMA 2.7. A graph Γ is bipartite if and only if $a_i = 0$ for all $0 \le i \le D$.

Proof. ($\Rightarrow$) Suppose Γ is bipartite with bipartition $X = A \sqcup B$. Suppose that there exists a j, $0 \le j \le D$, such that $a_j \ne 0$. Then, for $x \in X$, there exists a vertex $y \in \Gamma_j(x)$ and a vertex $z \in \Gamma_j(x)$ such that $\partial(y, z) = 1$. That is, the edge (y, z) lies in the set $\Gamma_j(x)$, and we have $y, z \in A$ or $y, z \in B$. In either case, one of the two sets A or B contains the edge (y, z), contradicting the fact that Γ is bipartite. Hence, it must be the case that $a_j = 0$ for all $j = 0, 1, \cdots, D$.

($\Leftarrow$) Now let $a_i = 0$, for all $0 \le i \le D$. For $x \in X$, we partition X into

$$X = \bigsqcup_{i=0}^{D} \Gamma_i(x) = \Gamma_0(x) \cup \Gamma_1(x) \cup \cdots \cup \Gamma_D(x).$$

Then we can construct a bipartition of X (cf. Figure 2(c))

$$A = \{\Gamma_0(x), \Gamma_2(x), \cdots\}$$

$$B = \{\Gamma_1(x), \Gamma_3(x), \cdots\}.$$

Let m, n be any even integers such that $0 \leq m, n \leq D$. Then there are no edges between $\Gamma_n(x)$ and $\Gamma_m(x)$. Similarly, for any odd integers r, s such that $0 \leq r, s \leq D$, there are no edges between $\Gamma_r(x)$ and $\Gamma_s(x)$. Since by assumption the set $\Gamma_i(x)$ does not contain an edge, for all $0 \leq i \leq D$, then $A \sqcup B$ must form a bipartition of Γ. $\square$

For a graph Γ and vertex $x \in X$, we define k_i such that

$$k_i = |\Gamma_i(x)|,$$

and we refer to k_i as the *ith* valency of Γ.

Lemma 2.8. For $0 \leq i \leq D$,

$$k_i = \frac{b_{i-1}b_{i-2}\cdots b_2 b_1 b_0}{c_i c_{i-1}\cdots c_2 c_1}.$$

Proof. Fix a vertex $x \in X$. For $1 \leq i \leq D$, let $y \in \Gamma_{i-1}(x)$.

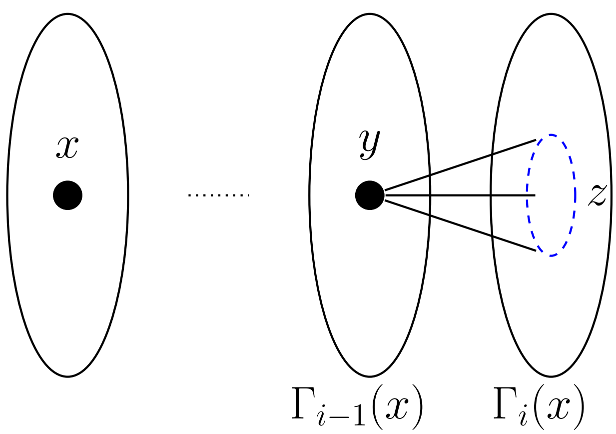


Figure 2(d)

Let z denote a neighbor of y that lies in $\Gamma_i(x)$ (cf. Figure 2(d)). Notice that (by definition) the number of such vertices z is equal to b_{i-1}. That is,

$$|\Gamma_1(y) \cap \Gamma_i(x)| = b_{i-1}.$$

Since all vertices $y \in \Gamma_{i-1}(x)$ have the same number of neighbors in $\Gamma_i(x)$, it follows that the number of edges between $\Gamma_{i-1}(x)$ and $\Gamma_i(x)$ is equal to $k_{i-1}b_{i-1}$. Similarly, the number of neighbors in $\Gamma_{i-1}(x)$ of each vertex $z \in \Gamma_i(x)$ is equal to c_i. That is,

$$|\Gamma_1(z) \cap \Gamma_{i-1}(x)| = c_i.$$

And hence the number of edges between $\Gamma_i(x)$ and $\Gamma_{i-1}(x)$ is equal to $k_i c_i$. Then it follows by double counting that

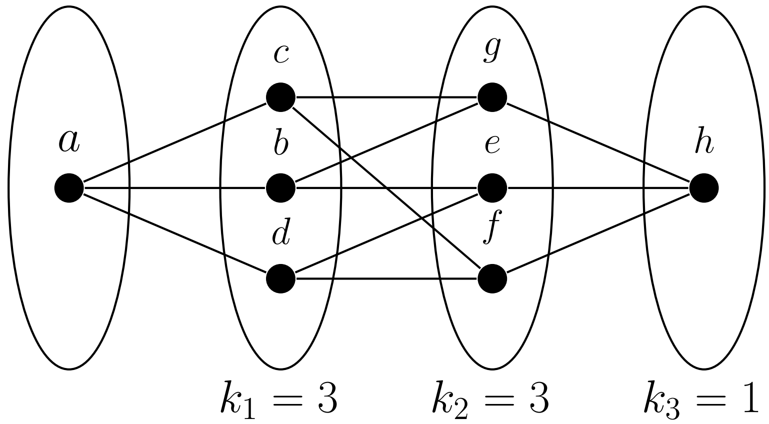
$$k_{i-1}b_{i-1} = k_i c_i,$$

so that, inductively, we obtain

$$\begin{aligned} k_i &= \frac{b_{i-1}}{c_i} k_{i-1} \\ &= \frac{b_{i-1}b_{i-2}}{c_i c_{i-1}} k_{i-2} \\ &\vdots \\ &= \frac{b_{i-1} \cdots b_1 b_0}{c_i \cdots c_2 c_1}. \end{aligned}$$

□

EXAMPLE 2.9. Recall the cubic graph from Example 2.2. We fix vertex a and construct the following associated distance diagram.



By (4), we have

$$
\begin{aligned}
k_1 &= \frac{b_0}{c_1} = \frac{3}{1} = 3, \\
k_2 &= \frac{b_1 b_0}{c_2 c_1} = \frac{3 \cdot 2}{2 \cdot 1} = 3, \\
k_3 &= \frac{b_2 b_1 b_0}{c_3 c_2 c_1} = \frac{1 \cdot 3 \cdot 3}{3 \cdot 3 \cdot 1} = 1.
\end{aligned}
$$

Chapter 3: The Subconstituent Algebra

Consider a graph Γ as discussed in Chapter 2. For the rest of the thesis, we assume that Γ is distance-regular with diameter D. In this chapter, we define the subconstituent algebra that is associated with Γ, and discuss its properties. Throughout this thesis we denote by $Mat_X(\mathbb{C})$ the set of complex matrices indexed by the set X.

Definition 3.1. (Adjacency Matrix) Let Γ be distance-regular with vertex set X and diameter D. For each integer $0 \le i \le D$, we define the matrix $A_i \in Mat_X(\mathbb{C})$ by

$$(A_i)_{xy} = \begin{cases} 1 & \partial(x,y) = i \\ 0 & \partial(x,y) \neq i \end{cases}$$

for any $x, y \in X$. We call A_i the *ith distance matrix* of Γ. In particular, $A_1 = A$ is called the *adjacency matrix* of Γ. Observe that A_i is a real symmetric matrix.

Example 3.2. Recall the cubic graph of Example 2.2. The adjacency matrix for this graph is given by

$$A = \begin{bmatrix} 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

LEMMA 3.3 The following properties hold for the distance matrices of Γ.

$$A_0 = I \tag{5}$$

$$\sum_{i=1}^{D} A_i = J, \qquad J \text{ denotes the all 1's matrix} \tag{6}$$

$$\overline{A_i} = A_i \quad 0 \leq i \leq D \tag{7}$$

$$A_i^T = A_i \quad 0 \leq i \leq D \tag{8}$$

$$A_i A_j = \sum_{h=0}^{D} p_{ij}^h A_h \quad 0 \leq i \leq D. \tag{9}$$

Proof. Property (5) is straightforward from the definition of the A_i's. For $x, y \in X$ with $\partial(x, y) = i$, we have

$$\begin{aligned}
\sum_{i=0}^{D} (A_i)_{xy} &= (A_0 + A_1 + \cdots + A_i + \cdots + A_D)_{xy} \\
&= (A_0)_{xy} + (A_1)_{xy} + \cdots + (A_i)_{xy} + \cdots + (A_D)_{xy} \\
&= 1.
\end{aligned}$$

By the arbitrary choice of x and y, each entry of the sum of the distance matrices is equal to 1, which shows property (6). Properties (7) and (8) follow from the fact that the A_i's are real-symmetric matrices. To prove property (9), consider the left-hand side of the equation. For vertices $x, y \in X$ with $\partial(x, y) = h$, we have

$$(A_i A_j)_{xy} = \sum_{w \in X} (A_i)_{xw} (A_j)_{wy}.$$

But by the definition of the distance matrices,

$$(A_i)_{xw}(A_j)_{wy} = \begin{cases} 1, & \text{if } \partial(x,w) = i \text{ and } \partial(w,y) = j \\ 0, & \text{otherwise} \end{cases}$$

Hence, the sum is exactly the number p^h_{ij}. Now consider the right-hand side of (9). We have

$$\sum_{k=0}^{D}(p^k_{ij}A_k)_{xy} = (p^0_{ij}A_0 + p^1_{ij}A_1 + \cdots + p^h_{ij}A_h + \cdots + p^D_{ij}A_D)_{xy} = p^h_{ij},$$

which follows from the fact that since $\partial(x,y) = h$, we have $(A_i)_{xy} = 0$, $i \neq h$, and $(A_h)_{xy} = 1$. $\square$

By (1), it follows from property (9) that

$$A_iA_j = A_jA_i, \qquad 0 \leq i,j \leq D. \tag{10}$$

Lemma 3.4. Let M be a subspace of $Mat_X(\mathbb{C})$ spanned by $\{A_0, A_1, \cdots, A_D\}$. Then $\{A_i\}_{i=0}^D$ is a basis for M.

Proof. It suffices to show that $\{A_i\}_{i=0}^D$ is linearly independent. Suppose for scalars $\alpha_i \in \mathbb{C}$, we have

$$\alpha_0A_0 + \alpha_1A_1 + \cdots + \alpha_DA_D = \mathbf{O}.$$

where $\mathbf{O}$ denotes the zero matrix. Then, for an arbitrary (x,y)-entry with $\partial(x,y) = i$, we must have

$$\alpha_0(A_0)_{xy} + \alpha_1(A_1)_{xy} + \cdots + \alpha_D(A_D)_{xy} = \alpha_i(A_i)_{xy} = \alpha_i = 0$$

since $(A_i)_{xy} = 1$. By the arbitrary choice of x, y and i, it follows that $\alpha_0 = \alpha_1 = \cdots = \alpha_D = 0$, so that the A_i's are linearly independent. $\square$

Lemma 3.5. The distance matrices satisfy the following 3-term recurrence relation:

$$AA_j = b_{j-1}A_{j-1} + a_jA_j + c_{j+1}A_{j+1}, \qquad (0 \leq j \leq D)$$

where a_j, b_j, c_j are the intersection numbers of Γ with $A_{-1} := 0$ and $A_{D+1} := 0$.

Proof. Setting $i = 1$ in Lemma 3.3 (9) yields

$$AA_j = \sum_{h=0}^{D} p_{1j}^h A_h = p_{1j}^0 A_0 + p_{1j}^1 A_1 + \cdots + p_{1j}^{j-1} A_{j-1} + p_{1j}^j A_j + p_{1j}^{j+1} A_{j+1} + \cdots + p_{1j}^D A_D.$$

By (2), we obtain

$$AA_j = p_{1j}^{j-1} A_{j-1} + p_{1j}^j A_j + p_{1j}^{j+1} A_{j+1} = b_{j-1}A_{j-1} + a_jA_j + c_{j+1}A_{j+1},$$

where the expression on the far right follows from the definition of b_i, a_i, and c_i. $\square$

Thus, AA_j can be written as a linear combination of A_{j-1}, A_j, A_{j+1}. Now, we are ready for the following lemma.

Lemma 3.6. Let A_i be the *ith* distance matrix, $0 \leq i \leq D$. Then there exists a polynomial $f_i \in \mathbb{C}[x]$ of degree i such that $f_i(A) = A_i$.

Proof. We proceed by induction on i. First note that

$$A_0 = I = A^0, \qquad A_1 = A = A^1,$$

so that $f_0(x) = 1$ and $f_1(x) = x$. These are the trivial cases. We need to look at the cases where $r > 1$. By Lemma 3.5, we have

$$AA_r = b_{r-1}A_{r-1} + a_rA_r + c_{r+1}A_{r+1}$$

which implies that

$$c_{r+1}A_{r+1} = AA_r - b_{r-1}A_{r-1} - a_rA_r.$$

Now, let us drop our indices by 1, so that we now have

$$c_rA_r = AA_{r-1} - b_{r-2}A_{r-2} - a_{r-1}A_{r-1}.$$

Now, using $r = 2$ as the base case for our induction yields

$$\begin{aligned} c_2A_2 =& AA_1 - b_0A_0 - a_1A_1 \\ =& AA - b_0I - a_1A \\ =& A^2 - a_1A - b_0I, \end{aligned}$$

which implies

$$A_2 = \frac{1}{c_2}(A^2 - a_1A - b_0),$$

and hence we define $f_2(x) = \frac{1}{c_2}(x^2 - a_1x - b_0)$. Then $f_2(A) = A_2$. Thus, statement holds for $r = 2$. Now assume the result holds up to the $r - 1$ case. By the induction hypothesis, we have

$$\begin{aligned} c_rA_r &= AA_{r-1} - b_{r-2}A_{r-2} - a_{r-1}A_{r-1} \\ &= Af_{r-1}(A) - b_{r-2}f_{r-2}(A) - a_{r-1}f_{r-1}(A). \end{aligned}$$

Since $Af_{r-1}(A)$ is a polynomial of degree r, $a_{r-1}f_{r-1}(A)$ is a polynomial of degree $r-1$, and $b_{r-1}f_{r-2}(A)$ is a polynomial of degree $r-2$, the above reduces into a single polynomial of degree r. That is,

$$c_rA_r = \gamma_r A^r + \cdots \gamma_1 A + \gamma_0 I,$$

which implies that

$$A_r = \frac{1}{c_r}(\gamma_r A^r + \cdots \gamma_1 A + \gamma_0 I),$$

and hence we define $f_r(x) = \frac{1}{c_r}(\gamma_r x^r + \cdots \gamma_1 x + \gamma_0 I)$, so that $f_r(A) = A_r$, as desired. □

Definition 3.7. (Adjacency Algebra) Recall the $(D+1)$-dimensional subspace M with a basis $\{A_0, A_1, \cdots, A_D\}$. By Lemma 3.3 (9), M is closed under matrix multiplication, so M is an (associative) $\mathbb{C}$-algebra. By (10), M is commutative and by Lemma 3.6, M is generated by the adjacency matrix A of Γ. We call M the *adjacency algebra* of Γ.

Consider the adjacency algebra M of Γ and its generator A. Since A is real symmetric, every eigenvalue of A is real. Moreover, A has $D+1$ mutually distinct eigenvalues, since A generates M. We denote the eigenvalues of A by

$$\theta_0, \theta_1, \cdots, \theta_D.$$

We call θ_i an *eigenvalue* of Γ. For $0 \le i \le D$, let E_i denote an orthogonal projection of A onto the eigenspace associated with the eigenvalue θ_i. Then by [1, p.45], $E_0, E_1, \cdots, E_D$ is a basis for M such that

$$E_0 = |X|^{-1}J, \tag{11}$$

$$\sum_{i=0}^{D} E_i = I, \tag{12}$$

$$\overline{E_i} = E_i \quad 0 \leq i \leq D, \tag{13}$$

$$E_i^T = E_i \quad 0 \leq i \leq D, \tag{14}$$

$$E_i E_j = \delta_{ij} E_i \quad 0 \leq i, j \leq D. \tag{15}$$

The elements $E_0, E_1, \cdots, E_D$ are unique and are called the *primitive idempotents* of Γ. We define $E_i = 0$ if $i < 0$ or $i > D$, and

$$m_i = \text{rank } E_i \quad (0 \leq i \leq D).$$

Since $\{A_0, A_1, \cdots, A_D\}$ and $\{E_0, E_1, \cdots, E_D\}$ are both bases for the adjacency algebra, we can write each one as a linear combination of the other. The scalars, which are complex, must depend on the matrices A_i and E_j, so we have

$$A_i = \sum_{j=0}^{D} p_i(j) E_j \quad (0 \leq i \leq D), \tag{16}$$

$$E_i = |X|^{-1} \sum_{j=0}^{D} q_i(j) A_j \quad (0 \leq i \leq D). \tag{17}$$

Since $\overline{A_i} = A_i$ and $\overline{E_i} = E_i$, it is easy to see that $p_i(j), q_i(j)$ are real.

Now, we discuss the dual adjacency algebra of Γ. For the rest of the section, let $x \in X$ be fixed. For $0 \leq i \leq D$, let $E_i^* = E_i^*(x)$ denote the diagonal matrix in $Mat_X(\mathbb{C})$ with the following property

$$(E_i^*)_{yy} = \begin{cases} 1, & \partial(x,y) = i \\ 0, & \partial(x,y) \neq i \end{cases} \tag{18}$$

for $y \in X$. Then we have

$$\sum_{i=0}^{D} E_i^* = I \tag{19}$$

$$\overline{E_i^*} = E_i^* \qquad 0 \leq i \leq D \tag{20}$$

$$E_i^{*T} = E_i^* \qquad 0 \leq i \leq D \tag{21}$$

$$E_i^* E_j^* = \delta_{ij} E_i^*, \qquad 0 \leq i, j \leq D. \tag{22}$$

We call E_i^* the *ith dual primitive idempotents* of Γ with respect to x. By (19)-(22), $\{E_i^*\}_{i=0}^{D}$ forms a basis for a commutative subalgebra $M^*(x)$ of $Mat_X(\mathbb{C})$. We call $M^* = M^*(x)$ the *dual adjacency algebra* of Γ with respect to x. For $0 \leq i \leq D$, let $A_i^* = A_i^*(x)$ denote the diagonal matrix in $Mat_X(\mathbb{C})$ with (y, y)-entry

$$(A_i^*)_{yy} = |X|(E_i)_{xy}, \qquad (y \in X).$$

Then we have the following properties.

$$A_0^* = I \tag{23}$$

$$\sum_{i=0}^{D} A_i^* = |X| E_0^* \tag{24}$$

$$\overline{A_i^*} = A_i^* \tag{25}$$

$$A_i^{*T} = A_i^* \tag{26}$$

$$A_i^* A_j^* = \sum_{h=0}^{D} q_{ij}^h A_h^* \tag{27}$$

We call A_i^* the *ith dual distance matrix* of Γ with respect to x. By (16), (17) and (18), we have

$$A_i^* = \sum_{j=0}^{D} q_i(j) E_j^* \tag{28}$$

$$E_i^* = |X|^{-1} \sum_{j=0}^{D} p_i(j) A_j^*. \tag{29}$$

Since $\{E_i^*\}_{i=0}^D$ is a basis for M^*, it follows from (29) that A_i^* is also a basis for M^*. We abbreviate $A^* = A_1^*$ and call this the *dual adjacency matrix* of Γ. By setting $i = 1$ in (28), we have

$$A^* = \sum_{j=0}^{D} q_1(j) E_j^*. \tag{30}$$

We abbreviate $q_1(j)$ by θ_j^*. Then θ_j^* $(0 \leq j \leq D)$ is an eigenvalue of A^*. We call θ_j^* a *dual eigenvalue* of Γ. From (22) and (30), we have

$$A^* E_i^* = E_i^* A^* = \theta_i^* A^*. \tag{31}$$

Let $T = T(x)$ denote the subalgebra of $Mat_X(\mathbb{C})$ generated by M and M^*.

T is called the *subconstituent algebra* of Γ with respect to x. Note that A and A^* generate T. For $0 \leq h, i, j \leq D$, the following relations in T hold.

$$E_h^* A_i E_j^* = 0 \text{ if and only if } p_{ij}^h = 0, \tag{32}$$

$$E_h A_i^* E_j = 0 \text{ if and only if } q_{ij}^h = 0. \tag{33}$$

To show (32), let $A_i E_j^* = B$ and note that, by definition of the usual matrix product, we have

$$(E_h^* B)_{yz} = \sum_{w \in X} (E_h^*)_{yw} B_{wz}.$$

Suppose that $\partial(x, y) = h$ for our fixed vertex x. Then each term in the above summand is zero, except for when $w = y$. Hence we obtain

$$(E_h^* B)_{yz} = \sum_{w \in X} (E_h^*)_{yw} B_{wz} = (E_h^*)_{yy} B_{yz} = B_{yz} = (A_i E_j^*)_{yz} = \sum_{w \in X} (A_i)_{yw} (E_j^*)_{wz}.$$

Now suppose $\partial(z, x) = j$ and $\partial(y, z) = i$. Then the last summation becomes

$$\sum_{w \in X} (A_i)_{yw} (E_j^*)_{wz} = (A_i)_{yz} (E_j^*)_{zz} = 1.$$

Hence $E_h^* A_i E_j^* \neq 0$ if $\partial(x, y) = h$, $\partial(z, x) = j$ and $\partial(y, z) = i$. That is, by definition, if $p_{ij}^h \neq 0$. The result follows.

Chapter 4: The Hypercube Q_D

We now extend the concepts presented in Chapters 2 and 3 to to a particular distance-regular graph known as a hypercube. In this section we introduce the hypercube Q_D of dimension D and discuss its properties.

DEFINITION 4.1. (Hypercube) Let D be a positive integer and let $\{1,0\}^D$ denote the set of sequences $\alpha_1\alpha_2\cdots\alpha_D$ such that $\alpha_i \in \{1,0\}$ for $1 \leq i \leq D$. We let Q_D denote the graph with vertex set

$$X = \{1,0\}^D,$$

and edge set

$$R = \{(x,y) \mid x, y \in X, x \text{ and } y \text{ differ in exactly one coordinate.}\}.$$

The graph Q_D is called the *hypercube* (or *Hamming cube*) of dimension D.

For $x, y \in X$, we define the distance $\partial(x,y)$ between x and y to be equal to the number of digits in which x and y differ. For example, consider three vertices $a, b, c \in X$ of Q_5 with $a = 10010, b = 11001, c = 01010$. Then we have $\partial(a,b) = 3$ and $\partial(b,c) = 4$.

Observe that Q_D is connected. Since for any vertex $x \in X$, x has D digits, each of which can take the value of 0 or 1, the hypercube graph Q_D has 2^D vertices and $2^{D-1}D$ edges. Additionally, Q_D has valency D. To see this, consider the graph Q_D and the vertex $x = 000\cdots000$ (D zeros). The vertex

x has the following neighbors:

$$100\cdots000,$$
$$010\cdots000,$$
$$001\cdots000,$$
$$\vdots$$
$$000\cdots001.$$

That is, the number of neighbors of x is equal to the number of ways that we can choose a digit and replace it with a 1. Since there are D digits, we have

$$k = k_1 = \binom{D}{1}.$$

A hypercube can be defined by increasing the numbers of dimensions of a shape. For example, a zero-dimensional hypercube is a point, a two-dimensional hypercube is a square, and so on. In the following example we consider the three-dimensional hypercube.

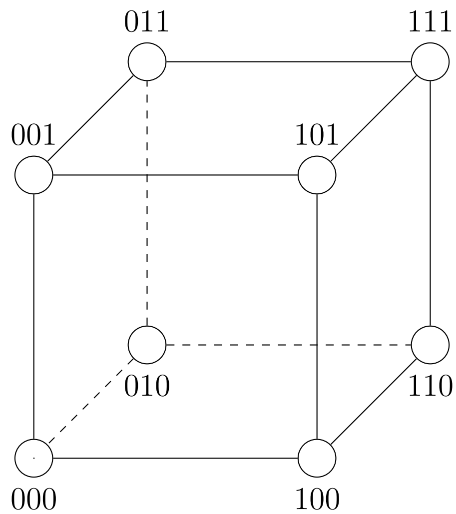
Example 4.2. Let $D = 3$, so that

$$X = \{1, 0\}^3 = \{000, 001, 010, 011, 100, 110, 101, 111\},$$

and

$$\begin{aligned} R = \{&\{000, 001\}, \{000, 010\}, \{000, 100\}, \{010, 001\}, \{010, 110\}, \{101, 001\}, \\ &\{101, 100\}, \{110, 010\}, \{110, 100\}, \{111, 011\}, \{111, 101\}, \{111, 101\}\}. \end{aligned}$$

We obtain the following 3-dimensional hypercube Q_3:



The graph Q_D is bipartite. We construct a bipartition as follows. Fix a vertex $x \in X$ and note that the set $\Gamma_1(x)$ contains vertices that differ from x in 1 coordinate, and hence there exist edges between $\Gamma_0(x)$ and $\Gamma_1(x)$. Similarly, $\Gamma_2(x)$ consists of vertices that differ from x in 2 coordinates and differ from each element of $\Gamma_1(x)$ in 1 coordinate. Hence, there exist edges between $\Gamma_1(x)$ and $\Gamma_2(x)$. Continuing this way, we find that there exist edges between Γ_i and Γ_k if $k = i+1$ or $k = i-1$ for $0 \leq i \leq D$. We construct the following sets U and V such that

$$U = \{\Gamma_0(x) \cup \Gamma_2(x) \cup \Gamma_4(x) \cdots\},$$
$$V = \{\Gamma_1(x) \cup \Gamma_3(x) \cup \Gamma_5(x) \cdots\}.$$

Then $X = U \cup V$ is a bipartition of Q_D.

Lemma 4.3. The *ith* valency of Q_D is given by

$$k_i = \binom{D}{i} \qquad (0 \leq i \leq D).$$

Proof. Without loss of generality, fix the vertex $x \in X$ where x consists of D 0's. Consider a vertex $y \in \Gamma_i(x)$. Then $\partial(x, y) = i$, so that x and y differ in i coordinates. To obtain such a vertex, we swap i of the D coordinates of x to

1. The number of ways to do this is the number of vertices that lie in $\Gamma_i(x)$. The result follows. $\square$

PROPOSITION 4.4. The graph Q_D has intersection numbers

$$c_i = i, \quad a_i = 0, \quad b_i = D - i \qquad (0 \leq i \leq D).$$

Therefore, Q_D is distance-regular.

Proof. Since Q_D is bipartite, it follows from Lemma 2.6 that $a_i = 0$ for all i. It remains to verify the claim for c_i and b_i. We fix an arbitrary vertex x in Q_D, denoted by $x = 000 \cdots 000$ (D zeros). Choose any arbitrary vertex $y \in \Gamma_i(x)$. Without loss of generality, we denote the vertex y by

$$y = \underbrace{111 \cdots 11}_{i}\underbrace{000 \cdots 00}_{D-i}.$$

which has i 1's and $D - i$ 0's. We will first consider the neighbors of y which lie in $\Gamma_{i-1}(x)$. Such a vertex has the following two properties: it must differ from y in one digit and simultaneously contain $i - 1$ 1's (since it is at distance $i - 1$ from x). Therefore, each neighbor of y lying in $\Gamma_{i-1}(x)$ must have a 0 in one of the digits where y contains a 1. Hence, the neighbors of y that lie in $\Gamma_{i-1}(x)$ have the form

$$\begin{gathered} 011 \cdots 11000 \cdots 00 \\ 101 \cdots 11000 \cdots 00, \\ 110 \cdots 11000 \cdots 00, \\ \vdots \\ 111 \cdots 10000 \cdots 00, \end{gathered}$$

consisting each of $i-1$ digits of 1's. Thus, starting with vertex y, there are i digits that we can change in order to obtain a neighbor of y that lies in $\Gamma_{i-1}(x)$. By the arbitrary choice of vertices x and y, it follows that $c_i = |\Gamma_i(x) \cap \Gamma(y)| = i$. We now consider the neighbors of y which lie in $\Gamma_{i+}(x)$. Each neighbor of y that lies in $\Gamma_{i+1}(x)$ has $i+1$ digits of 1 and also differs from y in exactly one digit. The only way to obtain such a vertex is to change one of the 0's of y to a 1. Thus, since y contains $D-i$ zeros, we have $D-i$ neighbors of y lying in $\Gamma_{i+1}(x)$. Again by the arbitrary choice of x and y, it follows that $b_i = |\Gamma_{i+}(x) \cap \Gamma(y)| = D - i$. □

LEMMA 4.5. Recall that the eigenvalues of a graph are the eigenvalues of its adjacency matrix. Let $\theta_0, \theta_1, \cdots, \theta_D$ denote the eigenvalues of Q_D, and for $0 \le i \le D$, let m_i denote the multiplicity of θ_i. Then we have,

$$\begin{aligned} \theta_i &= D - 2i \quad (0 \le i \le D), \\ m_i &= \binom{D}{i} \quad (0 \le i \le D). \end{aligned}$$

In order to provide a sketch of the proof, we first need the following definition.

DEFINITION 4.6. (Cross Product for Graphs) Let G, H be two graphs with vertex sets X_G and X_H. The vertex set of $G \times H$ is the Cartesian product $X_G \times X_H$. Additionally, two vertices (u, u') and (v, v') are adjacent in $G \times H$ if and only if either

$u = v$ and u' is adjacent to v' in H, **or**

$u' = v'$ and u is adjacent to v in G.

For example, let us consider the complete graph[1] $G = K_2$ with vertices u, v.

[1] a graph G is called *complete* if there is an edge between any pair of vertices.

Let $H = K_2$ with vertices u', v'. The vertex set of $G \times H$ would then be

$$X_G \times X_H = \{(u, u'), (u, v'), (v, u'), (v, v')\}.$$

Additionally, by definition, we have that (u, u') is adjacent to (u, v') and (v, u'), with (u, v') and (v, u') also both being adjacent to (v, v'). Hence we arrive at the fact that $K_2 \times K_2$ is isomorphic to Q_2, which we denote by $K_2 \times K_2 \cong Q_2$. Continuing the above example, consider $K_2 \times K_2 \times K_2 \cong Q_2 \times K_2$. As above, the vertex set of Q_2 is $X = \{(u, u'), (u, v'), (v, u'), (v, v')\}$. Let this copy of K_2 have vertex set $Y = \{x, y\}$. Thus, $X \times Y$ consists of the following vertices

$$\begin{array}{cc}
((u, u'), x), & ((u, u'), y), \\
((u, v'), x), & ((u, v'), y), \\
((v, u'), x), & ((v, u'), y), \\
((v, v'), x), & ((v, v'), y).
\end{array}$$

Again using Definition 4.6, we find that each vertex listed above is adjacent to three others, and hence we have constructed Q_3. That is, $Q_2 \times K_2 \cong K_2 \times K_2 \times K_2 \cong Q_3$. Inductively, we have

$$Q_D \cong K_2 \times K_2 \times \cdots \times K_2 \qquad (D \text{ copies}).$$

We are now ready for the following sketch of the proof of Lemma 4.5.

Proof. (sketch) It can be shown that if G and H are graphs, then the eigenvalues of the Cartesian product $G \times H$ are $\theta_G + \theta_H$, where θ_G and θ_H range over all eigenvalues of G and H, respectively. As we observed, Q_D can be viewed as the Cartesian product $K_2 \times K_2 \times \cdots \times K_2$ (D copies). Since the adjacency matrix of K_2 is

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

which has eigenvalues 1 and -1, each with multiplicity 1, we can say that the eigenvalues of Q_D are given by the sums

$$\lambda_1 + \lambda_2 + \cdots + \lambda_D, \tag{34}$$

with $\lambda_i \in \{1, -1\}$ for $1 \leq i \leq D$. Now, let us define θ_i as the eigenvalue of Q_D that has i copies of -1 in the sum. Thus, we have

$$\begin{aligned}
&\theta_0 = 1 + 1 + \cdots + 1 + 1 = D \\
&\theta_1 = -1 + 1 + \cdots + 1 + 1 = D - 2 \\
&\theta_2 = -1 - 1 + 1 + \cdots + 1 + 1 = D - 4 \\
&\vdots \\
&\theta_i = \underbrace{-1 - 1 - \cdots - 1 - 1}_{i} + 1 + 1 = D - 2i.
\end{aligned}$$

Since the number of ways to replace i digits by a -1 is simply $\binom{D}{i}$, $\theta_i = D - 2i$ appears $\binom{D}{i}$ times for $0 \leq i \leq D$. The result follows. $\square$

We finish this section with the following lemma.

LEMMA 4.7. **[6]** Let $\theta_0^*, \theta_1^*, \cdots \theta_D^*$ denote the dual eigenvalues of Q_D, and let m_i^* denote the multiplicity of θ_i^*. Then,

$$\begin{aligned}
\theta_i^* &= D - 2i, \\
m_i^* &= \binom{D}{i}.
\end{aligned}$$

COROLLARY 4.8. Let A be the adjacency matrix of Q_D. Then for $y, z \in X$, the (y, z)-entry of A^2 is equal to the number of paths of length 2 that join y and z. This follows from Lemma 2.6.

Chapter 5: The Fundamental Relations

Following the notation of Chapter 4, we now fix $\Gamma = Q_D$ for the rest of the thesis. Recall that in Chapter 3 we defined the subconstituent algebra $T = T(x)$ of Γ with respect to x. In this chapter we show two fundamental relations in T associated with its generators A and A^*.

LEMMA 5.1. Take any vertices x, y, z of Q_D such that $\partial(x, y) = \partial(x, z)$ and $\partial(y, z) = 2$. Then the following two statements are true:

(i) There exists a unique vertex u of Q_D such that $\partial(y, u) = 1$, $\partial(z, u) = 1$ and $\partial(x, u) = \partial(x, y) - 1$.

(ii) There exists a unique vertex v of Q_D such that $\partial(y, v) = 1, \partial(z, v) = 1$ and $\partial(x, v) = \partial(x, y) + 1$.

Proof. By the structure of Q_D; see Figure 4(a) below.

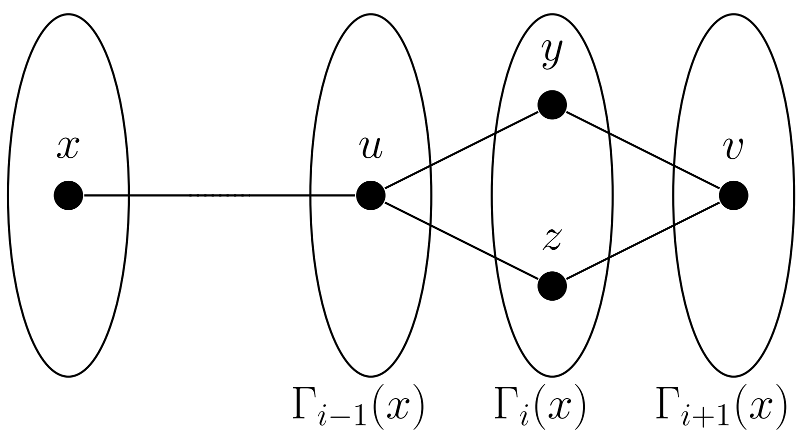


Figure 4(a)

$\square$

THEOREM 5.2. Fix a vertex x of Q_D, and write $A^* = A^*(x)$. Then we have the following two relations:

(i) $A^2A^* - 2AA^*A + A^*A^2 = 4A^*$,

(ii) $A^{*2}A - 2A^*AA^* + AA^{*2} = 4A$.

Proof. (i) Take vertices y, z of Q_D. We compute the (y,z)-entry of the left-hand side of equation (i), and the right-hand side of equation (i), to show that they are equal. To this end, we consider four cases.

Case 1)	$y = z,$	$\partial(x,y) = i$	
Case 2)	$y \neq z,$	$\partial(x,y) = i,$	$\partial(x,z) = i-2$
Case 3)	$y \neq z,$	$\partial(x,y) = i,$	$\partial(x,z) = i$
Case 4)	$y \neq z,$	$\partial(x,y) = i,$	$\partial(x,z) = i+2.$

All other cases have a 0 entry.

Case 1: Suppose $\partial(x,y) = i$ and $y = z$. Then we have the following:

$$\begin{cases} \left(A^2A^*\right)_{yz} = \left(A^2A^*\right)_{yy} = \sum\limits_{w\in X}(A^2)_{yw}(A^*)_{wy} = (A^2)_{yy}(A^*)_{yy} = D\sum\limits_{j=1}^{D}(\theta_j^*E_j^*)_{yy} = D\theta_i^*. \\ \left(A^*A^2\right)_{yz} = \left(A^*A^2\right)_{yy} = \sum\limits_{w\in X}A^*_{yw}(A^2)_{wy} = (A^*)_{yy}(A^2)_{yy} = D\sum\limits_{j=1}^{D}(\theta_j^*E_j^*)_{yy} = D\theta_i^*. \\ (A^*)_{yz} = (A^*)_{yy} = \sum\limits_{j=1}^{D}(\theta_j^*E_j^*)_{yy} = \theta_i^*. \end{cases}$$

The middle term of the left-hand side of (i) requires some explanation. Let $B = A^*A$. Then, for a given vertex w, we have

$$B_{wy} = \sum_{t\in X}(A^*)_{wt}(A)_{ty} = (A^*)_{ww}(A)_{wy} = (\sum_{i=0}^{D}\theta_i^*E_i^*)_{ww}(A)_{wy}.$$

Now note that

$$(\sum_{i=0}^{D}\theta_i^*E_i^*)_{ww}(A)_{wy} = \begin{cases} 1, & y \sim w \\ 0, & \text{otherwise} \end{cases}.$$

Thus, w is either a member of Γ_{i+1} or a member of Γ_{i-1}. If the former is the case, then

$$(\sum_{i=0}^{D} \theta_i^* E_i^*)_{ww}(A)_{wy} = \theta_{i+1}^*,$$

and we know that there are b_i such w. If the latter is the case, we have

$$(\sum_{i=0}^{D} \theta_i^* E_i^*)_{ww}(A)_{wy} = \theta_{i-1}^*,$$

and we know that there are c_i such w. Now, it follows that

$$(AA^*A)_{yz} = (AB)_{yy} = \sum_{w \in X} (A)_{yw}(B)_{wy} = c_i\theta_{i-1}^* + b_i\theta_{i+1}^*.$$

Using Lemma 4.7 and Proposition 4.4 we obtain

$$\begin{aligned}
c_i(\theta_{i-1}^*) + b_i(\theta_{i+1}^*) &= i(D - 2(i-1)) + (D-i)(D - 2(i+1)) \\
&= i(D - 2i + 2) + (D-i)(D - 2i - 2) \\
&= Di - 2i^2 + 2i + D^2 - 2D_i - 2D - D_i + 2i^2 + 2i \\
&= D^2 + 4i - 2D_i - 2D.
\end{aligned}$$

Then the following table shows each term of the equation for this case:

term	A^2A^*	AA^*A	A^*A^2	A^*
(y, z)-entry	$D\theta_i^*$	$D^2 + 4i - 2Di - 2D$	$D\theta_i^*$	θ_i^*

Then, we see that

$$\begin{aligned}(A^2A^* - 2AA^*A + A^*A^2)_{yz} &= D\theta_i^* - 2(D^2 + 4i - 2Di - 2D) + D\theta_i^* \\ &= 4(D - 2i) \\ &= 4\theta_i^* \\ &= 4(A^*)_{yz}.\end{aligned}$$

For the following three cases, we recall that A^* is a diagonal matrix and hence the last term in (i) is zero on the off-diagonal entries.

Case 2: Suppose $\partial(x, z) = i - 2$ and $\partial(y, z) = 2$. Then we have the following:

$$\begin{cases} \left(A^2A^*\right)_{yz} = \sum_{w\in X}(A^2)_{yw}(A^*)_{wz} = (A^2)_{yz}(A^*)_{zz} = 2\sum_{j=1}^{D}(\theta_j^*E_j^*)_{zz} = 2\theta_{i-2}^*. \\ \left(A^*A^2\right)_{yz} = \sum_{w\in X}(A^*_{yw})(A^2)_{wz} = (A^*)_{yy}(A^2)_{yz} = 2\sum_{j=1}^{D}(\theta_j^*E_j^*)_{yy} = 2\theta_i^*. \\ (A^*)_{yz} = 0. \end{cases}$$

We now compute the middle term. We have

$$(AA^*A)_{yz} = \sum_{w\in X}(AA^*)_{yw}A_{wz}.$$

Since A^* is a diagonal matrix, we have

$$(AA^*)_{yw} = \sum_{w\in X}(AA^*)_{yw} = \sum_{u\in X}A_{yu}A^*_{uw} = A_{yw}A^*_{ww}.$$

Thus we have

$$(AA^*A)_{yz} = \sum_{w \in X} A_{yw} A^*_{ww} A_{wz} = 2\theta^*_{i-1},$$

which follows because $w \in \Gamma_{i-1}(x)$, and there are only two vertices w such that $\partial(y, w) = 1$ and $\partial(z, w) = 1$. The following table shows each term of the equation for this case:

term	A^2A^*	AA^*A	A^*A^2	A^*
(y, z)-entry	$2\theta^*_{i-2}$	$2\theta^*_{i-1}$	$2\theta^*_i$	0

Then, we see that

$$\begin{aligned}(A^2A^* - 2AA^*A + A^*A^2)_{yz} &= 2(D - 2i + 4) - 2(2D - 4i + 4) + 2D - 4i \\ &= 0 \\ &= 4(A^*)_{yz}.\end{aligned}$$

Case 3: Suppose $\partial(x, z) = i$ and $\partial(x, y) = i$. Then we have the following:

$$\begin{cases} \left(A^2A^*\right)_{yz} = \sum_{w \in X} (A^2)_{yw}(A^*)_{wz} = (A^2)_{yz}(A^*)_{zz} = 2\sum_{j=1}^{D} (\theta^*_j E^*_j)_{zz} = 2\theta^*_i. \\ \left(A^*A^2\right)_{yz} = \sum_{w \in X} (A^*_{yw})(A^2)_{wz} = (A^*)_{yy}(A^2)_{yz} = 2\sum_{j=1}^{D} (\theta^*_j E^*_j)_{yy} = 2\theta^*_i. \\ (A^*)_{yz} = 0. \end{cases}$$

We now compute the middle term. We have

$$(AA^*A)_{yz} = \sum_{w \in X} (AA^*)_{yw} A_{wz}.$$

Since A^* is a diagonal matrix, we have

$$(AA^*)_{yw} = \sum_{u \in X} A_{yu} A^*_{uw} = A_{yw} A^*_{ww}.$$

Thus we have

$$(AA^*A)_{yz} = \sum_{w \in X} A_{yw} A^*_{ww} A_{wz} = \theta^*_{i-1} + \theta^*_{i+1},$$

which follows from Lemma 5.1. That is, there exist only two vertices joining y and z in by a path of length two; one vertex $v \in \Gamma_{i+1}(x)$ and the other vertex $u \in \Gamma_{i-1}(x)$. The following table shows each term of the equation for this case:

term	A^2A^*	AA^*A	A^*A^2	A^*
(y,z)-entry	$2\theta^*_i$	$\theta^*_{i-1} + \theta^*_{i+1}$	$2\theta^*_i$	0

Then, we see that

$$\begin{aligned}(A^2A^* - 2AA^*A + A^*A^2)_{yz} &= 4(D-2i) - 2(D-2i+2+D-2i-2) \\ &= 4D - 8i - 2D + 4i - 4 - 2D + 4i + 4 \\ &= 0 = 4A^*.\end{aligned}$$

Case 4: Suppose $\partial(x,z) = i+2$ and $\partial(y,z) = 2$. Then we have the following:

$$\begin{cases} \left(A^2A^*\right)_{yz} = \sum_{w \in X} (A^2)_{yw}(A^*)_{wz} = (A^2)_{yz}(A^*)_{zz} = 2\sum_{j=1}^{D} (\theta^*_j E^*_j)_{zz} = 2\theta^*_{i+2}. \\ \left(A^*A^2\right)_{yz} = \sum_{w \in X} (A^*_{yw})(A^2)_{wz} = (A^*)_{yy}(A^2)_{yz} = 2\sum_{j=1}^{D} (\theta^*_j E^*_j)_{yy} = 2\theta^*_i. \\ (A^*)_{yz} = 0. \end{cases}$$

Notice that case 4 has the same set up as case 2, except here y and z are joined by a vertex $w \in \Gamma_{i+1}(x)$, of which there are only two choices. Hence we obtain

$AA^*A = 2\theta^*_{i+1}$. The following table shows each term of the equation for this case:

term	A^2A^*	AA^*A	A^*A^2	A^*
(y,z)-entry	$2\theta^*_{i+2}$	$2\theta_{i+1}$	$2\theta^*_i$	0

Then we have

$$(A^2A^* - 2AA^*A + A^*A^2)_{yz} = 2(D-2i-4) - 4(D-2i-2) + 2D - 4i \tag{35}$$

$$= 2D - 4i - 8 - 4D + 8i + 8 + 2D - 4i \tag{36}$$

$$= 0 = 4A^*. \tag{37}$$

We see that in all other cases, each of the four entries are equal to zero. For the four cases we have shown above, the equation (i) holds.

We now show equation (ii). Take vertices y, z of Q_D. We compute the (y,z)-entry of the left-hand side of equation (ii), and the right-hand side of equation (i), to show that they are equal. To this end, we consider two cases.

Case 1) $\quad \partial(x,z) = i-1 \quad \partial(x,y) = i \quad \partial(y,z) = 1$

Case 2) $\quad \partial(x,z) = i+1 \quad \partial(x,y) = i \quad \partial(y,z) = 1.$

Case 1: Suppose $\partial(x,z) = i-1$ and $\partial(y,z) = 1$. Then we have the following:

$$\begin{cases} \left(A^{*2}A\right)_{yz} = \sum_{w\in X}(A^{*2})_{yw}(A)_{wz} = (A^{*2})_{yy}(A)_{yz} = \sum_{j=1}^{D}(\theta^*_j E^*_j)_{yy} = \theta^{*2}_i. \\ \left(AA^{*2}\right)_{yz} = \sum_{w\in X}(A_{yw})(A^{*2})_{wz} = (A)_{yz}(A^{*2})_{zz} = \sum_{j=1}^{D}(\theta^*_j E^*_j)_{yy} = \theta^{*2}_{i-1}. \\ (A)_{yz} = 1. \end{cases}$$

We now compute the middle term. We have

$$(A^*AA^*)_{yz} = \sum_{w \in X} (A^*A)_{yw} A^*_{wz}.$$

Since A^* is a diagonal matrix, we have

$$(A^*A)_{yw} = \sum_{u \in X} A^*_{yu} A_{uw} = A^*_{yy} A_{yw} = \theta^*_i A_{yw}.$$

Thus, we have

$$(A^*AA^*)_{yz} = \theta^*_i \sum_{w \in X} A_{yw} A^*_{wz} = \theta^*_i A_{yz} A^*_{zz} = \theta^*_i \theta^*_{i-1}.$$

The following table shows each term of the equation for this case:

term	$A^{*2}A$	A^*AA^*	AA^{*2}	A
(y, z)-entry	θ^{*2}_i	$\theta^*_i \theta^*_{i-1}$	θ^{*2}_{i-1}	1

Thus, we obtain

$$\begin{aligned}
(A^{*2}A - 2A^*AA^* + AA^{*2})_{yz} &= \theta^{*2}_i - 2\theta^*_i \theta^*_{i-1} + \theta^{*2}_{i-1} \\
&= (D - 2i)^2 - 2(D - 2i)(D - 2(i - 1)) + (D - 2(i - 1))^2 \\
&= 1 \\
&= 4A_{yz}.
\end{aligned}$$

Case 2: Suppose $\partial(x,z) = i+1$ and $\partial(y,z) = 1$. Then we have the following:

$$\begin{cases} \left(A^{*2}A\right)_{yz} = \sum_{w\in X}(A^{*2})_{yw}(A)_{wz} = (A^{*2})_{yy}(A)_{yz} = \sum_{j=1}^{D}(\theta_j^{*2}E_j^*)_{yy} = \theta_i^{*2}. \\ \left(AA^{*2}\right)_{yz} = \sum_{w\in X}(A_{yw})(A^{*2})_{wz} = (A)_{yz}(A^{*2})_{zz} = \sum_{j=1}^{D}(\theta_j^{*2}E_j^*)_{yy} = \theta_{i+1}^{*2}. \\ (A)_{yz} = 1. \end{cases}$$

The computation of the middle term for case 2 is identical to that of case 1; the only difference is now we have $\partial(x,z) = i+1$. Hence, we obtain

$$(A^*AA^*)_{yz} = \theta_i^*\theta_{i+1}^*.$$

The following table shows each term of the equation for this case:

term	$A^{*2}A$	A^*AA^*	AA^{*2}	A
(y,z)-entry	θ_i^{*2}	$\theta_i^*\theta_{i+1}^*$	θ_{i+1}^{*2}	1

Thus, we obtain

$$\begin{aligned} (A^{*2}A - 2A^*AA^* + AA^{*2})_{yz} &= \theta_i^{*2} - 2\theta_i^*\theta_{i+1}^* + \theta_{i+1}^{*2} \\ &= (D-2i)^2 - 2(D-2i)(D-2(i+1)) + (D-2(i+1))^2 \\ &= 1 \\ &= 4A_{yz}. \end{aligned}$$

We see that in all other cases, each of the four entries are equal to zero. For the two cases we have shown above, the equation (ii) holds. □

Chapter 6: The Raising and Lowering Matrices

In this section, we define the matrices R and L and show their connections with the adjacency and dual adjacency matrices A and A^* of Q_D.

DEFINITION 6.1. (Raising and lowering matrices) For a fixed vertex x of Q_D, with $E_i^* = E_i^*(x), 0 \le i \le D$, we define $R = R(x)$ and $L = L(x)$ by

$$R = \sum_{i=0}^{D} E_{i+1}^* A E_i^*, \qquad L = \sum_{i=0}^{D} E_{i-1}^* A E_i^*.$$

We refer to R and L as the *raising matrix* and *lowering matrix* respectively (and with respect to x). We observe that

$$\bar{R} = R, \qquad \bar{L} = L, \qquad R^t = L,$$

$$RE_i^*V \subseteq E_{i+1}^*V, \qquad LE_i^*V \subseteq E_{i-1}^*V, \qquad 0 \le i \le D.$$

and

$$R + L = A. \tag{38}$$

where $V = \mathbb{C}^X$.

LEMMA 6.2. Fix a vertex x of Q_D, write $A^* = A^*(x)$ and let $R = R(x)$, $L = L(x)$ be as in Definition 6.1. Then

(i) $R = \dfrac{AA^* - A^*A + 2A}{4}$

(ii) $L = \dfrac{A^*A - AA^* + 2A}{4}$.

Proof. (i) Abbreviate

$$C = \frac{AA^* - A^*A + 2A}{4}.$$

We will show that

$$C = \sum_{i=0}^{D} E^*_{i+1} A E^*_i.$$

Using (19), we have

$$\begin{aligned}
C = ICI &= (\sum_{j=0}^{D} E^*_j) C (\sum_{k=0}^{D} E^*_k) \\
&= (E^*_0 + E^*_1 + \cdots + E^*_D) C (E^*_0 + E^*_1 + \cdots + E^*_D) \\
&= \sum_{j=0}^{D} \sum_{k=0}^{D} (E^*_j C E^*_k).
\end{aligned}$$

Observe that a single term of the above summation can be written as

$$\begin{aligned}
E^*_j C E^*_k &= E^*_j \left(\frac{AA^* - A^*A + 2A}{4} \right) E^*_k \\
&= \frac{E^*_j AA^* E^*_k - E^*_j A^* A E^*_k + 2E^*_j A E^*_k}{4} \\
&= \frac{E^*_j A \theta^*_k E^*_k - \theta^*_j E^*_j A E^*_k + 2E^*_j A E^*_k}{4} \\
&= \frac{\theta^*_k - \theta^*_j + 2}{4} (E^*_j A E^*_k),
\end{aligned}$$

where the third equality follows from (31). Note from (32) that

$$E^*_j A_i E^*_k = 0 \qquad \text{if } |j - k| \neq 1,$$

which implies that the only non-zero terms are when $k = j + 1$ or $k = j - 1$. Then we have

$$\begin{aligned}
\sum_{j=0}^{D}\sum_{k=0}^{D}(E_j^*CE_k^*) &= \sum_{j=0}^{D}(E_j^*CE_{j+1}^* + E_j^*CE_{j-1}^*)\\
&= \sum_{j=0}^{D}\left(\frac{\theta_{j+1}^* - \theta_j^* + 2}{4}(E_j^*AE_{j+1}^*) + \frac{\theta_{j-1}^* - \theta_j^* + 2}{4}(E_j^*AE_{j-1}^*)\right)
\end{aligned}$$

By Lemma 4.7 we have

$$\frac{\theta_{j+1}^* - \theta_j^* + 2}{4} = \frac{D - 2(j+1) - D + 2j + 2}{4} = 0,$$

and

$$\frac{\theta_{j-1}^* - \theta_j^* + 2}{4} = \frac{D - 2(j-1) - D + 2j + 2}{4} = 1.$$

Therefore, we see that

$$\begin{aligned}
C &= \sum_{j=0}^{D}\sum_{k=0}^{D}(E_j^*CE_k^*)\\
&= \sum_{j=0}^{D} E_j^*AE_{j-1}^*.
\end{aligned}$$

By shifting the indices, we have

$$C = \sum_{i=0}^{D} E_{i+1}^*AE_i^* = R,$$

as desired. The proof of (ii) is similar. $\square$

Lemma 6.3. Fix a vertex x of Q_D, and let $A^* = A^*(x)$ and let $R = R(x)$ and $L = L(x)$. Then the following equations hold:

(i) $LR - RL = A^*$,

(ii) $RA^* - A^*R = 2R$,

(iii) $LA^* - A^*L = -2L$.

Proof. (i) We have from Lemma 6.2 that

$$LR - RL = \frac{(A^*A - AA^* + 2A)(AA^* - A^*A + 2A)}{16} - \frac{(AA^* - A^*A + 2A)(A^*A - AA^* + 2A)}{16}.$$

Evaluating each term separately yields

$$LR = \frac{A^*AAA^* - A^*AA^*A + 2A^*AA - AA^*AA^*}{16} + \frac{AA^*A^*A - 2AA^*A + 2AAA^* - 2AA^*A + 4A^2}{16},$$

and

$$RL = \frac{AA^*A^*A - AA^*AA^* + 2AA^*A - A^*AA^*A}{16} + \frac{A^*AAA^* - 2A^*AA + 2AA^*A - 2AAA^* + 4A^2}{16}.$$

After several cancellations, we are left with

$$\begin{aligned} LR - RL = \frac{4A^*AA - 8AA^*A + 4AAA^*}{16} &= \frac{4A^*A^2 - 8AA^*A + 4A^2A^*}{16} \\ &= \frac{4 \cdot 4A^*}{16} \\ &= A^*. \end{aligned}$$

We used Theorem 5.2 in simplifying the last string of equalities.

(ii) We have

$$\begin{aligned}
RA^* - A^*R &= \frac{(AA^* - A^*A + 2A)A^*}{4} - \frac{A^*(AA^* - A^*A + 2A)}{4} \\
&= \frac{AA^*A^* - A^*AA^* + 2AA^* - A^*AA^* + A^*A^*A - 2A^*A}{4} \\
&= \frac{AA^{*2} - 2A^*AA^* + 2AA^* - 2A^*A + A^{*2}A}{4} \\
&= \frac{2AA^* - 2A^*A + (A^{*2}A - 2A^*AA^* + AA^{*2})}{4} \\
&= \frac{2AA^* - 2A^*A + 4A}{4} = \frac{2(AA^* - A^*A + 2A)}{4} = 2R.
\end{aligned}$$

(iii) Similar to the proof of (ii), we obtain

$$\begin{aligned}
LA^* - A^*L &= \frac{(A^*A - AA^* + 2A)A^*}{4} - \frac{A^*(A^*A - AA^* + 2A)}{4} \\
&= \frac{A^*AA^* - AA^{*2} + 2AA^* - A^{*2}A + A^*AA^* - 2A^*A}{4} \\
&= \frac{-2A^*A + 2AA^* - (A^{*2}A - 2A^*AA^* + AA^{*2})}{4} \\
&= \frac{-2A^*A + 2AA^* - 4A}{4} = \frac{-2(A^* - AA^* + 2A)}{4} = -2L.
\end{aligned}$$

□

Chapter 7: The Lie Algebra $Sl_2(\mathbb{C})$

In this chapter, we define the Lie algebra $sl_2(\mathbb{C})$ and construct a homomorphism from the universal enveloping algebra of $sl_2(\mathbb{C})$ to T. Using the notation of Chapter 6, we present our second main result. We denote by $Mat_2(\mathbb{C})$ the $\mathbb{C}$-algebra consisting of all 2×2 matrices with entries in $\mathbb{C}$. Let $\mathbf{L}$ denote the vector space consisting of elements that are subsets of $Mat_2(\mathbb{C})$ that have trace 0. That is,

$$\mathbf{L} = \{A \in Mat_2(\mathbb{C}) : \mathrm{tr}(A) = 0\}.$$

By the *Lie algebra* $sl_2(\mathbb{C})$ we mean the vector space $\mathbf{L}$ together with a bilinear map [,]

$$\begin{aligned}
&\mathbf{L} \times \mathbf{L} \to \mathbf{L} \\
&[x, y] \to xy - yx \qquad (x, y \in \mathbf{L}).
\end{aligned}$$

We refer to the map [,] as the *Lie bracket*.

LEMMA 7.1. The Lie algebra $sl_2(\mathbb{C})$ has a (standard) basis

$$r = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad l = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad h = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Proof. Let $A = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ denote an element of $\mathbf{L}$. Then

$$A = cr + bl + ah.$$

Since r, h and l are linearly independent, the result follows. $\square$

LEMMA 7.2. The elements of the standard basis $\{r, l, h\}$ are related as follows.

$$[l, r] = h$$

$$[r, h] = 2r$$

$$[l, h] = -2l.$$

Proof.

$$[l, r] = lr - rl = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = h,$$

$$[r, h] = rh - hr = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = 2r,$$

$$[l, h] = lh - hl = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 0 \end{bmatrix} - 2l.$$

□

DEFINITION 7.3. (Universal enveloping algebra of $sl_2(\mathbb{C})$) Let U denote the associative $\mathbb{C}$-algebra with 1 generated by symbols $\mathcal{R}, \mathcal{L}, \mathcal{H}$ subject to the relations

$$\mathcal{L}\mathcal{R} - \mathcal{R}\mathcal{L} = \mathcal{H}, \qquad \mathcal{R}\mathcal{H} - \mathcal{H}\mathcal{R} = 2\mathcal{R}, \qquad \mathcal{L}\mathcal{H} - \mathcal{H}\mathcal{L} = -2\mathcal{L}.$$

We call U the universal enveloping algebra of $sl_2(\mathbb{C})$.

The following theorem is the second main result in the paper. We recall the hypercube Q_D of dimension D.

THEOREM 7.4. Fix a vertex x of Q_D, and let $T = T(x)$ be the subconstituent algebra of Q_D. Then the following two conditions hold:

(i) There exists a unique homomorphism of $\mathbb{C}$-algebras $\rho = \rho_x$ from U to T satisfying

$$\rho(\mathcal{L}) = L, \qquad \rho(\mathcal{R}) = R, \qquad \rho(\mathcal{H}) = A^*,$$

(ii) ρ is onto T.

Proof. (i) To show that the relations in U are preserved under ρ, we need only show that

$$\begin{aligned}
&\rho(\mathcal{L}\mathcal{R} - \mathcal{R}\mathcal{L}) = \rho(\mathcal{H}), \\
&\rho(\mathcal{R}\mathcal{H} - \mathcal{H}\mathcal{R}) = 2\rho(\mathcal{R}), \\
&\rho(\mathcal{L}\mathcal{H} - \mathcal{H}\mathcal{L}) = -2\rho(\mathcal{L}).
\end{aligned}$$

Using the linearity of ρ with the definitions of R and L:

$$\begin{aligned}
\rho(\mathcal{L}\mathcal{R} - \mathcal{R}\mathcal{L}) &= \rho(\mathcal{L})\rho(\mathcal{R}) - \rho(\mathcal{R})\rho(\mathcal{L}) \\
&= LR - RL \\
&= A^* \\
&= \rho(\mathcal{H}).
\end{aligned}$$

Where the third equality follows from Lemma 6.3(i). The other two equalities can be shown in a similar manner.

(ii) Since T is generated by A and A^*, it suffices to show that the preimage of A

and A^* under ρ lies in U. Thus,

$$\begin{aligned}\rho(\mathcal{R}+\mathcal{L}) &= \rho(\mathcal{R})+\rho(\mathcal{L}) \\ &= R+L \\ &= A,\end{aligned}$$

where the last equality follows from (38). Also,

$$\rho(\mathcal{H}) = A^*.$$

$\square$

Conclusion

The main results of this thesis are the relationship in T between the adjacency matrix A and the dual adjacency matrix A^* of the hypercube Q_D, as well as the unique homomorphism mapping U to T. These main results involve the underlying algebra T of Q_D, which is the primary focus of our study. However, as mentioned, the subconstituent algebra T has several connections to other areas of mathematics, such as coding theory, representation theory, the theory of double affine Hecke algebras, and more. The subconstituent algebra is a useful tool in the study of these fields of mathematics. For further study of these connections, we refer the reader to [3], [4], [6], and [8].

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