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Harmonic Morphisms with One-dimensional Fibres and Milnor Fibrations

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by

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Abstract

We study a problem at the intersection of harmonic morphisms and real analytic Milnor fibrations. In [9], Baird and Ou establish that a harmonic morphism from $G : \mathbb{R}^m \setminus V_G \to \mathbb{R}^n \setminus \{0\}$ defined by homogeneous polynomials of order p retracts to a harmonic morphism $\psi| : S^{m-1} \setminus K_\epsilon \to S^{n-1}$ that induces a Milnor fibration over the sphere. In seeking to relax the homogeneity assumption on the map G, we determine that the only harmonic morphism $\varphi : \mathbb{R}^m \setminus V_G \to S_\epsilon^{m-1}$ that preserves $\arg G$ is radial projection. Due to this limitation, we confirm Baird and Ou's result, yet establish further that in fact only homogeneous polynomial harmonic morphisms retract to harmonic-morphism Milnor maps over the sphere.

1 Introduction

Recall that a function f that mapsan n-dimensional Riemannian manifold N to $\mathbb{R}$ is considered *harmonic* if it satisfies the Laplace equation: $\Delta f = 0$. A *harmonic morphism* $\varphi : (M^m, g) \to (N^n, h)$ between Riemannian manifolds M and N is a map that pulls back germs of harmonic functions on an open subset of N to germs of harmonic functions on an open subset of M. In other words, harmonic morphisms are maps that preserve the harmonic structure of a given harmonic function: $\Delta(f \circ \varphi) = 0$.

Baird and Ou in [9] prove that a harmonic morphism $G : \mathbb{R}^m \setminus V_G \to \mathbb{R}^n \setminus \{0\}$ that is both defined by homogeneous maps and which possesses an isolated singularity ($\mathrm{d}G = 0$) at the origin can be retracted to a harmonic morphism that constitutes a Milnor map from $S^{m-1} \setminus K_\epsilon$ to S^{n-1}, where V_G is the variety of G and $K_\epsilon = S_\epsilon^{m-1} \cap V_G$. It was our hope in this work to potentially relax the requirement that the maps defining the harmonic morphism G need be homogeneous. The viability of such a weakening of the homogeneity assumption we discovered was not, however, tenable and have thus solidified that the class of harmonic morphisms which also retract to Milnor maps is limited to those defined by homogeneous maps.

Before addressing the relationship between harmonic morphisms and Milnor fibrations, we must start by thoroughly laying out the requisite theory of harmonic morphisms. We first look at harmonic morphisms between Euclidean manifolds. We then look at the historically significant case of harmonic morphisms from $\mathbb{R}^3$ to $\mathbb{C}$ which was originally investigated by Jacobi in [6]. We are particularly interested in harmonic morphisms of codimension 1, of which the harmonic morphisms that arise in Jacobi's investigation are prime examples. In [6] Jacobi completely characterized harmonic morphisms from $\mathbb{R}^3$ to $\mathbb{C}$. His characterization can be shown to have an interesting overlap with the theory of minimal surfaces developed by Weierstrass and Enneper. Ultimately, all harmonic morphisms from $\mathbb{R}^3$ to $\mathbb{C}$ are parameterized by the same meromorphic functions that turn up in the Weierstrass-Enneper parameterization of minimal surfaces.

Following our discussion of Jacobi's problem, we then generalize the theory of harmonic morphisms as maps between Riemannian manifolds. This requires a close look at the basic theory of distributions, curvature, vector bundles, and connections. This discussion culminates in the derivation of the *fundamental equation of harmonic morphisms* (see [13]) which can be used to completely characterize harmonic morphisms with one-dimensional fibres. In the following section, we survey the three types of harmonic morphisms with one-dimensional fibres: Killing type, warped product type, and T type, respectively.

Finally, we return to the original question concerning the relationship between harmonic morphisms and Milnor fibrations, confirming that the class of harmonic morphisms which retract to Milnor fibrations from the sphere to the sphere are limited to those defined by homogeneous maps.

2 Harmonic Morphisms between Euclidean Spaces

To serve as an introduction to harmonic morphisms, we start by considering harmonic morphisms between Euclidean spaces. The mathematics in the Euclidean case falls out more simply, making it easier to immediately appreciate the implications.

2.1 Definitions and Characterization

Definition 2.1. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a $\mathcal{C}^2$ function. We say f is a *harmonic function* if the Laplacian of f is identically zero:

$$\Delta f = 0$$

Definition 2.2. (Harmonic Morphism on Euclidean Spaces) A map $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is called a *harmonic morphism* if for any harmonic function $f : U \to \mathbb{R}$, defined on an open subset U of $\mathbb{R}^n$ with $\varphi^{-1}(U) \subseteq \mathbb{R}^m$ non-empty, $f \circ \varphi : \varphi^{-1}(U) \to \mathbb{R}$ is a harmonic function.

Thus, let $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ be a harmonic morphism and let $f : \mathbb{R}^n \to \mathbb{R}$ be a harmonic function. Then one basic characterization of φ as a harmonic morphism is that it should satisfy the following PDE:

$$\Delta^m(f \circ \varphi) = 0 \tag{2.1.1}$$

where Δ^m represents the m-dimensional Laplacian. Note that on a Euclidean space the Laplacian takes the familar form:

$$\Delta^m f = \sum_{k=0}^{m} \frac{\partial^2 f}{\partial x_i^2} \tag{2.1.2}$$

otherwise it depends expressly on the metric associated to the underlying manifold (see Remark 5.1.1).

What Equation 2.1.1 essentially means is that φ pulls back harmonic functions on $\mathbb{R}^n$ to harmonic functions on $\mathbb{R}^m$. In general, precomposition of a harmonic function with any given map does not preserve the harmonic structure of f. Thus, a map that does preserve the harmonic structure is special and is given the name "harmonic morphism".

For further insight into the defining properties of a harmonic morphism which maps between Euclidean spaces, lets work out a closed form expression for Equation 2.1.1.

First, we assume that $\varphi = (\varphi_1, \ldots, \varphi_n)$, where each component function is such that $\varphi_i : \mathbb{R}^m \to \mathbb{R}$. Then,

$$\begin{aligned}
\Delta^m(f \circ \varphi) &= \sum_{i=1}^{m} \frac{\partial^2}{\partial x_i^2} (f \circ \varphi) \\
&= \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \left(\frac{\partial}{\partial x_i} (f \circ \varphi) \right) \\
&= \sum_{i=1}^{m} \frac{\partial}{\partial x_i} \left(\sum_{k=1}^{n} \frac{\partial (f \circ \varphi)}{\partial \varphi_k} \frac{\partial \varphi_k}{\partial x_i} \right) \\
&= \sum_{i=1}^{m} \left(\sum_{k=1}^{n} \frac{\partial}{\partial x_i} \left[\frac{\partial (f \circ \varphi)}{\partial \varphi_k} \frac{\partial \varphi_k}{\partial x_i} \right] \right) \\
&= \sum_{i=1}^{m} \left(\sum_{k=1}^{n} \sum_{j=1}^{n} \left[\frac{\partial \left(\frac{\partial (f \circ \varphi)}{\partial \varphi_k} \right)}{\partial \varphi_j} \frac{\partial \varphi_j}{\partial x_i} \frac{\partial \varphi_k}{\partial x_i} \right] + \sum_{k=1}^{n} \frac{\partial (f \circ \varphi)}{\partial \varphi_k} \frac{\partial^2 \varphi_k}{\partial x_i^2} \right) \\
&= \sum_{i=1}^{m} \sum_{k,j=1}^{n} \frac{\partial^2 (f \circ \varphi)}{\partial \varphi_j \partial \varphi_k} \frac{\partial \varphi_j}{\partial x_i} \frac{\partial \varphi_k}{\partial x_i} + \sum_{i=1}^{m} \sum_{k=1}^{n} \frac{\partial (f \circ \varphi)}{\partial \varphi_k} \frac{\partial^2 \varphi_k}{\partial x_i^2}
\end{aligned}$$

Thus Equation 2.1.1 gives us the following:

$$\sum_{i=1}^{m}\sum_{k,j=1}^{n}\frac{\partial^2(f\circ\varphi)}{\partial\varphi_j\partial\varphi_k}\frac{\partial\varphi_j}{\partial x_i}\frac{\partial\varphi_k}{\partial x_i}+\sum_{i=1}^{m}\sum_{k=1}^{n}\frac{\partial(f\circ\varphi)}{\partial\varphi_k}\frac{\partial^2\varphi_k}{\partial x_i^2}=0 \tag{2.1.3}$$

In and of itself Equation 2.1.3 does not immediately grant any insight, however, through an appropriate selection of basic harmonic functions, we may arrive at the following theorem:

Theorem 2.1. *Let $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ such that $\varphi = (\varphi_1, \dots, \varphi_n)$ and let $\lambda : \mathbb{R}^m \to \mathbb{R}^+$ be the dilation of function of the map φ. Then φ is a harmonic morphism if and only if the following conditions are satisfied:*

(a) $\Delta^m\varphi_i = 0$ for each component function of φ

(b) $\nabla\varphi_i \cdot \nabla\varphi_j = \lambda(\mathbf{x})\delta_{ij}$ where $\mathbf{x} \in \mathbb{R}^m$

Proof. $(\Rightarrow)$: (a) Let φ be an harmonic morphism. Then $\Delta^m(f\circ\varphi) = 0$ for every f a harmonic function. Consider the following harmonic function:

$$f : \mathbb{R}^n \to \mathbb{R}$$

$$(x_1, \dots, x_i, \dots, x_n) \mapsto x_i$$

where $1 \le i \le n$. Hence,

$$\Delta^m(f\circ\varphi) = \Delta^m\varphi_i$$

therefore

$$\Delta^m\varphi_i = 0$$

since i is arbitrary, this must hold for all the component functions of φ. Thus, all component functions are harmonic.

$(\Rightarrow)$: (b) Now, consider a new harmonic function f, defined as follows:

$$f : \mathbb{R}^n \to \mathbb{R}$$

$$(x_1, \dots, x_l, \dots, x_p, \dots, x_n) \mapsto x_l x_p$$

where $1 \leq l, p \leq n$ and $l \neq p$. So observe,

$$\Delta^m(f \circ \varphi) = \sum_{i=1}^{m} \sum_{k,j=1}^{n} \left[\frac{\partial^2 (\varphi_l \varphi_p)}{\partial \varphi_j \partial \varphi_k} \frac{\partial \varphi_j}{\partial x_i} \frac{\partial \varphi_k}{\partial x_i} \right] + \sum_{i=1}^{m} \sum_{k=1}^{n} \left[\frac{\partial (\varphi_l \varphi_p)}{\partial \varphi_k} \frac{\partial^2 \varphi_k}{\partial x_i^2} \right]$$

The double sum over k and j will only produce non-zero terms whenever k and j equal some combination of l and n. Thus,

$$\sum_{i=1}^{m} \left[\frac{\partial \varphi_l}{\partial x_i} \frac{\partial \varphi_p}{\partial x_i} + \varphi_l \frac{\partial^2 \varphi_p}{\partial x_i^2} + \frac{\partial \varphi_p}{\partial x_i} \frac{\partial \varphi_l}{\partial x_i} + \varphi_p \frac{\partial^2 \varphi_l}{\partial x_i^2} + \varphi_l \frac{\partial^2 \varphi_p}{\partial x_i^2} + \varphi_p \frac{\partial^2 \varphi_l}{\partial x_i^2} \right]$$
$$= 2 \sum_{i=1}^{m} \frac{\partial \varphi_l}{\partial x_i} \frac{\partial \varphi_p}{\partial x_i} + 2\varphi_l \sum_{i=1}^{m} \frac{\partial^2 \varphi_p}{\partial x_i^2} + 2\varphi_p \sum_{i=1}^{m} \frac{\partial^2 \varphi_l}{\partial x_i^2}$$
$$= 2 (\nabla \varphi_l \cdot \nabla \varphi_p) + 2\varphi_l (\Delta^m \varphi_p) + 2\varphi_p (\Delta^m \varphi_l)$$
$$= 2 (\nabla \varphi_l \cdot \nabla \varphi_p) + 0 + 0$$
$$= 2 (\nabla \varphi_l \cdot \nabla \varphi_p) \overset{!}{=} 0$$

therefore

$$\nabla \varphi_l \cdot \nabla \varphi_p = 0$$

Next, we consider yet another harmonic function

$$f : \mathbb{R}^n \to \mathbb{R}$$
$$(x_1, \ldots, x_l, \ldots, x_p, \ldots, x_n) \mapsto x_l^2 - x_p^2$$

where $1 \leq l, p \leq n$ and $l \neq p$.

Also, notice from (i) the general equation for $\Delta^m(f \circ \varphi)$ simplifies:

$$\Delta^m(f \circ \varphi) = \sum_{i=1}^{m} \sum_{k,j=1}^{n} \frac{\partial^2 \left(\varphi_l^2 - \varphi_p^2\right)}{\partial \varphi_j \partial \varphi_k} \frac{\partial \varphi_j}{\partial x_i} \frac{\partial \varphi_k}{\partial x_i}$$

Again, the double sum over k and j only produces non-zero terms whenever k and j equal some combination of l and n. Thus,

$$\sum_{i=1}^{m} \left[(2\varphi_l - 2\varphi_p) \frac{\partial \varphi_l}{\partial x_i} \frac{\partial \varphi_p}{\partial x_i} - 2\varphi_p \frac{\partial^2 \varphi_p}{\partial x_i^2} + (2\varphi_l - 2\varphi_p) \frac{\partial \varphi_p}{\partial x_i} \frac{\partial \varphi_l}{\partial x_i} + 2\varphi_l \frac{\partial^2 \varphi_l}{\partial x_i^2} + 2 \left(\frac{\partial \varphi_l}{\partial x_i} \right)^2 - 2 \left(\frac{\partial \varphi_p}{\partial x_i} \right)^2 \right]$$
$$= 4(\varphi_l - \varphi_p) (\nabla \varphi_l \cdot \nabla \varphi_p) + 2 (\nabla \varphi_l \cdot \nabla \varphi_l - \nabla \varphi_p \cdot \nabla \varphi_p)$$
$$= 0 + 2 \left(|\nabla \varphi_l|^2 - |\nabla \varphi_p|^2 \right) \overset{!}{=} 0$$

therefore

$$|\nabla\varphi_l|^2 = |\nabla\varphi_p|^2$$

The implication of the above result is important. Since, by assumption $l \neq p$, the above result says that for every $\mathbf{x} \in \mathbb{R}^m$ and for each $i, j \in \mathbb{N},\ 1 \leq i, j \leq n$

$$|\nabla\varphi_i(\mathbf{x})|^2 = |\nabla\varphi_j(\mathbf{x})|^2$$

Notice that this is indeed equivalent to the statement:

$$\nabla\varphi_i \cdot \nabla\varphi_i = \lambda(\mathbf{x})$$

Since by definition $\nabla\varphi_i \cdot \nabla\varphi_i = |\nabla\varphi_i|^2$, and $|\nabla\varphi_j|^2$ can be viewed as a scalar function of $\mathbf{x} \in \mathbb{R}^m$. Since again i is arbitrary the above statement says that for any given $\mathbf{x} \in \mathbb{R}^m$, all gradients for all component functions of φ have the same length.

So the implication in total means that while the lengths of the gradients may change as we pass from one point to another, they all change together in such a way that their lengths remain the same and their directions remain mutually orthogonal. As will be seen in later sections, these properties of the component functions of φ characterize it as a *horizontally weakly conformal map* with λ the so-called *dilation function.*

($\Leftarrow$) : Assume the component functions of φ are harmonic $\Delta^m\varphi_i = 0$ and assume they are mutually orthogonal $\nabla\varphi_i \cdot \nabla\varphi_j = \lambda(\mathbf{x})\delta_{ij}$

Consider:

$$\Delta^m(f \circ \varphi) = \sum_{i=1}^{m}\sum_{k,j=1}^{n}\left[\frac{\partial^2\left(f \circ \varphi\right)}{\partial\varphi_j\partial\varphi_k}\frac{\partial\varphi_j}{\partial x_i}\frac{\partial\varphi_k}{\partial x_i}\right] + \sum_{i=1}^{m}\sum_{k}^{n}\left[\frac{\partial(f \circ \varphi)}{\partial\varphi_k}\frac{\partial^2\varphi_k}{\partial x_i^2}\right]$$

Commuting the sums,

$$\begin{aligned}
\Delta^m(f\circ\varphi) &= \sum_{k,j=1}^{n}\sum_{i=1}^{m}\left[\frac{\partial^2(f\circ\varphi)}{\partial\varphi_j\partial\varphi_k}\frac{\partial\varphi_j}{\partial x_i}\frac{\partial\varphi_k}{\partial x_i}\right] + \sum_{k}^{n}\sum_{i=1}^{m}\left[\frac{\partial(f\circ\varphi)}{\partial\varphi_k}\frac{\partial^2\varphi_k}{\partial x_i^2}\right] \\
&= \sum_{k,j=1}^{n}\left[\frac{\partial^2(f\circ\varphi)}{\partial\varphi_j\partial\varphi_k}\sum_{i=1}^{m}\frac{\partial\varphi_j}{\partial x_i}\frac{\partial\varphi_k}{\partial x_i}\right] + \sum_{k=1}^{n}\left[\frac{\partial(f\circ\varphi)}{\partial\varphi_k}\sum_{i=1}^{m}\frac{\partial^2\varphi_k}{\partial x_i^2}\right] \\
&= \sum_{k,j=1}^{n}\left[\frac{\partial^2(f\circ\varphi)}{\partial\varphi_j\partial\varphi_k}(\nabla\varphi_j\cdot\nabla\varphi_k)\right] + \sum_{k=1}^{n}\left[\frac{\partial(f\circ\varphi)}{\partial\varphi_k}(\Delta^m\varphi_k)\right] \\
&= \sum_{j=1}^{n}\frac{\partial^2(f\circ\varphi)}{\partial\varphi_j^2}|\nabla\varphi_j|^2 + \sum_{k=1}^{p}\frac{\partial^2(f\circ\varphi)}{\partial\varphi_k^2}|\nabla\varphi_k|^2 + 0
\end{aligned}$$

Then for a fixed $\mathbf{x}\in\mathbb{R}^n$, we have $|\nabla\varphi_i|^2 = \lambda\in\mathbb{R}$ constant, for each $i\in\mathbb{N}$, $1\le i\le n$. Thus,

$$\Delta^m(f\circ\varphi) = 2\lambda\sum_{j=1}^{n}\frac{\partial^2(f\circ\varphi)}{\partial\varphi_j^2}$$

Since $f\circ\varphi = f(\varphi_1,\ldots,\varphi_n)$ means

$$\Delta^m(f\circ\varphi) = 2\lambda\Delta^n f$$

Then since by assumption f is harmonic, we finally get:

$$\Delta^m(f\circ\varphi) = 0$$

Hence, φ is a harmonic morphism.

□

One of the immediate consequences of Theorem 2.1 is a that the possible dimensions m and p are limited when looking for non-trivial (non-constant) harmonic morphisms.

Theorem 2.2. *Let $\varphi:\mathbb{R}^m\to\mathbb{R}^n$ be a harmonic morphism with $n>m$. Then φ is a constant function.*

Proof. Let $\varphi:\mathbb{R}^m\to\mathbb{R}^n$ be an harmonic morphism with $n>m$. Then for any $i,j\in\mathbb{N}$ such that $1\le i,j\le n$ with $i\ne j$, we have $\nabla\varphi_i\cdot\nabla\varphi_j = 0$. Hence, $\nabla\varphi_i\perp\nabla\varphi_j$. Thus, we have

n-many linearly independent vectors in $\mathbb{R}^m$. But since, $n > m$ means at least $(n-m)$-many of them must be $\mathbf{0}$, for each $\mathbf{x} \in \mathbb{R}^m$. However, since φ is an harmonic morphism, means that $|\nabla\varphi_i|^2 = |\nabla\varphi_j|^2$, for all $i, j \in \mathbb{N}$ such that $1 \leq i, j \leq n$ and for each $\mathbf{x} \in \mathbb{R}^m$. Thus, if at least one $\nabla\varphi_i = \mathbf{0}$ for every $\mathbf{x} \in \mathbb{R}^m$, means this must be true for each φ_i. Thus $\nabla\varphi_i = \mathbf{0}$ for $1 \leq i \leq n$. This means $\varphi_i \overset{!}{=} \alpha_i$ a constant, for each i. Therefore, $\varphi = (\alpha_1, \ldots, \alpha_n)$ is a constant function. □

2.2 Examples

Below are provided a few essential and informative examples of harmonic morphisms between Euclidean spaces.

Example 2.2.1. Let φ be a constant function. Then φ is a harmonic morphism.

Proof. Let $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ be a constant function. Then this implies all of its components are constant, $\varphi = (\alpha_1, \ldots, \alpha_n)$. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a harmonic function (or really any function), then $f \circ \varphi = f(\alpha_1, \ldots, \alpha_n)$ a constant. Hence, $\Delta(f \circ \varphi) = 0$. □

Example 2.2.2. Let $\varphi : \mathbb{C} \to \mathbb{C}$ be $\pm$-holomorphic. Then φ is a harmonic morphism.

Proof. In order for $\varphi = \big(u(x,y), v(x,y)\big)$ to be a harmonic morphism, it must constitute a solution to the following system of PDE's:

$$(a)\ \nabla u \cdot \nabla v = u_x v_x + u_y v_y \overset{!}{=} 0$$

$$(b)\ |\nabla u|^2 = |\nabla v|^2$$

$$(c)\ \Delta u = 0 \ \text{ and } \ \Delta v = 0$$

Now, notice as a special case, that holomorphic maps from $\mathbb{C}$ to $\mathbb{C}$ satisfy the above system of PDE's (bearing in mind the standard isomorphism $\mathbb{C} \cong \mathbb{R}^2$).

So, if φ is holomorphic, then it must satisfy the Cauchy-Riemann equations

$$u_x = v_y$$
$$u_y = -v_x$$

Thus, the PDE (a) is satisfied:

$$\begin{aligned} u_x v_x + u_y v_y &= (v_y)v_x + (-v_x)v_y \\ &= v_y v_x - v_x v_y \\ &= 0 \end{aligned}$$

and PDE (b)

$$\begin{aligned} u_x^2 + u_y^2 &= (v_y)^2 + (-v_x)^2 \\ &= v_y^2 + v_x^2 \end{aligned}$$

Lastly, since φ is holomorphic means its components are harmonic conjugates: $\Delta u = 0$ and $\Delta v = 0$. Hence, PDE (c) is also satisfied. This shows that all holomorphic maps from $\mathbb{C}$ to $\mathbb{C}$ are harmonic morphisms. Finally, notice that the proof follows exactly the same if we assume φ is anti-holomorphic. □

Example 2.2.3. As a generalization of Example 2.2.2, any ±-holomorphic map from $\mathbb{C}^n$ to $\mathbb{C}$ is a harmonic morphism.

Example 2.2.4. Let $f : \mathbb{R}^m \to \mathbb{R}^n$ and $g : \mathbb{R}^p \to \mathbb{R}^m$ be a harmonic morphisms. Then $f \circ g : \mathbb{R}^p \to \mathbb{R}^n$ is a harmonic morphism.

3 Jacobi's Problem

The original study of those maps that we now call harmonic morphisms is traced back to Jacobi's study of the following question: Let $\varphi : U \to \mathbb{C}$ by a $\mathcal{C}^2$ function on an open subset of Euclidean 3-space $\mathbb{R}^3$ which is harmonic (satisfies Laplace's equation):

$$\Delta\varphi \equiv \sum_{i=1}^{3} \frac{\partial^2\varphi}{\partial x_i^2} = 0$$

Then the questions is: Under what conditions on φ is the composition $f \circ \varphi$ harmonic for an arbitrary holomorphic map $f : V \to \mathbb{C}$ defined on an open subset of $\mathbb{C}$?

3.1 A Solution

Now, notice that if we let $\varphi = \varphi_1 + i\varphi_2$ where $\varphi_i : \mathbb{R}^3 \to \mathbb{R}$ then we can derive a formula that is essentially equivalent to that previously derived in Equation 2.1.3 only simplifying the notation to reflect that we are now looking at maps over $\mathbb{C}$. Thus, using the chain rule to expand $\Delta(f \circ \varphi)$ we get:

$$\frac{\partial}{\partial x_i}(f \circ \varphi) = \sum_{j=1}^{2} \frac{\partial(f \circ \varphi)}{\partial \varphi_j} \frac{\partial \varphi_j}{\partial x_i}$$

then letting

$$\frac{\mathrm{d}f}{\mathrm{d}z} := \sum_{j=1}^{2} \frac{\partial(f \circ \varphi)}{\partial \varphi_j}$$

we get

$$\Delta(f \circ \varphi) = \frac{\mathrm{d}f}{\mathrm{d}z}\Delta\varphi + \frac{\mathrm{d}^2 f}{\mathrm{d}z^2} \sum_{i=1}^{3} \left(\frac{\partial \varphi}{\partial x_i}\right)^2 \tag{3.1.1}$$

It is clear from Equation 3.1.1 that if we require $f \circ \varphi$ to be harmonic, then we must require that φ itself be harmonic (which we have assumed) and that

$$\sum_{i=1}^{3} \left(\frac{\partial \varphi}{\partial x_i}\right)^2 = 0 \tag{3.1.2}$$

Thus, it follows that for any harmonic map $\varphi : \mathbb{R}^3 \to \mathbb{C}$, satisfying Equation 3.1.2 we get $\Delta(f \circ \varphi) = 0$. Such a φ we call a harmonic morphism.

Remark 3.1.1. Notice that Equation 3.1.1 and Equations 3.1.2 are invariant under isometries of the domain, meaning they are independent of choice of coordinate system for $\mathbb{R}^3$ and can be written more generally in terms of gradients of the component functions φ_1 and φ_2. Thus:

$$|\nabla\varphi_1| = |\nabla\varphi_2| \text{ and } \langle\nabla\varphi_1, \nabla\varphi_2\rangle = 0 \tag{3.1.3}$$

A Euclidean map that satisfies Equation 3.1.3 is called *horizontally weakly conformal*. Hence, a map φ is *horizontally weakly conformal* (see § 5.2) if and only if the gradients of its real and imaginary parts are mutually orthogonal and have the same norm for each $x \in \mathbb{R}^m$.

3.2 An Implicit Definition

Theorem 3.1. *Let $G : A \to \mathbb{C}$ be a smooth function on an open subset of $\mathbb{R}^3 \times \mathbb{C}$ which is holomorphic in the second variable. Suppose that*

$$\nabla G \equiv \left(\frac{\partial G}{\partial x_1}, \frac{\partial G}{\partial x_2}, \frac{\partial G}{\partial x_3}\right) \neq 0$$

$\forall(\boldsymbol{x}, z) \in A$ with $G(\boldsymbol{x}, z) = 0$. Then a smooth solution $\varphi : U \to \mathbb{C}$ on an open subset of $\mathbb{R}^3$ to the equation

$$G(\boldsymbol{x}, \varphi(\boldsymbol{x})) = 0 \tag{3.2.1}$$

for $\boldsymbol{x} \in U$, satisfies

$$\text{(a)} \sum_{i=1}^{3} \frac{\partial^2 \varphi}{\partial x_i^2} = 0, \quad \text{and} \quad \text{(b)} \sum_{i=1}^{3} \left(\frac{\partial \varphi}{\partial x_i}\right)^2 = 0 \tag{3.2.2}$$

if and only if G satisfies the corresponding equations:

$$\text{(a)} \sum_{i=1}^{3} \frac{\partial^2 G}{\partial x_i^2} = 0, \quad \text{and} \quad \text{(b)} \sum_{i=1}^{3} \left(\frac{\partial G}{\partial x_i}\right)^2 = 0 \tag{3.2.3}$$

Proof. Suppose that $\varphi : U \to \mathbb{C}$ is a solution to Equation 3.2.1 on some open subset $U \subset \mathbb{R}^3$. Then by the chain rule, we have at all point $(\boldsymbol{x}, z) = (\boldsymbol{x}, \varphi(\boldsymbol{x}))$ with $\boldsymbol{x} \in U$,

$$\frac{\partial G}{\partial z}\frac{\partial \varphi}{\partial x_i} + \frac{\partial G}{\partial x_i} = 0 \tag{3.2.4}$$

for $i = 1, 2, 3$. Then since $\nabla G \neq 0$ implies $\frac{\partial G}{\partial z} \neq 0$. Then squaring and adding the three cases $i = 1, 2, 3$, we get

$$\sum_{i=1}^{3} \left(\frac{\partial G}{\partial x_i}\right)^2 = \left(\frac{\partial G}{\partial z}\right)^2 \sum_{i=1}^{3} \left(\frac{\partial \varphi}{\partial x_i}\right)^2 fr5 \tag{3.2.5}$$

Then differentiating Equation 3.2.4 with respect to x_i gives us

$$\frac{\partial^2 G}{\partial z^2}\left(\frac{\partial \varphi}{\partial x_i}\right)^2 + \frac{\partial^2 G}{\partial x_i \partial z}\frac{\partial \varphi}{\partial x_i} + \frac{\partial G}{\partial z}\frac{\partial^2 \varphi}{\partial x_i^2} + \frac{\partial^2 G}{\partial x_i^2} = 0 \tag{3.2.6}$$

for $i = 1, 2, 3$. Now, adding up the three cases and applying 3.2.2, we get:

$$\sum_{i=1}^{3} \frac{\partial^2 G}{\partial x_i \partial z}\frac{\partial \varphi}{\partial x_i} = 0 \tag{3.2.7}$$

Then differentiation with respect to z in congress with Equation 3.2.4 gives:

$$\frac{\partial G}{\partial z}\sum_{i=1}^{3} \frac{\partial^2 G}{\partial x_i \partial z}\frac{\partial \varphi}{\partial x_i} = -\sum_{i=1}^{3} \frac{\partial G}{\partial x_i}\frac{\partial^2 G}{\partial x_i \partial z} = -\frac{1}{2}\frac{\partial}{\partial z}\sum_{i=1}^{3}\left(\frac{\partial G}{\partial x_i}\right)^2 = 0 \tag{3.2.8}$$

So Equation 3.2.7 along with Equation 3.2.5 shows that Equation 3.2.2(b) holds $\Leftrightarrow$ Equation 3.2.3(b) holds. Summing Equation 3.2.6 for $i = 1, 2, 3$ and using Equation 3.2.2(b) and the fact that since $\frac{\partial G}{\partial z} \neq 0$ implies $\sum_{i=1}^{3} \frac{\partial^2 G}{\partial x_i \partial z}\frac{\partial \varphi}{\partial x_i} = 0$ from Equation 3.2.8.

Then summing Equation 3.2.6

$$\sum_{i=1}^{3} \frac{\partial^2 G}{\partial x_i^2} = -\frac{\partial G}{\partial z}\sum_{i=1}^{3} \frac{\partial^2 \varphi}{\partial x_i^2} \tag{3.2.9}$$

Which shows Equation 3.2.2(a) $\Leftrightarrow$ Equation 3.2.3(a). □

Remark 3.2.1. Note that the properties (a) and (b) in Theorem 3.1 correspond to G being harmonic and horizontally weakly conformal, respectively. These directly correspond to the properties of general harmonics on Euclidean spaces (see Theorem 2.1).

3.3 A Very Special G Map

It can be shown that the fibres of the map G of Theorem 3.1 must be straight lines. In this section, we start by presuming this is the case, however, this is in fact the only case. We call this G *very special* because it serves to characterize all harmonic morphisms from $\mathbb{R}^3$ to $\mathbb{C}$.

Definition 3.1. Let V be an open subset of $\mathbb{C}$. By a (nowhere zero) null holomorphic map $\boldsymbol{\xi} : V \to \mathbb{C}^3$, we mean a triple $\boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3)$ of holomorphic function $\xi_i : V \to \mathbb{C}$ such that

$$\text{(a)} \sum_{i=1}^{3} \xi_i(z)^2 = 0, \quad \text{and} \quad \text{(b)} \sum_{i=1}^{3} |\xi_i(z)|^2 \neq 0 \tag{3.3.1}$$

for all $z \in V$.

Given such a triple, consider the equation

$$\xi_1(z)x_1 + \xi_2(z)x_2 + \xi_3(z)x_3 = 1 \tag{3.3.2}$$

Now define G as follows:

$$G(\boldsymbol{x}, z) \equiv G(x_1, x_2, x_3, z) = \xi_1(z)x_1 + \xi_2(z)x_2 + \xi_3(z)x_3 - 1 \tag{3.3.3}$$

Note that G satisfies the conditions of Theorem 3.1. Then notice that as z varies in V, Equation 3.3.2 defines a two parameter family of straight lines, the two parameters corresponding to $\mathfrak{Re}(z)$ and $\mathfrak{Im}(z)$, respectively. if $\partial G/\partial z \neq 0$ at a point $(\boldsymbol{x}, z)$, the congruence forms a smooth foliation in a neighbourhood of that point. Any smooth local solution $\varphi : U \to \mathbb{C}$ to Equation 3.2.4 has these lines as fibres. Furthermore, by taking the modulus of Equation 3.2.4, the dilation λ of φ can be shown to be,

$$\lambda = \frac{|\nabla G|}{\sqrt{2}|\partial G/\partial z|} \tag{3.3.4}$$

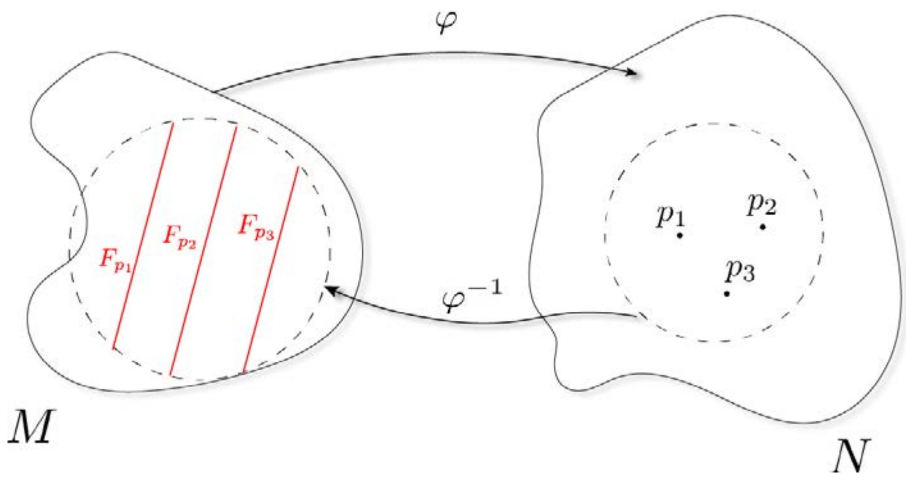

Figure 3.1: **Foliation of the Domain**: Here you can see that the preimages (fibres of φ) of points in N are disjoint lines in M.

3.4 Harmonic morphisms from $\mathbb{R}^3$ to $\mathbb{R}^2$

In developing a complete characterization of the harmonic morphisms between $\mathbb{R}^3$ to $\mathbb{C}$, an intersection occurs with Weierstrass and Enneper's theory of minimal surfaces. The same canonical meromorphic functions that parameterize minimal surfaces in $\mathbb{R}^3$ can be used to define harmonic morphisms between $\mathbb{R}^3$ to $\mathbb{C}$.

First, we must loosen the conditions on our special G of § 3.3 to the case when the ξ_i are meromorphic functions instead of strict holomorphic functions. In this case, each $\xi_i = \frac{\eta_i}{\zeta}$ where both η_i and ζ are holomorphic and ζ is zero exactly at all the points where ξ_i have poles, a set we call Z. Then restricting ourselves to $V \setminus Z$ and going back to Equation 3.3.2, we get

$$\eta_1(z)x_1 + \eta_2(z)x_2 + \eta_3(z)x_3 = \zeta(z) \tag{3.4.1}$$

Triples of meromorphic functions ξ_i satisfying Equation 3.4.1 occured in the Enneper-Weierstrass representation of minimal surfaces. This is classically given as follows:

$$\boldsymbol{\xi} = \frac{1}{2h}\big(-2g, 1-g^2, i(1+g^2)\big) \tag{3.4.2}$$

So letting $\xi_1(z) = -2g(z), \xi_2(z) = 1 - g(z)^2$ and $\xi_3(z) = i(1 + g(z)^2)$, we get the following variant of Equation 3.3.2

$$G(\boldsymbol{x}, z) = -2g(z)x_1 + (1 - g(z)^2)x_2 + i(1 + g(z)^2)x_3 - 2h(z) \tag{3.4.3}$$

Theorem 3.2. *(Local representations of harmonic morphisms on $\mathbb{R}^3$) Let g and h be meromorphic functions on V which are not identically infinite and which satisfy:*

$$\lim_{z \to z_0} h(z)/g(z)^2 = L \tag{3.4.4}$$

Where z_0 is a pole of h and $L \in \mathbb{R}$. Then

1. *any smooth local solution $\varphi : U \to \mathbb{C}, z = \varphi(x)$ to the equation*

$$-2g(z)x_1 + (1 - g(z)^2)x_2 + i(1 + g(z)^2)x_3 = 2h(z) \tag{3.4.5}$$

   on a convex open set is a submersive harmonic morphism with connected fibres not all in the direction of the negative x_1 axis;

2. *each such harmonic morphism is given in this way for unique g and h;*

3. *let $(\boldsymbol{x}_0, z_0) \in \mathbb{R}^3 \times V$. Then local solution $z = \varphi(\mathbf{x})$ to Equation 3.4.5 exists if and only if at $(\boldsymbol{x}_0, z_0)$*

$$\partial G/\partial z \equiv g'(z)(-2x_1 - 2g(z)x_2 + i2g(z)x_3) - 2h'(z) \neq 0 \tag{3.4.6}$$

3.4.1 Examples

Example 3.4.1. (Orthogonal Projection) Let $g \equiv 0$ and $h = \frac{1}{2}z$, then Equation 3.4.5 looks like:

$$x_2 + ix_3 = z \tag{3.4.7}$$

The solution $\varphi : \mathbb{R}^3 \to \mathbb{C}$ is clearly

$$z = \varphi(x_1, x_2, x_3) = x_2 + ix_3$$

As a map, φ constitues orthogonal projection of points in $\mathbb{R}^3$ on the $x_1 = 0$ plane.

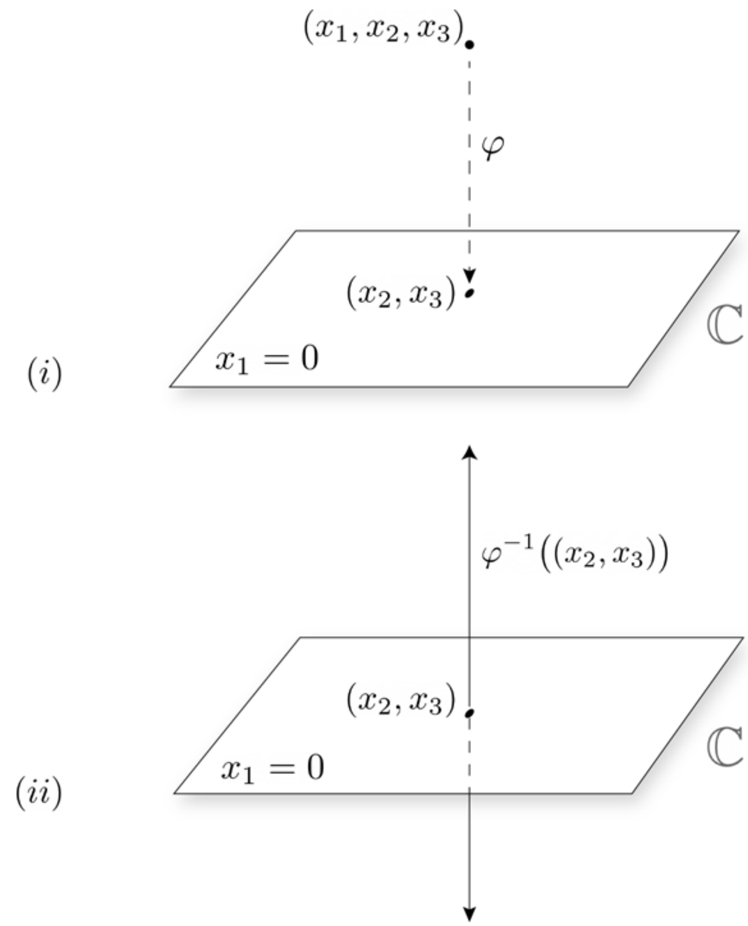

Figure 3.2: **Orthogonal projection** (i) The map φ mapping a point (x_1, x_2, x_3) orthogonally onto the $x_1 = 0$ plane. (ii) The preimage (fibre) under φ of a point (x_2, x_3) is a line in $\mathbb{R}^3$

Example 3.4.2. (Radial Projection) Let $g, h : \mathbb{C} \cup \{\infty\} \to \mathbb{C} \cup \{\infty\}$ such that

$$g(z) = z \ , \ \ h \equiv 0$$

Then equation 1.3.5 looks like

$$-2zx_1 + (1 - z^2)x_2 + i(1 + z^2)x^3 = 0 \tag{3.4.8}$$

This can be written as a quadratic in z, yielding

$$(x_2 - ix_3)z^2 + 2x_1 z - (x_2 + ix_3) = 0 \tag{3.4.9}$$

The solution $\varphi(\boldsymbol{x})$ is

$$
\begin{aligned}
\varphi(\boldsymbol{x}) &= \frac{-2x_1 \pm \sqrt{4x_1^2 + 4(x_2 + ix_3)(x_2 - ix_3)}}{2(x_2 - ix_3)} \\
&= \frac{-x_1 \pm \sqrt{x_1^2 + x_2^2 + x_3^2}}{x_2 - ix_3} \\
&= \frac{-x_1 \pm |\boldsymbol{x}|}{x_2 - ix_3}
\end{aligned}
$$

Choosing the positive solution, it can be seen that

$$
\varphi(\boldsymbol{x}) = \frac{-x_1 + |\boldsymbol{x}|}{x_2 - ix_3} = \sigma\left(\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\right) \tag{3.4.10}
$$

where $\sigma : S^2 \to \mathbb{C} \cup \{\infty\}$ is stereographic projection from the south pole. Since σ^{-1} is a conformal map it is clear $\sigma^{-1} \circ \varphi$ is a harmonic morphism and $\boldsymbol{x} \mapsto \frac{\boldsymbol{x}}{|\boldsymbol{x}|}$ i.e. $\sigma^{-1} \circ \varphi$ is equivalent to radial projection to the origin.

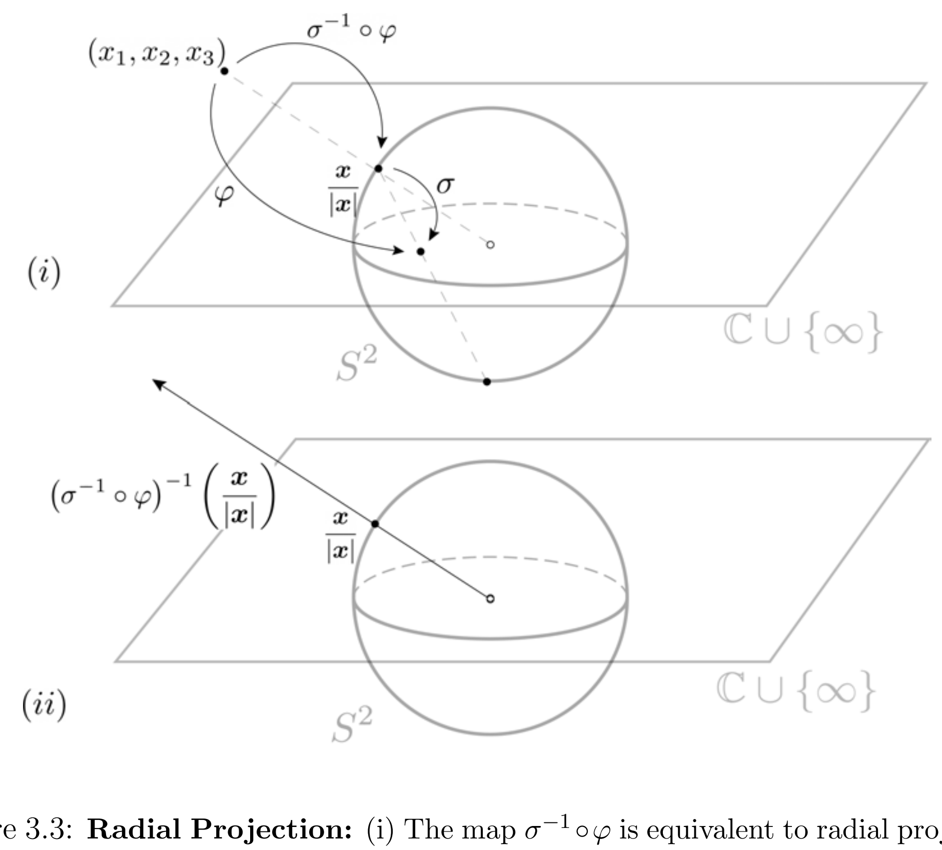

Figure 3.3: **Radial Projection:** (i) The map $\sigma^{-1} \circ \varphi$ is equivalent to radial projection about the origin (ii) The preimage (fibre) under $\sigma^{-1} \circ \varphi$ of a point $\frac{\boldsymbol{x}}{|\boldsymbol{x}|} \in S^2$ is a line in $\mathbb{R}^3$ projecting radially from the origin and containing the lift of the point $\frac{\boldsymbol{x}}{|\boldsymbol{x}|}$.

4 Distributions and Vector Bundles

Here we develop some of the machinery of manifold calculus, addressing the theory of distributions, curvature, vector bundles, and induced connections. A good deal of the theory of distributions is laid out in [8], here only modified to suit its application to harmonic morphisms and their associated horizontal and vertical distributions. As seen in [8], all the theory below can be developed for a more general distribution $\mathcal{D}$ and its orthogonal complement $\mathcal{D}^{\perp}$. The discussion of vector bundles and their induced connections furthermore is drawn primarily from [13].

4.1 Distributions

Definition 4.1. (Distribution) Let M be a smooth manifold of dimension m. Let $n \leq m$. For each $x \in M$ we assigned a subspace $\mathcal{D}_x \subseteq \mathrm{T}_x M$. Choose a neighborhood U of x and n-many linearly independent vector fields $X_1, \ldots, X_n$ such that for each $y \in U$ $\mathrm{span}\{X_1(y), \ldots, X_n(y)\} = \mathcal{D}_y$. Define $\mathcal{D}$ as the collection of all such $\mathcal{D}_x$ for $x \in U$. We call $\mathcal{D}$ an n-dimensional *distribution* of M. Furthermore, the set $\{X_1, \ldots, X_n\}$ is called a local basis of $\mathcal{D}$.

Remark 4.1.1. $\mathcal{D}$ is essentially a subbundle of the tangent bundle $\mathrm{T}M$.

An important property of distributions is involutivity.

Definition 4.2. A distribution $\mathcal{D}$ on M is called *involutive* if for each $x \in M$ there exists a local basis $\{X_1, \ldots, X_m\}$ in a neighborhood of x such that for all $1 \leq i, j \leq m$ the Lie bracket $[X_i, X_j]$ is in the span $\{X_1, \ldots, X_m\}$. This is often denoted $[\mathcal{D}, \mathcal{D}] \subset \mathcal{D}$.

Remark 4.1.2. The involutivity of a distribution is in some sense synonymous with its *integrability*. The Frobenius theorem links the integrability (involutivity) of a distribution (subbundle) to the fact that the distribution arises from a *regular foliation*. A foliation is essentially a decomposition of a manifold into submanifolds called "leaves" which are all of equal dimension. A *regular foliation* of a manifold M is a foliation where each leaf is the fibre of a map $f : M \to N$ where N is generally a submanifold of M.

In the context of harmonic morphisms, there are two distributions of key importance:

Definition 4.3. (Horizontal and Vertical Distributions) Let $\varphi : (M, g) \to (N, h)$ be a harmonic morphism between Riemannian manifolds M and N. First, notice that there exists an orthogonal splitting $\mathrm{T}_x M = \ker \mathrm{d}\varphi_x \oplus (\ker \mathrm{d}\varphi_x)^\perp$ of the tangent space at each $x \in M$. Let $\mathcal{H}_x = (\ker \mathrm{d}\varphi_x)^\perp$ be the horizontal space at x and $\mathcal{V}_x = \ker \mathrm{d}\varphi_x$ the vertical space at x. Then the horizontal distribution $\mathcal{H}$ is the collection of all such $\mathcal{H}_x$ and the vertical distribution the collection of all such $\mathcal{V}_x$ for each $x \in M$, where $\mathcal{H} = \mathcal{V}^\perp$ is a distribution orthogonal to $\mathcal{V}$.

Moving forward in this section, the definitions are mostly phrased in reference to the horizontal and vertical distributions so as to make their connection to harmonic morphisms more explicit.

Definition 4.4. (Affine Connection) Let M be a smooth manifold and let $\Gamma(\mathrm{T}M)$ denote the sections (vector fields) of the tangent bundle over M. An *affine connection* is a bilinear map

$$\Gamma(\mathrm{T}M) \times \Gamma(\mathrm{T}M) \to \Gamma(\mathrm{T}M)$$
$$(X, Y) \mapsto \nabla_X Y$$

with the following properties:

1. $\nabla_{fX} Y = f \nabla_X Y$
2. $\nabla_X (fY) = \mathrm{d}f(X)Y + f\nabla_X Y$

Remark 4.1.3. The first property indicates that the affine connection is $\mathcal{C}^\infty(M, \mathbb{R})$-linear in the first slot and the second property indicates that the affine connection follows a Leibniz rule.

Definition 4.5. (Levi-Civita Connection) Let (M, g) be a Riemannian manifold with metric g. An affine connection is called a *Levi-Civita connection* if it satisfies the following two properties:

1. the connection preserves the metric: $\nabla g = 0$

2. the connection is torsion-free: $\nabla_X Y - \nabla_Y X = [X, Y]$, where $X, Y \in \Gamma(\mathrm{T}M)$ and $[\cdot, \cdot]$ is the Lie bracket.

4.1.1 Connection induced on $\mathcal{H}$ by (M, g, ∇)

Using the Levi-Civita connection ∇ in (M, g) and the projection $\pi^{\mathcal{H}} : \mathrm{T}M \to \mathcal{H}$, one can define a covariant derivative between sections of $\mathcal{H}$ in the following way:

$$\nabla^{\mathcal{H}} : \Gamma(\mathcal{H}) \times \Gamma(\mathcal{H}) \to \Gamma(\mathcal{H})$$
$$(X, Y) \mapsto \pi^{\mathcal{H}}(\nabla_X Y)$$

this covariant derivative is called the *intrinsic connection* of the distribution $\mathcal{H}$ in the Riemannian manifold (M, g). The intrinsic connection satisfies the standard properties of an affine connection

1. $\nabla^{\mathcal{H}}_{fX} Y = f \nabla^{\mathcal{H}}_X Y$

2. $\nabla^{\mathcal{H}}_X (fY) = df(X)Y + f\nabla^{\mathcal{H}}_X Y$

Remark 4.1.4. Notice that the intrinsic connection is not torsion-free with respect to the Lie bracket in the manifold M. The torsion of $\nabla^{\mathcal{H}}$ for $X, Y \in \Gamma(\mathcal{H})$ is given by:

$$T(X, Y) = \nabla^{\mathcal{H}}_X Y - \nabla^{\mathcal{H}}_Y X - [X, Y] = -\pi^{\mathcal{V}}([X, Y])$$

4.2 The Second Fundamental Form of $\mathcal{H}$

Definition 4.6. The *second fundamental form* of $\mathcal{H}$ is the map

$$B : \Gamma(\mathcal{H}) \times \Gamma(\mathcal{H}) \to \Gamma(\mathrm{T}M) \tag{4.2.1}$$
$$(X, Y) \mapsto \nabla_X Y - \nabla^{\mathcal{H}}_X Y = \pi^{\mathcal{V}}(\nabla_X Y) \tag{4.2.2}$$

The second fundamental form satisfies the following properties:

1. B is $\mathcal{C}^\infty(M)-$bilinear and takes values in $\Gamma(\mathcal{V})$

2. By the above definition, we get the *Gauss formula*

$$\nabla_X Y = \nabla^{\mathcal{H}}_X Y + B(X, Y)$$

3. Let $\{e_i\}_{i=1}^{m-n}$ be a local orthonormal basis of $\Gamma(\mathcal{V})$, then:

$$B(X, Y) = \sum_{i=1}^{m-n} g\big(B(X, Y), e_i\big)e_i$$

4.3 Curvature

4.3.1 Symmetry of $B(X, Y)$ and Integrability

The second fundamental form of $\mathcal{H}$ given by Equation 4.2.1 can be decomposed into symmetric and anti-symmetric components:

$$B^s(X, Y) = \frac{1}{2}(B(X, Y) + B(Y, X)) \quad (4.3.1)$$

$$B^a(X, Y) = \frac{1}{2}(B(X, Y) - B(Y, X)) \quad (4.3.2)$$

where $X, Y \in \Gamma(\mathcal{H})$. Thus, $B(X, Y) = B^s(X, Y) + B^a(X, Y)$

Corollary 4.0.1. *The distribution $\mathcal{H}$ is involutive if and only if its second fundamental form B is symmetric.*

Proof. ($\Rightarrow$) Let $\mathcal{H}$ be involutive. Then it's closed under the Lie bracket. Hence, $\forall X, Y \in \Gamma(\mathcal{H})$, we have $[X, Y] \in \Gamma(\mathcal{H})$. Then observe that

$$B^a(X, Y) = \frac{1}{2}\pi^{\mathcal{V}}([X, Y])$$

Hence, since $[X, Y]$ is horizontal, gives $\pi^{\mathcal{V}}([X, Y]) = 0$. Thus $B(X, Y)$ is symmetric.

($\Leftarrow$) Similarly, assuming $B(X, Y)$ is symmetric means $[X, Y] \in \Gamma(\mathcal{H})$ for all $X, Y \in \Gamma(\mathcal{H})$. Thus, $\mathcal{H}$ is involutive by definition. □

Thus, by virtue of the Frobenius theorem, we can see that the distribution $\mathcal{H}$ is integrable $\iff B(X,Y)$ is symmetric.

4.3.2 Mean Curvature

Definition 4.7. Let $B^{\mathcal{V}}$ be the second fundamental form of the distribution $\mathcal{V}$ (cf: 4.2.1, 4.2.2). Then by the *mean curvature of* $\mathcal{V}$, is meant the vector field

$$\mu^{\mathcal{V}} = \frac{1}{q}\mathrm{Tr}\ B^{\mathcal{V}} = \frac{1}{q}\sum_{r=1}^{q}\pi^{\mathcal{H}}(\nabla_{e_r}e_r) \tag{4.3.3}$$

where $\{e_1,\ldots,e_r\}$ is a *local moving frame* for $\mathcal{V}$.

Definition 4.8. A distribution $\mathcal{V}$ on M is said to be

(i) *minimal*, if, for each $x \in M$, the mean curvature vanishes.

(ii) *totally geodesic*, if, for each $x \in M$, the symmetric component of the second fundamental form $B_x^{\mathcal{V},s}$ vanishes.

(iii) *umbilic* if, for each $x \in M$, the normal curvature $B_x^{\mathcal{V}}(V,V)$ in direction V is independent of $V \in \mathcal{V}_x$ for $|V| = 1$.

Definition 4.9. A distribution $\mathcal{V}$ on M is said to be *conformal* if for each $x \in M$,

$$g(\nabla_V X, Y) + g(X, \nabla_V Y) = f(V)\ g(X,Y) \tag{4.3.4}$$

where $X, Y \in \mathcal{H}_x$ and $V \in \mathcal{V}_x$ and $f(V)$ is a real number that depends on V.
if $f \equiv 0$ then $\mathcal{V}$ is called *Riemannian.*

The following proposition proves useful in typifing harmonic morphisms:

Proposition 4.3.1. (Orthogonal Distribution) A distribution $\mathcal{V}$ on a Riemannian manifold M is *conformal* (*Riemannian*, respectively) as accordingly the distribution $\mathcal{H} = \mathcal{V}^{\perp}$ is *umbilic* (*totally geodesic*, respectively).

4.4 Connections Over Vector Bundles

Now that we have presented the theory of connections over the tangent bundle $\mathrm{T}M \xrightarrow{\pi} M$ as well as those induced on the distributions $\mathcal{H}$ and $\mathcal{H}^{\perp} = \mathcal{V}$, we now aim to generalize the process even further, looking at connections induced over more general vector bundles.

Definition 4.10. Let $E \xrightarrow{\pi} M$ be a smooth vector bundle over a smooth manifold M (i.e. E has a vector space structure). A connection $\nabla = \nabla^E$ on E is a map $\nabla : \Gamma(\mathrm{T}M) \times \Gamma(E) \to \Gamma(E)$ such that for $X \in \Gamma(\mathrm{T}M)$ and $\sigma \in \Gamma(E)$:

$$(X, \sigma) \mapsto \nabla_X \sigma \tag{4.4.1}$$

This connection over the vector bundle has the usual properties of a connection. Thus, given $f \in \mathcal{C}^{\infty}(M)$ we have:

$$\nabla_{fX}\sigma = f\nabla_X \sigma \quad , \quad \nabla_X(f\sigma) = X(f)\sigma + f\nabla_X\sigma \tag{4.4.2}$$

4.4.1 Induced Connections

Given connections ∇^E and ∇^F on vector bundles $E \xrightarrow{\pi^E} M$ and $F \xrightarrow{\pi^F} M$, we can define connections over various other vector bundles which are derived from these original bundles.

Example 4.4.1. (Connection on dual bundle) Let $E^* = \hom(E, \mathbb{R})$ be the dual space of E. Then we can define the dual bundle as $E^* \xrightarrow{\pi^{E^*}} M$ and its associated connection ∇^{E^*} as follows: Let $\theta \in \Gamma(E^*)$ and $\sigma \in \Gamma(E)$, then

$$(\nabla_X\theta)\sigma = X(\theta(\sigma)) - \theta(\nabla^E_X\sigma) \tag{4.4.3}$$

Example 4.4.2. (Connection on *bundle of linear maps*) Consider the vector bundle $\hom(E, F) \xrightarrow{\pi} M$. Then for $\theta \in \Gamma(\hom(E, F))$ and $\sigma \in \Gamma(E)$, the induced connection is as follows:

$$(\nabla_X\theta)\sigma = \nabla^F_X(\theta(\sigma)) - \theta(\nabla^E_X\sigma) \tag{4.4.4}$$

Remark 4.4.1. Notice that equation Equation 4.4.4 follows naturally from the product rule:

$$\nabla^F_X(\theta(\sigma)) = (\nabla_X\theta)\sigma + \theta(\nabla^E_X\sigma)$$

Example 4.4.3. (Connection on the pull-back bundle) Let $\varphi : M \to N$ be a smooth map between smooth manifolds and let $W \xrightarrow{\pi} N$ be a vector bundle over N. Then the pull-back bundle is the vector bundle $\varphi^{-1}W \xrightarrow{\pi} M$ with typical fibre $(\varphi^{-1}W)(x) = (W \circ \varphi)(x)$ for $x \in M$. The induced *pull-back connection* ∇^{φ} is the unique linear connection on the pull-back bundle such that, for each $\sigma \in \Gamma(W)$,

$$\nabla^{\varphi}_X(\varphi^*\sigma) = \nabla^W_{\mathrm{d}\varphi(X)}(\sigma) \tag{4.4.5}$$

where $\varphi^*(\sigma) = \sigma \circ \varphi \in \Gamma(\varphi^{-1}W)$.

4.5 Second Fundamental Form of a Map and the Tension Field

Using the connections developed in § 4.4.1, we can derive a corresponding second fundamental form for the map φ.

Definition 4.11. Let $M := (M^m, g)$ and $N := (N^n, h)$ be Riemannian manifolds and let $\varphi : M \to N$ be a smooth map. Note that the differential of φ can be viewed as a section of $T^*M \otimes \varphi^{-1}TN$ (tensor product bundle) over M. There is a connection associated to this bundle derived from the Levi-Civita connection ∇^M on M and the pull-back connection ∇^{φ} (see § 4.4.1), which we simply refer to as ∇. After applying this connection to the differential of φ, we obtain *the second fundamental form of* φ. So let $X, Y \in \Gamma(\mathrm{T}M)$ then:

$$\nabla\mathrm{d}\varphi(X, Y) = \nabla^{\varphi}_X(\mathrm{d}\varphi(Y)) - \mathrm{d}\varphi(\nabla^M_X Y) \tag{4.5.1}$$

With the second fundamental form of φ, we can now define the *tension field* of φ, which in some sense is a generalization of the Laplacian, but now defined for a general higher dimensional manifold map φ as opposed to just a scalar function.

Definition 4.12. Let $\varphi : M \to N$ be a smooth map between Riemannian manifolds. Then *the tension field* $\tau(\varphi)$ viewed as a section of the pull-back bundle $\varphi^{-1}TN$ is defined as

$$\tau(\varphi) = \mathrm{Tr}\ \nabla\mathrm{d}\varphi = \sum_{i=1}^{m} \nabla\mathrm{d}\varphi(E_i, E_i) \tag{4.5.2}$$

where $\{E_i\}$ is an orthonormal frame of M, Tr is the trace, and $\nabla\mathrm{d}\varphi$ is the second fundamental form of φ.

5 Harmonic Morphisms over Riemannian Manifolds

In this section we generalize the theory laid out in § 2 to maps between Riemannian manifolds. An important distinction between the Euclidean and Riemannian cases is that, due to underlying curvature of manifolds involved, the conformality (angle-perserving property) of the map φ plays a more conspicuous role. We look at horizontally weakly conformal maps and harmonic maps, detailing how they relate to harmonic morphisms over Riemannian manifolds. We then finally arrive at the fundamental equation of harmonic morphisms, which allows us to posit a few insightful characterizations of harmonic morphisms.

5.1 Basic Definitions

Definition 5.1. (Harmonic Morphism between Riemannian manifolds) Let (M, g) and (N, h) be manifolds with their associated Riemannian metrics (symmetric, positive-definite). A map $\varphi : M \to N$ is called a *harmonic morphism* if for any harmonic function $f : U \subseteq N \to \mathbb{R}$, defined on an open subset U of N with $\varphi^{-1}(U) \subseteq M$ non-empty, $f \circ \varphi : \varphi^{-1}(U) \to \mathbb{R}$ is a harmonic function.

This is precisely the same definition seen in § 2, simply generalized to Riemannian manifolds. Despite the seeming lack of reference to the metrics on M and N, it's important to note that the Laplacian is metric dependent:

Remark 5.1.1. For instance, if $M = \mathbb{R}^m$ then the expression for the Laplacian is the familiar:

$$\Delta f = \sum_{k=0}^{m} \frac{\partial^2 f}{\partial x_i^2}$$

Otherwise, the expression for the Laplacian depends on the metric g on M. Thus given a coordinatization of M $(x_1, \ldots, x_m)$, the Laplacian can be generally expressed (using the Einstein summation convention):

$$\Delta f = \frac{1}{\sqrt{|g|}} \partial_i \left(\sqrt{|g|} g^{ij} \partial_j f \right)$$

where $\partial_i := \frac{\partial}{\partial x_i}$, $|\cdot|$ denotes the determinant, and g^{ij} is the inverse metric.

5.2 Horizontally Weakly Conformal Maps (HWC)

As mentioned above, the conformality of $\varphi : M \to N$ plays a more conspicuous role in the Riemannian case. Nonetheless, we do not require φ be conformal with respect to the entirety of the tanget bundle $\mathrm{T}M$, but rather only on the horizontal subbundle $\mathcal{H} =: (\ker \mathrm{d}\varphi)^{\perp}$ which we identify with the orthogonal complement of the kernel of the differential map.

Definition 5.2. (Horizontally Weakly Conformal) For $m \geq n$, a non-constant map $\varphi : (M^m, g) \to (N^n, h)$ and $x \in M$, let $\mathcal{V}_x := \ker \mathrm{d}\varphi_x \subset T_xM$ be the vertical distribution and $\mathcal{H}_x := \mathcal{V}_x^{\perp} \subset T_xM$ be the horizontal distribution. If $C_\varphi := \{x \in M \mid \mathrm{d}\varphi_x = 0\}$ and $\hat{M}^m := M - C_\varphi$, then $\varphi : (M, g) \to (N, h)$ is said to be *horizontally (weakly) conformal* if there exists a function $\lambda : \hat{M} \to \mathbb{R}^+$ such that

$$\lambda^2(x) g(X, Y) = h(\mathrm{d}\varphi(X), \mathrm{d}\varphi(Y))$$

for all $X, Y \in \mathcal{H}_x$ and $x \in \hat{M}$. The function λ is then extended to the whole of M by setting $\lambda|_{C_\varphi} \equiv 0$. The extended function $\lambda : M \to \mathbb{R}_0^+$ is called the *dilation* of φ.

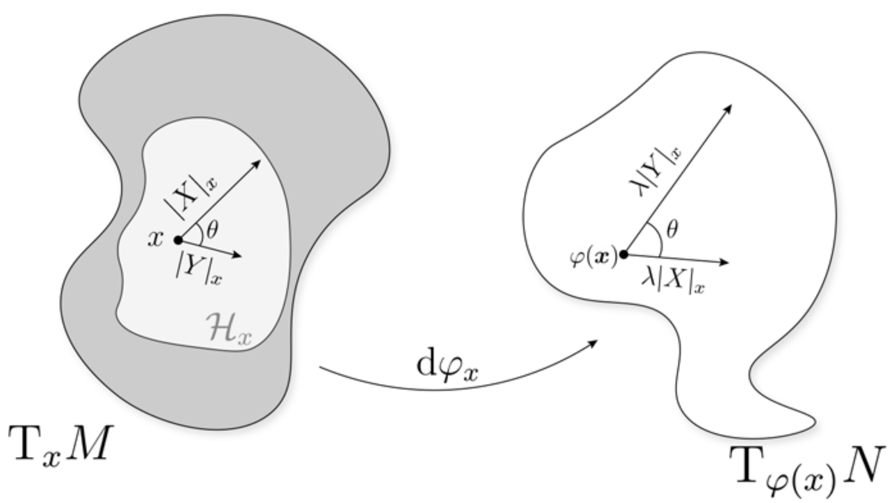

Figure 5.1: **HWC Map:** A horizontally weakly conformal map $\varphi : M \to N$ is shown. Here $X, Y \in \Gamma(\mathcal{H})$ are horizontal vector fields. Notice first the orthogonal splitting of $\mathrm{T}M = \ker \mathrm{d}\varphi \oplus (\ker \mathrm{d}\varphi)^{\perp}$. Next, notice that while the lengths of the vectors in $\mathrm{T}_{\varphi(x)}N$ are scaled by $1/\lambda$ that the angle θ is preserved under the differential map $\mathrm{d}\varphi$.

Remark 5.2.1. $\vec{\nabla}\lambda^2 \in \Gamma(\mathrm{T}M)$, where by $\Gamma(\mathrm{T}M)$ is meant the set of sections of the tangent bundle. On $(\hat{M}^m, g), \mathcal{V} := \{\mathcal{V}_x \mid x \in \hat{M}\}$ and $\mathcal{H} := \{\mathcal{H}_x \mid x \in \hat{M}\}$ are smooth distributions or subbundles of $T\hat{M}$. $\pi^{\mathcal{V}}$ and $\pi^{\mathcal{H}}$ are used to denote the natural projections onto $\mathcal{V}$ and $\mathcal{H}$ at each point $x \in \hat{M}$. On $\hat{M}$ there exists a unique orthogonal splitting of the $\vec{\nabla}\lambda^2$ into vertical and horizontal components:

$$\vec{\nabla}\lambda^2 = \pi^{\mathcal{H}}(\vec{\nabla}\lambda^2) + \pi^{\mathcal{V}}(\vec{\nabla}\lambda^2)$$

A map φ whose dilation λ is such that its level sets are horizontal submanifolds are special in the typification of harmonic morphisms (see § 6.3).

Definition 5.3. (Horizontally Homothetic) A non-constant map $\varphi : (M, g) \to (N, h)$ is said to be *horizontally homothetic* if it is horizontally conformal and $\pi^{\mathcal{H}}(\vec{\nabla}\lambda^2) \equiv 0$ on $\hat{M}$.

5.3 Harmonic Maps

In this section, we outline the basic theory of harmonic maps. Moving forward, it is necessary to make a distinction between harmonic functions and harmonic maps. In particular, *harmonic functions* are scalar maps which solve the Laplace equation, whereas *harmonic maps* are maps to higher dimensional manifolds that in a sense solve a higher dimensional analog to the Laplace equation. The problem, of course, is that the Laplacian of a higher dimensional map is not a scalar, but a tensor. This issue is obviated by looking rather at the trace of the second fundamental form of the map as opposed to the Laplacian itself (see § 4.5).

Definition 5.4. Let $\varphi : M \to N$ be a smooth map between Riemannian manifolds and let $x \in M$. Then the *Hilbert-Schmidt norm* $||\mathrm{d}\varphi_x||$ of its differential at x is defined by

$$||\mathrm{d}\varphi_x||^2 = \sum_{i=1}^{m} h(\mathrm{d}\varphi_x(E_i), \mathrm{d}\varphi_x(E_i)) \tag{5.3.1}$$

where $\{E_i\}$ is an orthonormal basis for $\mathrm{T}_x M$.

Alternatively, if we define the *pull-back* $\varphi^* h$ of the metric h by

$$\varphi^* h(E, F) = h(\mathrm{d}\varphi(E), \mathrm{d}\varphi(F)) \tag{5.3.2}$$

where $E, F \in \Gamma(\mathrm{T}M)$, then we can rewrite the Hilbert-Schmidt norm as

$$||\mathrm{d}\varphi_x||^2 = \mathrm{Tr}\ \varphi^* h = \sum_{i=1}^{m} \varphi^* h(E_i, E_i) \tag{5.3.3}$$

With this theory in place, we now work toward a definition of a harmonic map.

Definition 5.5. The *energy density* of φ at a point $x \in M$ is given by the following:

$$e(\varphi) = \frac{1}{2}\|\mathrm{d}\varphi\|^2 \tag{5.3.4}$$

where $\mathrm{d}\varphi$ is the differential map of φ and where again $\|\cdot\|^2$ is the Hilbert-Schmidt norm with respect to the induced metric on the bundle $\mathrm{T}^*M \otimes \varphi^{-1}(\mathrm{T}N)$

Definition 5.6. The *total energy* of φ is given by integration over M of the energy density of φ

$$E(\varphi) = \frac{1}{2}\int_M \|\mathrm{d}\varphi\|^2 dV \tag{5.3.5}$$

where dV is the volume element over M.

Let $\mathcal{C}^\infty(M, N)$ denote the space of all smooth maps from M to N. A map $\varphi \in \mathcal{C}^\infty(M, N)$ is said to be *harmonic* if it is an *extremal* of the energy functional $E(\cdot; D) : \mathcal{C}^\infty(M, N) \to \mathbb{R}$ over any compact domain D of M.

More specifically, let $\{\varphi_t\}$ be a family of smooth mappings from M to N which depends smoothly on a parameter $t \in (-\epsilon, \epsilon)$ for some $\epsilon \in \mathbb{R}$ such that $\varphi_0 = \varphi$. Then the following defines a harmonic map:

Definition 5.7. Let $\varphi : (M, g) \to (N, h)$ be a smooth map between Riemannian manifolds. Then φ is *harmonic* if

$$\frac{\mathrm{d}}{\mathrm{d}t} E(\varphi_t; D)\Big|_{t=0} = 0 \tag{5.3.6}$$

for all compact domains D and all smooth variations $\{\varphi_t\}$ of φ supported in D.

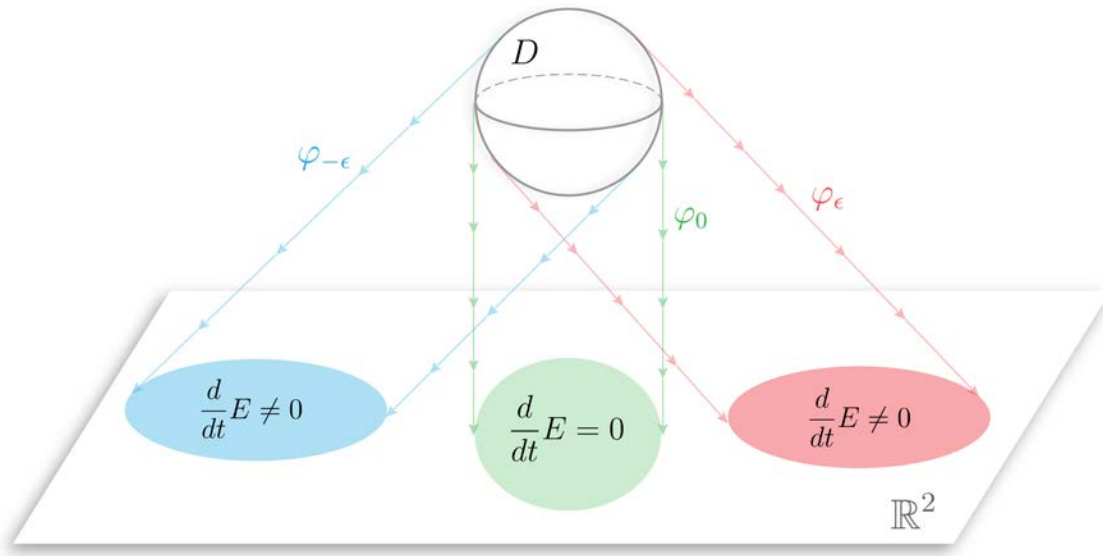

Figure 5.2: **Minimizing the Energy Functional** A simplified depiction of how the variation of a map φ on a compact domain D relates to the total energy E.

Now, it can be shown that the left hand side of Equation 5.3.6 can be written equivalently as follows:

Proposition 5.3.1. (First variation of the energy) Let $\varphi : M \to N$ be a smooth map and $\{\varphi_t\}$ be a smooth variation of φ supported in D. Then

$$\frac{\mathrm{d}}{\mathrm{d}t}E(\varphi_t; D)\bigg|_{t=0} = -\int_M \langle v, \tau(\varphi)\rangle \, dV \tag{5.3.7}$$

where $v(x) := \frac{\partial \varphi_t}{\partial t}(x)\big|_{t=0} \in \Gamma(\varphi^{-1}TN)$ and $\langle \cdot, \cdot \rangle$ is the metric on the pullback bundle.

Thus, observing that in Proposition 5.3.1, $\langle \cdot, \cdot \rangle$ as a metric is positive-definite, and assuming φ non-constant implies v is non-zero, means the only way the energy functional is identically zero over all compact domains $D \subseteq M$ is for $\tau(\varphi)$ to be identically zero. This leads to the following theorem:

Theorem 5.1. *(Harmonic Equation) Let $\varphi : M \to N$ be a smooth map. Then φ is harmonic if and only if*

$$\tau(\varphi) = 0 \tag{5.3.8}$$

The result of Theorem 5.1 is the most useful characterization of harmonic maps and is used extensively in the derivation of the fundamental equation of harmonic morphisms (see Equation 5.5.5).

5.4 Characterization of Harmonic Morphisms over Riemannian Manifolds

The following theorem attributed to Fuglede and Ishihara (see [3] and [5]) we state here without proof. This characterization of harmonic morphisms over Riemannian manifolds is incredibly useful and ties together all the abovementioned theory of HWC maps and harmonic maps.

Theorem 5.2. *A map $\varphi : (M^m, g) \to (N^n, h)$ is a harmonic morphism if and only if both of the following conditions hold:*

1. *φ is an harmonic map*
2. *φ is horizontally weakly conformal*

Remark 5.4.1. The proof essentially follows the same track as that of Theorem 2.1 in the Euclidean case, but requires slightly more advanced mathematical machinery beyond the scope of this thesis.

The following theorem, again stated without proof, relates the dimensions of the manifolds M and N to the submersivity of the map φ.

Theorem 5.3. *Let $\varphi : M^m \to N^n$ be a non-constant horizontally weakly conformal mapping of finite order. Then,*

(i) if $m < 2n - 2$ then φ is submersive.

(ii) if $m = 2n-2$ then either φ is submersive, or $(m, n) = (2, 2), (4, 3), (8, 5)$ or $(16, 9)$ and φ has isolated critical points, and the first non-constant term in its Taylor expansion is a Hopf polynomial map (up to homothety).

5.5 The Fundamental Equation of Harmonic Morphisms

Lemma 5.4. *(Second fundamental form of an HC submersion [13]) Let $\varphi : M \to N$ be a horizontally conformal submersion. Then, for any horizontal vector field $X, Y \in \Gamma(\mathcal{H})$,*

$$\nabla d\varphi(X,Y) = X(\ln\lambda)d\varphi(Y) + Y(\ln\lambda)d\varphi(X) - g(X,Y)d\varphi(\nabla\ln\lambda) \tag{5.5.1}$$

Proof. Let $\{\overline{E_i}\}$ be an orthonormal frame on an open set of N; lift each $\overline{E_i}$ to a horizontal vector field E_i on M, then λE_i is an orthonormal frame for the horizontal distribution of M. Let $\overline{X}$ and $\overline{Y}$ be vector fields on an open subset of N, and X and Y their associated horizontal lifts to M. Then, looking at the horizontal projection of the Levi-Civita connection on M, we have:

$$\begin{aligned}\pi^{\mathcal{H}}(\nabla_X Y) &= \sum_{i=1}^{n} g(\nabla_X Y, \lambda E_i)\lambda E_i \\ &= \lambda^2 \sum_{i=1}^{n} g(\nabla_X Y, E_i)E_i\end{aligned}$$

We first develop the right hand side of this equation. So, we use the Koszul formula to rewrite the metric expression:

$$\begin{aligned}\frac{\lambda^2}{2}\sum_{i=1}^{n}\big\{&X\big(g(Y,E_i)\big) + Y\big(g(E_i,X)\big) - E_i\big(g(X,Y)\big) \\ &- g(X,[Y,E_i]) - g(Y,[X,E_i]) + g(E_i[X,Y])\big\}E_i\end{aligned}$$

Now, using the fact that φ is horizontally conformal (i.e. $g(X,Y) = (1/\lambda^2)h(\overline{X},\overline{Y})$) and the naturality of the Lie Bracket (i.e. $\mathrm{d}\varphi([X,Y]_x) = [\mathrm{d}\varphi(X), \mathrm{d}\varphi(Y)]_{\varphi(x)} = [\overline{X},\overline{Y}]_{\varphi(x)}$), we get:

$$\begin{aligned}\frac{\lambda^2}{2}\sum_{i=1}^{n}\Big\{&X\Big((1/\lambda^2)h(\overline{Y},\overline{E_i})\Big) + Y\Big((1/\lambda^2)h(\overline{E_i},\overline{X})\Big) - E_i\Big((1/\lambda^2)h(\overline{X},\overline{Y})\Big) \\ &- (1/\lambda^2)h(\overline{X},[\overline{Y},\overline{E_i}]) - (1/\lambda^2)h(\overline{Y},[\overline{X},\overline{E_i}]) + (1/\lambda^2)h(\overline{E_i},[\overline{X},\overline{Y}])\Big\}E_i\end{aligned}$$

Using the product rule on the first three terms, we get:

$$\frac{\lambda^2}{2}\sum_{i=1}^{n}\Big\{X\Big(1/\lambda^2\Big)h(\overline{Y},\overline{E_i}) + (1/\lambda^2)\overline{X}\Big(h(\overline{Y},\overline{E_i})\Big)$$
$$+ Y\Big(1/\lambda^2\Big)h(\overline{E_i},\overline{X}) + (1/\lambda^2)\overline{Y}\Big(h(\overline{E_i},\overline{X})\Big)$$
$$- E_i\Big(1/\lambda^2\Big)h(\overline{X},\overline{Y}) - (1/\lambda^2)\overline{E_i}\Big(h(\overline{X},\overline{Y})\Big)$$
$$- (1/\lambda^2)h(\overline{X},[\overline{Y},\overline{E_i}]) - (1/\lambda^2)h(\overline{Y},[\overline{X},\overline{E_i}]) + (1/\lambda^2)h(\overline{E_i},[\overline{X},\overline{Y}])\Big\}E_i$$

regathering the terms,

$$\frac{\lambda^2}{2}\sum_{i=1}^{n}\Big\{X\Big(1/\lambda^2\Big)h(\overline{Y},\overline{E_i}) + Y\Big(1/\lambda^2\Big)h(\overline{E_i},\overline{X}) - E_i\Big(1/\lambda^2\Big)h(\overline{X},\overline{Y})\Big\}E_i$$
$$+ \frac{1}{2}\sum_{i=1}^{n}\Big\{\overline{X}\Big(h(\overline{Y},\overline{E_i})\Big) + \overline{Y}\Big(h(\overline{E_i},\overline{X})\Big) - \overline{E_i}\Big(h(\overline{X},\overline{Y})\Big)$$
$$- h(\overline{X},[\overline{Y},\overline{E_i}]) - h(\overline{Y},[\overline{X},\overline{E_i}]) + h(\overline{E_i},[\overline{X},\overline{Y}])\Big\}E_i$$

Now, observing that $\frac{\lambda^2}{2}X(1/\lambda^2) = \frac{\lambda^2}{2}(-\frac{2}{\lambda^3}X) = -\frac{1}{\lambda}X = -X(\ln\lambda)$ and using the Koszul formula on the second portion, gives,

$$\sum_{i=1}^{n}\Big\{-X\big(\ln\lambda\big)h(\overline{Y},\overline{E_i}) - Y\big(\ln\lambda\big)h(\overline{E_i},\overline{X}) + E_i\big(\ln\lambda\big)h(\overline{X},\overline{Y})\Big\}E_i$$
$$+ \sum_{i=1}^{n} h(\nabla_{\overline{X}}\overline{Y},\overline{E_i})E_i$$

Using horizontal conformality once again,

$$\sum_{i=1}^{n}\Big\{-X\big(\ln\lambda\big)(\lambda^2 g(Y,E_i)) - Y\big(\ln\lambda\big)(\lambda^2 g(E_i,X)) + E_i\big(\ln\lambda\big)(\lambda^2 g(X,Y))\Big\}E_i$$
$$+ \sum_{i=1}^{n}\big\{\lambda^2 g\big((\nabla_{\overline{X}}\overline{Y})^{\wedge},E_i\big)\big\}E_i$$
$$= \sum_{i=1}^{n}\Big\{-X\big(\ln\lambda\big)g(Y,\lambda E_i) - Y\big(\ln\lambda\big)g(\lambda E_i,X) + (\lambda E_i)\big(\ln\lambda\big)g(X,Y)\Big\}(\lambda E_i)$$
$$+ \sum_{i=1}^{n} g\big((\nabla_{\overline{X}}\overline{Y})^{\wedge},\lambda E_i\big)(\lambda E_i)$$

Now performing the sums,

$$-X(\ln\lambda)Y - Y(\ln\lambda)X + g(X,Y)(\vec{\nabla}\ln\lambda) + (\nabla_{\overline{X}}\overline{Y})^{\wedge}$$

Finally, combining the horizontal lift of the connection in N with the horizontal projection of the connection in M, we arrive at:

$$(\nabla_{\overline{X}}\overline{Y})^{\wedge} - \pi^{\mathcal{H}}(\nabla_X Y) = X(\ln\lambda)Y + Y(\ln\lambda)X - g(X,Y)(\vec{\nabla}\ln\lambda)$$

The left hand side of the equation is equivalent to the horizontal lift of the second fundamental form:

$$(\nabla_{\overline{X}}\overline{Y})^{\wedge} - \pi^{\mathcal{H}}(\nabla_X Y) = (\nabla \mathrm{d}\varphi(X,Y))^{\wedge}$$

Hence applying the differential map $\mathrm{d}\varphi$ to the above equation gives the result:

$$\nabla \mathrm{d}\varphi(X,Y) = X(\ln\lambda)\,\mathrm{d}\varphi(Y) + Y(\ln\lambda)\,\mathrm{d}\varphi(X) - g(X,Y)\mathrm{d}\varphi(\vec{\nabla}\ln\lambda)$$

□

Proposition 5.5.1. Let $\varphi : M^m \to N^n$ be a smooth horizontally conformal submersion between Riemannian manifolds of dimensions $m, n \geq 1$. Let $\lambda : M \to (0,\infty)$ denote the dilation of φ and let $\mu^{\mathcal{V}}$ denote the mean curvature vector fields of it fibres. Then the tension field of φ is given by

$$\tau(\varphi) = -(n-2)\,\mathrm{d}\varphi(\vec{\nabla}\ln\lambda) - (m-n)\,\mathrm{d}\varphi(\mu^{\mathcal{V}}) \tag{5.5.2}$$

Proof. Let $\{E_i\}_{i=1}^n$ be a local orthonormal frame for the horizontal distribution $\mathcal{H}$. Then the horizontal trace (the trace restricted to $\mathcal{H}\times\mathcal{H}$) of the second fundamental form can be computed as follows (using Equation 5.5.1):

$$\begin{aligned}\mathrm{Tr}^{\mathcal{H}}\,\nabla \mathrm{d}\varphi &= \sum_{i=1}^{n} \nabla \mathrm{d}\varphi(E_i,E_i)\\ &= \sum_{i=1}^{n}\left\{E_i(\ln\lambda)\,\mathrm{d}\varphi(E_i) + E_i(\ln\lambda)\,\mathrm{d}\varphi(E_i) - g(E_i,E_i)\mathrm{d}\varphi(\vec{\nabla}\ln\lambda)\right\}\end{aligned}$$

Observe that since the E_i are orthonormal that $g(E_i,E_i) = 1$ and that by linearity of the differential map we may write:

$$\mathrm{d}\varphi\left(\sum_{i=1}^{n}\left\{2\,E_i(\ln\lambda)\,E_i - \vec{\nabla}\ln\lambda\right\}\right)$$

Observing that $\sum_{i=1}^{n} E_i(\ln\lambda)\, E_i = \vec{\nabla}\ln\lambda$, we get the result:

$$(2-n)\,\mathrm{d}\varphi(\vec{\nabla}\ln\lambda) = -(n-2)\,\mathrm{d}\varphi(\vec{\nabla}\ln\lambda) \tag{5.5.3}$$

Now, looking at the vertical trace (the trace restricted to $\mathcal{V}\times\mathcal{V}$) of the second fundamental form and letting $\{U_i\}_{i=1}^{m-n}$ be a local orthonormal frame for the vertical distribution $\mathcal{V}$:

$$\begin{aligned}\mathrm{Tr}^{\mathcal{V}}\,\nabla\mathrm{d}\varphi &= \sum_{i=1}^{m-n}\nabla\mathrm{d}\varphi(U_i,U_i)\\ &= \sum_{i=1}^{m-n}\left\{\nabla^{N}_{\mathrm{d}\varphi(U_i)}\mathrm{d}\varphi(U_i) - \mathrm{d}\varphi(\nabla^{M}_{U_i}U_i)\right\}\end{aligned}$$

Then, looking at the second term in the above summand, since by definition each $U_i \in \ker\mathrm{d}\varphi$ ($\mathrm{d}\varphi(U_i) = 0 \in \mathrm{T}N$) and since the differential is a linear map, we get:

$$-\mathrm{d}\varphi\left(\sum_{i=1}^{m-n}\nabla^{M}_{U_i}U_i\right)$$

Since only the horizontal components of the vector fields $\nabla^{M}_{U_i}U_i$ survive the differential map, means:

$$-\mathrm{d}\varphi\left(\sum_{i=1}^{m-n}\nabla^{M}_{U_i}U_i\right) = -\mathrm{d}\varphi\left(\sum_{i=1}^{m-n}\pi^{\mathcal{H}}(\nabla^{M}_{U_i}U_i)\right)$$

Then by Equation 4.3.3, we have

$$\sum_{i=1}^{m-n}\pi^{\mathcal{H}}(\nabla^{M}_{U_i}U_i) = (m-n)\mu^{\mathcal{V}}$$

Then the linearity of the differential map gives us our result:

$$-(m-n)\,\mathrm{d}\varphi(\mu^{\mathcal{V}}) \tag{5.5.4}$$

Finally, by summing Equation 5.5.3 and Equation 5.5.4, we get the trace of the second fundamental form in M which by Equation 5.5.2 gives us the final result:

$$\tau(\varphi) = -(n-2)\,\mathrm{d}\varphi(\vec{\nabla}\ln\lambda) - (m-n)\,\mathrm{d}\varphi(\mu^{\mathcal{V}})$$

$\square$

From the above expression for the tension field, we derive the so-called fundamental equation of harmonic morphisms:

Theorem 5.5. *(Fundamental Equation) Let $\varphi : M^m \to N^n$ be a smooth non-constant horizontally weakly conformal map between Riemannian manifolds of dimensions $m, n \geq 1$. Then φ is harmonic, and thus a harmonic morphism, if and only if, at every regular point, the mean curvature vector field $\mu^{\mathcal{V}}$ of the fibres and the gradient of the dilation λ of φ are related by*

$$(n-2)\,\pi^{\mathcal{H}}(\vec{\nabla}\ln\lambda) + (m-n)\,\mu^{\mathcal{V}} = 0 \tag{5.5.5}$$

Proof. ($\Rightarrow$): Let φ be a harmonic morphism. Then as stated above, φ is considered a harmonic map if the tension field vanishes. Notice that by linearity of the differential map, we may write Equation 5.5.2 equivalently as:

$$\tau(\varphi) = \mathrm{d}\varphi\Big\{-(n-2)\,\vec{\nabla}\ln\lambda - (m-n)\,\mu^{\mathcal{V}}\Big\}$$

Recalling that $\mu^{\mathcal{V}} \in \Gamma(\mathcal{H})$, notice that

$$\begin{aligned}\mathrm{d}\varphi\Big\{-(n-2)\,\vec{\nabla}\ln\lambda - (m-n)\,\mu^{\mathcal{V}}\Big\} &= \mathrm{d}\varphi\Big\{\pi^{\mathcal{H}}\Big(-(n-2)\,\vec{\nabla}\ln\lambda - (m-n)\,\mu^{\mathcal{V}}\Big)\Big\}\\ &= \mathrm{d}\varphi\Big\{-(n-2)\,\pi^{\mathcal{H}}(\vec{\nabla}\ln\lambda) - (m-n)\,\mu^{\mathcal{V}}\Big\}\end{aligned}$$

Since we are interested $\tau(\varphi) \equiv 0$ means that non-trivial solutions occur only on the horizontal component of the vector field. In addition, since the differential $\mathrm{d}\varphi$ is an isomorphism between $\mathcal{H}$ and $\mathrm{T}N$ implies that the differential $\mathrm{d}\varphi$ is only zero at zero. Hence, we are strictly interested in the case when:

$$-(n-2)\,\pi^{\mathcal{H}}(\vec{\nabla}\ln\lambda) - (m-n)\,\mu^{\mathcal{V}} \equiv 0$$

($\Leftarrow$): Let $(n-2)\,\pi^{\mathcal{H}}(\vec{\nabla}\ln\lambda) + (m-n)\,\mu^{\mathcal{V}} = 0$. Then by applying the differential map $\mathrm{d}\varphi$ we get $\tau(\varphi) = 0$ which follows from proposition Equation 5.5.2. Hence since the tension field vanishes implies by definition that φ is harmonic and hence by theorem Theorem Theorem 5.2 we have φ is a harmonic morphism. □

5.5.1 Additional Characterization of Harmonic Morphisms

An immediate consequence of the Equation 5.5.5 is the following theorem attributed to Baird and Eels (see [1]):

Theorem 5.6. *Let $m > n \geq 2$ and let $\varphi : (M^m, g) \to (N^n, h)$ be a horizontally conformal submersion. if*

1. *$n = 2$, then φ is a harmonic map if and only if φ has minimal fibres.*

2. *$n \geq 3$, then two of the following conditions imply the other:*

   (a) φ is a harmonic map,

   (b) φ has minimal fibres,

   (c) φ is horizontally homothetic.

Proof. The proof follows quite naturally from following the various consequences of Equation 5.5.5.

$$(n-2)\,\pi^{\mathcal{H}}(\vec{\nabla}\ln\lambda) + (m-n)\,\mu^{\mathcal{V}} = 0$$

Case 1: Let $n = 2$. Then the fundamental equation reduces to

$$(m-2)\,\mu^{\mathcal{V}} = 0$$

Since $m > 2$ implies the mean curvature of the fibres $\mu^{\mathcal{V}} = 0$ which means the fibres are minimal.

Case 2: $[(a)$ and $(b) \Rightarrow (c)]$ Let $n \geq 3$. Let φ be a harmonic map with minimal fibres. Then by Theorem 5.2 φ is a harmonic morphism. Thus, in applying Equation 5.5.5, it reduces to

$$(n-2)\,\pi^{\mathcal{H}}(\vec{\nabla}\ln\lambda) = 0$$

Thus $\pi^{\mathcal{H}}(\vec{\nabla}\ln\lambda) = 0$ which means the map is horizontally homothetic in accordance with Definition 5.3 since clearly $\pi^{\mathcal{H}}(\vec{\nabla}\ln\lambda) = 0 \Longleftrightarrow \pi^{\mathcal{H}}(\vec{\nabla}\lambda^2) = 0$.

Case 3: [(a) and $(c) \Rightarrow (b)$]: Let φ be a harmonic map and be horizontally homothetic. Then since φ is harmonic implies $\tau(\varphi) = 0$ by Theorem 5.1. Thus, Equation 5.5.5 must equal zero. Hence, since the map φ is horizontally homothetic implies the fundamental equation reduces to

$$(m-2)\,\mu^{\mathcal{V}} = 0$$

Hence, $\mu^{\mathcal{V}} = 0$ which again means φ has minimal fibres.

Case 4: [(b) and $(c) \Rightarrow (a)$] Let φ have minimal fibres and be horizontally homothetic. Then clearly, $\pi^{\mathcal{H}}(\vec{\nabla} \ln \lambda) = 0$ by definition of horizontal homothety, and $\mu^{\mathcal{V}} = 0$ by defintion of φ having minimal fibres. Thus,

$$(n-2)\,\pi^{\mathcal{H}}(\vec{\nabla} \ln \lambda) + (m-n)\,\mu^{\mathcal{V}} = 0$$

which implies φ is a harmonic morphism, and thus also a harmonic map. □

6 Harmonic Morphisms with One-Dimensional Fibres

We now turn to harmonic morphisms whose fibers have a one-dimensional relative dimension. While no complete characterization of harmonic morphisms with higher-dimensional fibres currently exists, harmonic morphisms with one-dimensional fibres have been completely typified. One-dimensional fibrations are particularly important for our investigation into the intersection between harmonic morphisms and Milnor fibrations, and so we develop all the required theory to lay out their characterization.

Thus, we take a cursory look at the three types of harmonic morphisms with one-dimensional fibres: Killing type, warped product type, and T type. It is important to appreciate from the outset that the three types are not mutually exclusive. There are examples of one-dimensionally fibred harmonic morphisms which are both of Killing and warped product type, or of warped product and T type, however, it is not feasible to have one that is both Killing and T type. Nevertheless, these three types are all three differentiated by the geometric relationship between the level sets of the dilation function, and the integral manifolds of the horizontal and vertical distributions induced by the map φ.

6.1 Definitions

Definition 6.1. (One-dimensional fibres) Let $\varphi : M^{n+1} \to N^n$ for $n \geq 1$ be a non-constant harmonic morphism. Then, at regular points, the fibres are of dimension 1. Thus, φ is a *harmonic morphism with one-dimensional fibres.*

Remark 6.1.1. Recall from Theorem 5.3, that the relationship between the dimensions of M and N indicate the submersivity of φ. So in the one-dimensional fibre case, if $n+1 < 2n-2 \Rightarrow n > 3$ then φ is submersive. If $n = 3$ then φ has at most isolated singularities.

A useful tool in classifying harmonic morphisms with one-dimensional fibres is by means of the so-called fundamental vertical vector field of φ.

Definition 6.2. (Fundamental vector field) Let $U \in \Gamma(\mathcal{V})$ be such that $|U| = 1$. Then the *fundamental (vertical) vector field of* φ is a a vector field $V \in \Gamma(\mathcal{V})$ such that $|V| = \lambda^{n-2}$. In other words, $V = \lambda^{n-2}U$.

6.2 Killing Type

The first type of harmonic morphism with one-dimensional fibres we discuss is the Killing type. In this type, the fundamental vertical vector fields are Killing vector fields.

Definition 6.3. Let $X \in \Gamma(\mathrm{T}M)$. We say X is *Killing* if for any $Y, Z \in \Gamma(\mathrm{T}M)$

$$g(\nabla_X Y, Z) + g(Y, \nabla_X Z) = 0 \tag{6.2.1}$$

In an intuitive sense, what it means for a vector field X to be Killing is that if we take two vectors at point p on the manifold M and displace them infinitesimally in the direction of X, that their geometric relationship to one another is preserved (angles and lengths).

Definition 6.4. Let $\varphi : M^{n+1} \to N^n$ be a non-constant harmonic morphism with dilation λ. Say that φ is of *Killing type* if, in a neighborhood of each regular point, the fibres are tangent to a Killing vector field

The following proposition makes the implications of the above definition more clear:

Proposition 6.2.1. A non-constant harmonic morphism is of Killing type if and only if one of the following equivalent conditions holds on the set of regular points:

(i) the fundamental vertical vector field V is a Killing vector field.

(ii) the gradient of the dilation is horizontal (i. e. $\pi^{\mathcal{V}}(\vec{\nabla}\lambda) = 0$).

(iii) the associated foliation is Riemannian.

Remark 6.2.1. Notice that in the one-dimensional fibre case the condition for a distribution to be *Riemannian* (see Equation 4.3.4) is equivalent to the condition that the fundamental vector field be *Killing* [Cf Equation 6.2.1].

Another equivalent way of thinking about $\pi^{\mathcal{V}}(\vec{\nabla}\lambda) = 0$ is to consider the level sets of the dilation function, which we here denote $\Lambda_\alpha := \{x \in M \mid \lambda(x) = \alpha, \alpha \in \mathbb{R}^+\}$. In this case, the level sets of λ define submanifolds of M. We can think of Λ_α as corresponding to a certain vector field. With Killing type harmonic morphisms, the fibres of the map are parallel with the level sets of the dilation function $\lambda(x)$, so that if you flow along a fibre of a Killing type harmonic morphism, you remain on the same level set of the dilation function. Hence, we can equivalently say that for Killing type harmonic morphisms, $\Lambda_\alpha \in \Gamma(\mathcal{V})$ when viewed as a vector field.

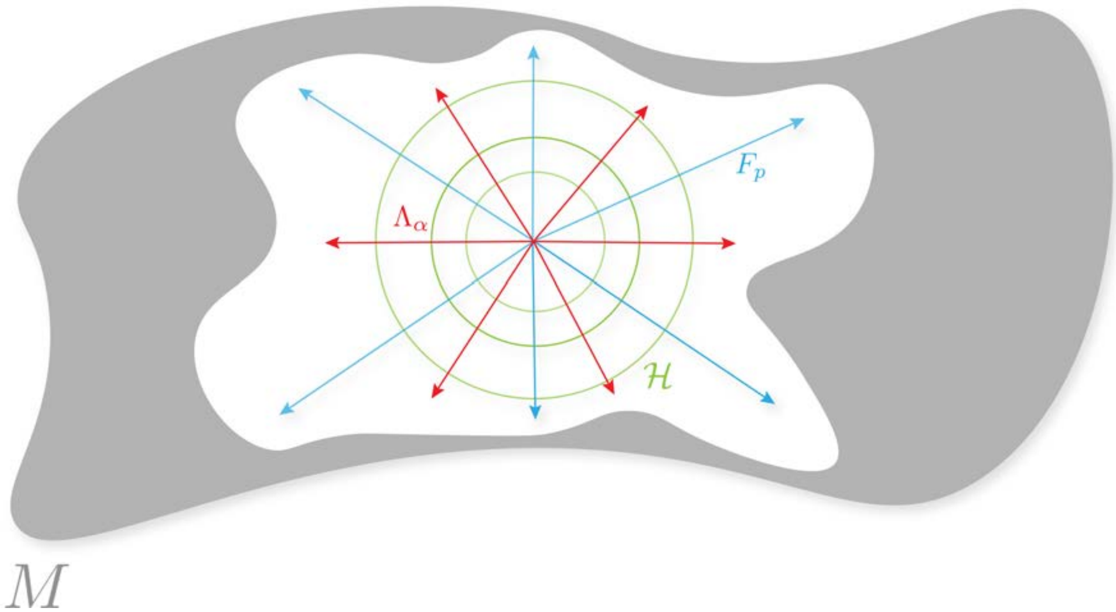

Figure 6.1: **Killing Type:** A simplified depiction of the geometric relationships between the level sets of the dilation function Λ_α, horizontal hypersurfaces $\mathcal{H}$, and fibres $F_p := \{x \in M \mid \varphi(x) = p\ ,\ p \in N\}$ for a Killing type harmonic morphism. Note that each individual ray corresponds to either a different fibre (blue) or a different level set of λ (red).

6.2.1 Examples

Example 6.2.1. Orthogonal projection is a canonical Killing type harmonic morphism (see Examples 3.4.1)

Example 6.2.2. The canonical Hopf fibration $S^3 \to S^2$ is of Killing type.

Example 6.2.3. The Hopf polynomial map $\varphi : \mathbb{R}^4 \to \mathbb{R}^3$:

$$(z_0, z_1) \mapsto (|z_0|^2 - |z_1|^2, 2\bar{z}_0 z_1)$$

is of a Killing type with dilation $\lambda = 2\sqrt{|z_0|^2 + |z_1|^2}$.

6.2.2 Through the Lens of the Fundamental Equation

If we look at the Killing type through the lens of Equation 5.5.5, we can make the following observation about the mean curvature of the fibres:

First, in order that we have a harmonic morphism, the fundamental equation must hold ($n \geq 3$):

$$(n-2)\, \pi^{\mathcal{H}}(\vec{\nabla} \ln \lambda) + \mu^{\mathcal{V}} = 0$$

Since, in the Killing type, $\pi^{\mathcal{V}}(\vec{\nabla} \ln \lambda) = 0$ implies $\pi^{\mathcal{H}}(\vec{\nabla} \ln \lambda) = \vec{\nabla} \ln \lambda$. Hence,

$$(n-2)\, \vec{\nabla} \ln \lambda + \mu^{\mathcal{V}} = 0$$

Which gives us,

$$\mu^{\mathcal{V}} = \vec{\nabla}(-\ln \lambda^{n-2})$$

This shows that in general the mean curvature of the fibres $\mu^{\mathcal{V}}$ for a Killing type harmonic morphism is not constant (and non-zero) and therefore the fibres are not in general minimal.

6.3 Warped Product Type

We now look at the so-called warped product type of harmonic morphisms. As will be shown, these arise in a natural way as the projection to the second factor of a *warped product manifold*, inheriting in turn a metric scaled by the associated warping function. In addition, warped product type harmonic morphisms are typified by being both horizontally homothetic and having minimal fibres. In the case of harmonic morphisms with one dimensional fibres, having minimal fibres requires that they be totally geodesic as well (straight lines with respect to the manifold).

6.3.1 Warped Product Manifolds

Definition 6.5. (warped product Manifold) Let (M, g) and (N, h) be Riemannian manifolds, and let f be a positive function on M. Consider the product manifold $M \times N$ with the projection maps $\pi^M : M \times N \to M$ and $\pi^N : M \times N \to N$. The warped product $W = M \times_f N$ is the manifold $M \times N$ equipped with the warped product metric given by

$$w = g + f^2 h \tag{6.3.1}$$

The function $f : M \to \mathbb{R}^+$ is called the *warping function* of the warped product and the pair (W, w) is called a *warped product manifold*.

6.3.2 Warped Product Type Harmonic Morphisms

Proposition 6.3.1. (Characterization of Warped Products) Let $M \times_f N$ be a warped product manifold with associated warping function $f : M \to \mathbb{R}^+$. Then,

(i) The projection π^N on to the second factor is a horizontally homothetic submersion with totally geodesic fibres and integrable horizontal distribution. Its dilation at $(x, y) \in M \times_f N$ is $1/f(y)$

(ii) Conversely, any horizontally homothetic submersion $(M, g) \to (N, h)$ with totally geodesic fibres and integrable horizontal distribution is locally the projection of a warped product. In fact, if (M, g) is complete, and M and N are simply connected, it is globally such a projection.

6.3.3 Examples

The following three examples are considered canonical examples. There are three more similar examples for warped product type harmonic morphisms that map from hyperbolic space. But while these are also considered canonical, they will not be discussed here.

Example 6.3.1. (Orthogonal projection) Consider the map $\varphi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ that for each $x_0 \in \mathbb{R}$ maps $(x_0, \tilde{x}) \in \mathbb{R}^{n+1}$ to $\tilde{x} \in \mathbb{R}^n$. Then φ may be viewed as the projection from the warped product $\mathbb{R} \times_f \mathbb{R}^{n-1}$ where $f \equiv 1$. Orthogonal projection is the one canonical example whose dilation is constant $1/f = 1$. It thus defines a Riemannian submersion and hence is also of Killing type.

Example 6.3.2. (Radial projection to a sphere) Next, consider $\varphi : \mathbb{R}^n \setminus \{0\} \to S^{n-1}$ that maps $\tilde{x} \in \mathbb{R}^n \setminus \{0\}$ to $\tilde{x}/|\tilde{x}| \in S^{n-1}$. Then φ may be viewed as the projection from the warped product $\mathbb{R}^+ \times_f S^{n-1}$ where $f = |\tilde{x}|$. Here the dilation is non-constant and hence φ is not Killing.

Example 6.3.3. (Radial projection from a sphere) Finally, consider $\varphi : S^n \backslash \{(\pm 1, 0, \ldots, 0)\} \to S^{n-1}$ that maps a point $(x_0, \tilde{x}) \in S^n \backslash \{(\pm 1, 0, \ldots, 0)\}$ to $\tilde{x}/|\tilde{x}| \in S^{n-1}$. Then φ may be viewed as the projection from the warped product $(-\pi, \pi) \times_f S^{n-1}$ where $f = \sin(\theta)$ where θ is the angle in $(-\pi, \pi)$ that corresponds to $\tilde{x}$.

Example 6.3.4. (Riemannian Product) A warped product with $f \equiv 1$ is called a Riemannian product. The projection of a Riemannian product onto either of its factors is a Riemannian submersion with totally geodesic fibers and integrable horizontal distribution. Conversely, each Riemannian submersion is locally of this form.

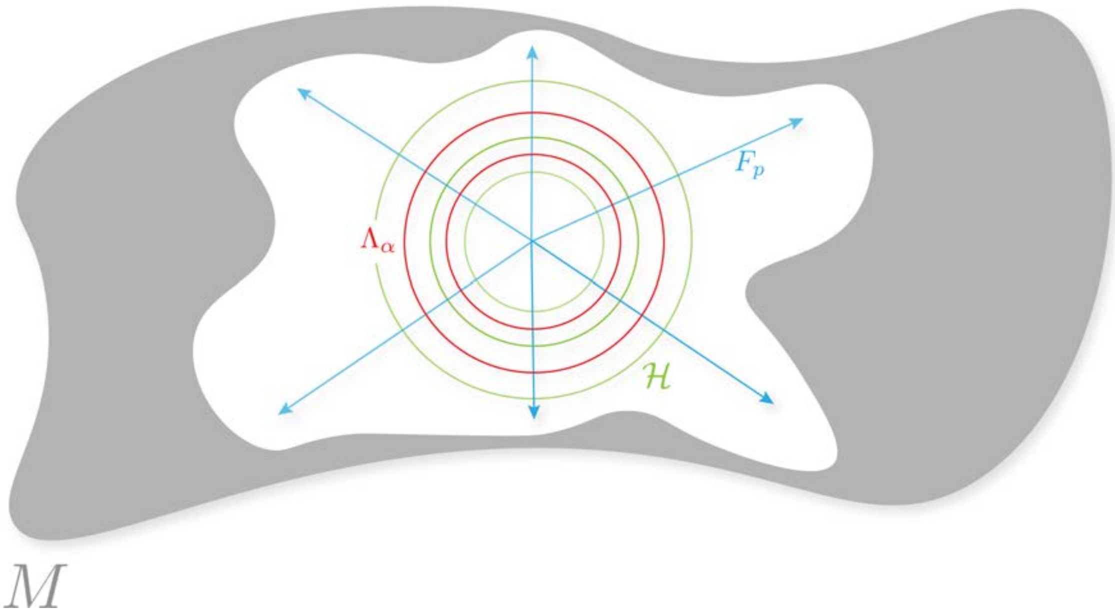

Figure 6.2: **Warped Product Type:** A simplified depiction of the geometric relationships between the level sets of the dilation function Λ_α, horizontal hypersurfaces $\mathcal{H}$, and fibres $F_p := \{x \in M \mid \varphi(x) = p \ , \ p \in N\}$ for a warped product type harmonic morphism. Note that each individual blue-colored ray corresponds to a different fibre and each separate red circle corresponds to a different level set of λ.

By virtue of being horizontally homothetic, another important property of warped product type harmonic morphisms (proven by Fuglede in [2]) is the following:

Proposition 6.3.2. A horizontally homothetic map has no critical points of finite order.

Remark 6.3.1. All warped product type harmonic morphisms are horizontally homothetic maps, hence, Proposition 6.3.2 could be read equivalently to say: *A warped product harmonic morphism has no critical points of finite order.*

Let $\varphi : M^{n+1} \to N^n$ be a non-constant horizontally homothetic harmonic morphism. If $\vec{\nabla}\lambda$ is non-zero on a dense subset of M, then φ is of warped product type.

6.3.4 Another Characterization of Warped Product Type Harmonic Morphisms

We may also relate warped product type harmonic morphisms to geometry of the integral submanifolds of the horizontal distribution $\mathcal{H}$.

Definition 6.6. We say that a family of oriented hypersurfaces is *parallel* if they form a Riemannian foliation

Proposition 6.3.3. A family of oriented hypersurfaces is parallel if any of the following equivalent conditions holds:

(i) any two nearby hypersufaces are a constant distance apart

(ii) moving along geodesics normal to one of the hypersurfaces by a (small enough) constant distance locally produces another hypersurface of the family

(iii) the integral curves of the unit vector field normal to the hypersufaces are geodesics

(iv) parallel transport along these integral curves maps the tangent space of one hypersurface to the tangent space of another.

Definition 6.7. A family of oriented hypersurfaces is called *isoparametric* if they are all parallel and each hypersurface has constant mean curvature.

Proposition 6.3.4. Let $\varphi : M^{n+1} \to N^n$ $(n \geq 1)$ be a harmonic morphism of warped product type. Then the leaves of $\mathcal{H}$ form an isoparametric family of hypersurfaces with each hypersurface umbilic.

A useful result that echoes the above conclusion of Proposition 6.3.4 is the following:

Theorem 6.1. *Let N^{n-1} be a submanifold of codimension 1 in $\mathbb{R}^n$. Let $\varphi : X \subset \mathbb{R}^n \to N$ be an HWC map and a submersion such that $\varphi(x) - x$ is orthogonal to N at $\varphi(x)$. Then N is either a hypersphere, hyperplane, or part of one.*

Proof. Let $\mathbf{s} : U \subset \mathbb{R}^{n-1}$ be a parametrization of N. Let $\mathbf{n}(\mathbf{u})$ be a normal vector field on N. Our subset X can be parametrized by $\mathbf{s}(\mathbf{u}) + t\mathbf{n}(\mathbf{u})$. With these parametrizations, φ is the map $\varphi(\mathbf{u}, t) = \mathbf{u}$, and so $\mathrm{d}\varphi = [I_{n-1}|\mathbf{0}]$. Clearly ∂_t spans the kernel of $\mathrm{d}\varphi$, and so the row vectors of $\mathrm{d}\mathbf{s}$ span the horizontal space. If $\mathbf{v}$ and $\mathbf{w}$ are vectors in $\mathbb{R}^{n-1}$, then the metric applied these vectors give us:

$$\begin{aligned} g(\mathbf{v}, \mathbf{w}) &= (\mathrm{d}\mathbf{s}\mathbf{v} + t\mathrm{d}\mathbf{n}\mathbf{v}) \cdot (\mathrm{d}\mathbf{s}\mathbf{w} + t\mathrm{d}\mathbf{n}\mathbf{w}) \\ &= (\mathrm{d}\mathbf{s}\mathbf{v} \cdot \mathrm{d}\mathbf{s}\mathbf{w}) + t(\mathrm{d}\mathbf{s}\mathbf{v} \cdot \mathrm{d}\mathbf{n}\mathbf{w} + \mathrm{d}\mathbf{n}\mathbf{v} \cdot \mathrm{d}\mathbf{s}\mathbf{w}) + t^2(\mathrm{d}\mathbf{n}\mathbf{v} \cdot \mathrm{d}\mathbf{n}\mathbf{w}) \\ &= (\mathrm{d}\mathbf{s}\mathbf{v} \cdot \mathrm{d}\mathbf{s}\mathbf{w}) - 2t(n \cdot \mathrm{d}^2\mathbf{s}\mathbf{v}\mathbf{w}) + t^2(\mathrm{d}\mathbf{n}\mathbf{v} \cdot \mathbf{n}\mathbf{w}) \\ &= \mathrm{I}(\mathbf{v}, \mathbf{w}) - 2t\mathrm{II}(\mathbf{v}, \mathbf{w}) + t^2(\mathrm{d}\mathbf{n}\mathbf{v} \cdot \mathrm{d}\mathbf{n}\mathbf{w}) \end{aligned}$$

On the hypersurface $\mathbf{s}$, the metric is just $h(\mathbf{v}, \mathbf{w}) = \mathrm{d}\mathbf{s}\mathbf{v} \cdot \mathrm{d}\mathbf{s}\mathbf{w} = \mathrm{I}(\mathbf{v}, \mathbf{w})$. Since I, II and $\mathrm{d}\mathbf{n} \cdot \mathrm{d}\mathbf{n}$ do not depend on t, the only way the metrics will be multiples is if I, II, and $\mathrm{d}\mathbf{n} \cdot \mathrm{d}\mathbf{n}$ are all multiples of each other. In particular, the second fundamental form is a multiple of the first fundamental form, and so we have an umbilic on N. Since this formula was arbitrary, all points on N are umbilics, and so N must be a hypersphere, a hyperplane, or a portion of one of these.

Note that if N is a hyperplane, then $\mathbf{n}$ is constant, so $\mathrm{d}\mathbf{n} \cdot \mathrm{d}\mathbf{n} = 0$, and our metrics are multiples with scaling factor always equal to one. If N is a hypershere, then $\mathbf{n} = \lambda\mathbf{s}$, and so our expression for g reduces to $(1 - \lambda t)^2\mathrm{I}$. In particular, if our hypersphere was the unit sphere at the origin, then the scaling factor would be the distance squared from the origin. □

6.3.5 Through the Lens of the Fundamental Equation

A defining property of warped product harmonic morphisms is that they constitute horizontally homothetic maps. Hence, starting with Equation 5.5.5,

$$(n-2)\,\pi^{\mathcal{H}}(\vec{\nabla} \ln \lambda) + \ \mu^{\mathcal{V}} = 0$$

we set $\pi^{\mathcal{H}}(\vec{\nabla} \ln \lambda) = 0$ since φ is horizontally homothetic. The fundamental equation reduces,

$$\mu^{\mathcal{V}} = 0$$

Hence, since the mean curvature must be zero implies the fibres are minimal. Then since the fibres are one-dimensional means they must be geodesics in M.

6.4 Transnormal/Third Type (T Type)

The third and final type of harmonic morphism is the so-called T type. Here the defining features are that the vertical component of gradient of the dilation function $\pi^{\mathcal{V}}(\vec{\nabla}\lambda)$ is non-zero and that the fibres are transversal to the level sets of λ. It's necessary to make a distinction between pure T type and T type/warped product type harmonic morphisms. Pure T type are relatively rare and do not exist for harmonic morphisms from space forms for $n \geq 4$, while there are many examples of T type/warped product type harmonic morphisms.

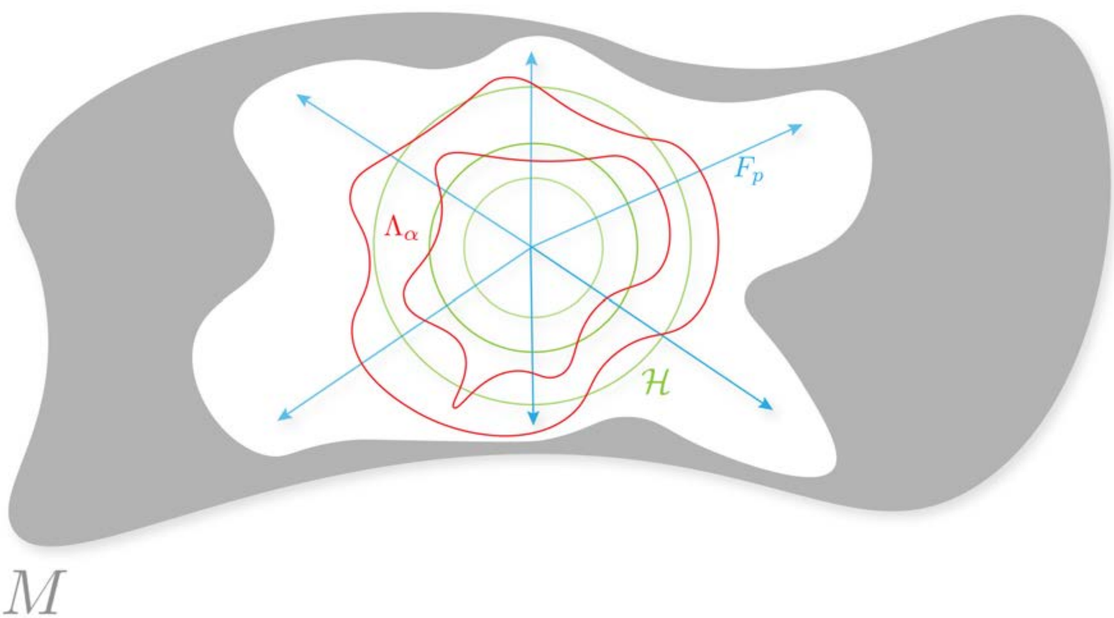

Figure 6.3: **T Type**: A simplified depiction of the geometric relationships between the level sets of the dilation function Λ_α, horizontal hypersurfaces $\mathcal{H}$, and fibres $F_p := \{x \in M \mid \varphi(x) = p \ , \ p \in N\}$ for a T type harmonic morphism. Note that each individual blue-colored ray corresponds to a different fibre and each separate closed red curve corresponds to a different level set of λ.

Definition 6.8. Let $\varphi : M^{n+1} \to N^n$ $(n \geq 1)$ be a non-constant harmonic morphism. We say that φ is of type T on M if, on $M \setminus C_\varphi$, $\left|\pi^{\mathcal{V}}(\vec{\nabla}\lambda)\right|$ is a non-zero constant along each component of the level surfaces of λ.

Remark 6.4.1. (i) The condition $\pi^{\mathcal{V}}(\vec{\nabla}\lambda) \neq 0$ implies that the level surfaces of λ are transversal to the fibres of φ

(ii) A harmonic morphism is simultaneously of warped product type and of type T if and only if $\vec{\nabla}\lambda \in \Gamma(\mathcal{V})$ and non-zero.

Lemma 6.2. *Let $\varphi : M^{n+1} \to N^n$ for $n \geq 1$ be a harmonic morphism with $\pi^{\mathcal{V}}(\vec{\nabla}(\lambda) \neq 0$ on $M \setminus C_\varphi$ where C_φ is the set of critical points of φ. Then φ has non-compact fibres. Further, if $n \geq 3$, then φ is submersive.*

6.4.1 Examples

As mentioned before, the examples of pure T type harmonic morphisms are rare. One well-studied example is the so-called Eguchi-Hanson metric.

Example 6.4.1. (Eguchi-Hanson metric) Let $i : S^3 \hookrightarrow \mathbb{R}^4$ be the canonical inclusion of the 3-sphere in $\mathbb{R}^4$. Define $A = i^*(-y_2 \mathrm{d}y_1 + y_1 \mathrm{d}y_2 - y_3 \mathrm{d}y_4 + y_4 \mathrm{d}y_3)$, where (y_1, y_2, y_3, y_4) are the basic coordinates on $\mathbb{R}^4$. Now, define a Riemannian metric on $S^3 \times (0, \infty)$

$$g_a = s^2 h(\tilde{y}) + (\mathrm{d}s + s^{-1} a A(\tilde{y}))^2 \tag{6.4.1}$$

where h is the canonical metric on S^3 and $(\tilde{y}, s) \in S^3 \times (0, \infty)$. Then for any $a \neq 0$, the canonical projection $\varphi : (S^3 \times (0, \infty), g_a) \to (S^3, h)$ is a harmonic morphisms of T type.

6.5 Global Extensions

An important aspect to the above typification of harmonic morphisms is that once you determine that a given harmonic morphism is one of the three types locally, then it must remain of that type globally. A harmonic morphism can not be of one type in one region of the manifold, and then of another type elsewhere. This result is summed up in the following proposition:

Proposition 6.5.1. (Global extension) Let $\varphi : M^m \to N^n$ for $n \geq 1$ be a non-constant harmonic morphism from a real analytic manifold.

(i) if φ is of Killing type on a open subset of M, then φ is of Killing type on all of M.

(ii) if φ is of warped product type on an open subset of $M \setminus C_\varphi$ where C_φ is the set of critical point of φ, then C_φ is empty and φ is of warped product type on all of M.

(iii) If φ is of T type on an open subset of M, then φ is of T type on $\widetilde{M} = M \setminus \{x \in M \setminus C_\varphi \mid \pi^{\mathcal{V}}(\vec{\nabla}\lambda)_x = 0\}$.

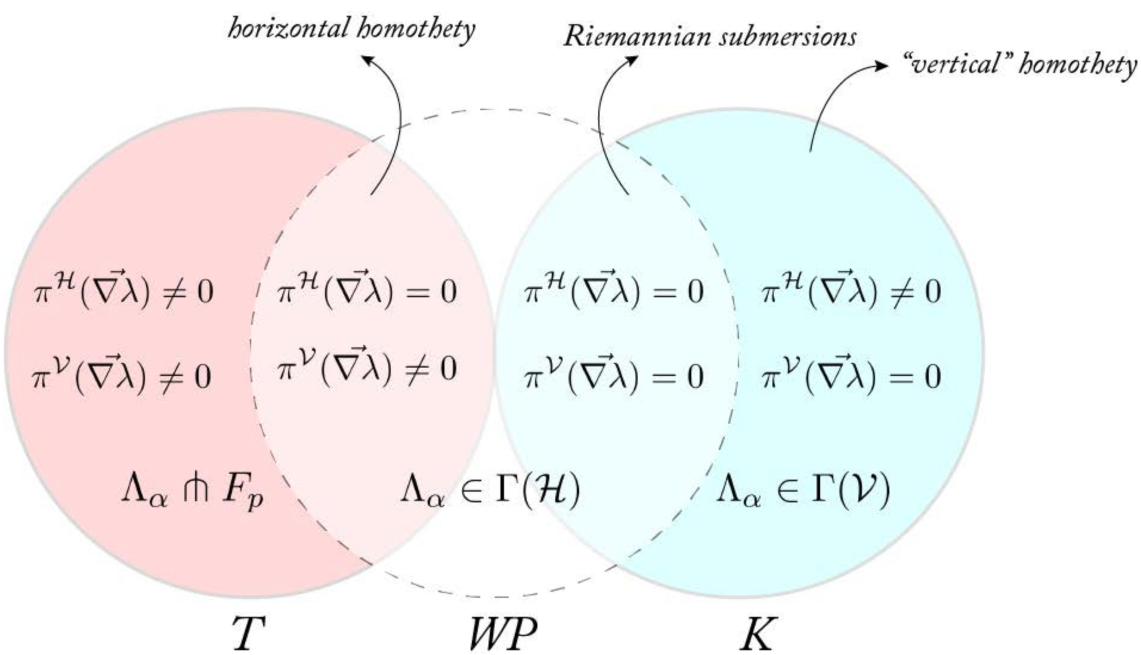

Figure 6.4: **Interrelation of Three Types**: Here notice that harmonic that warped product type harmonic morphisms can be essentially viewed as two seperate groups. The warped product-T type with $\pi^{\mathcal{H}}(\vec{\nabla}\lambda) = 0$ and $\pi^{\mathcal{V}}(\vec{\nabla}\lambda) \neq 0$ or the warped product-Killing type with $\pi^{\mathcal{H}}(\vec{\nabla}\lambda) = 0$ and $\pi^{\mathcal{V}}(\vec{\nabla}\lambda) = 0$. Otherwise, there it is not feasible to be both of T type and Killing type simultaneously.

6.6 Useful Results

As will be seen in § 7, the manifolds involved in the setup for Milnor's fibration theorems are $\mathbb{C}^n$ and S^n, both of which qualify as as *space forms*.

Definition 6.9. A *space form* is a complete Riemannian manifold with constant sectional curvature

The three canonical examples of space forms corresponding to when the sectional curvature is $-1, 0$ and 1, are H^n (hyperbolic space), $\mathbb{R}^n$, and S^n, respectively.

The following theorem shows that for space forms, the possible types of harmonic morphisms are limited:

Theorem 6.3. *Let $\varphi : M^{n+1} \to N^n$ be a harmonic morphism with M a space form. Then for $n \geq 3$, φ is either of Killing type or warped product type.*

Furthermore, if we restrict our attention to those one-dimensonally fibred harmonic morphisms of warped product type which are defined over space forms, then the following theorem shown in [4] provides a full characterization:

Theorem 6.4. *Let U be a domain of a space form $S^{n+1}, \mathbb{R}^{n+1}$, or H^{n+1} for $n \geq 1$. Let $\varphi : U \to N^n$ be a submersive horizontally homothetic harmonic morphism which has integrable horizontal distribution. Then, up to isometry, φ is one of the six canonical examples alluded to in § 6.3 [Cf Examples 6.3.1, 6.3.2 and 6.3.3].*

7 Milnor Fibration Application

This section addresses the overlap between Milnor fibrations and harmonic morphisms. In [9] it is shown that a homogeneous harmonic morphism G of degree p retracts to a harmonic morphism Milnor map over the sphere (described in § 7.4). In § 7.1 we outline our original aim in studying the overlap of Milnor fibrations and harmonic morphisms, seeking ultimately to relax the homogeneity assumption on G which is defined in § 7.4. We then in § 7.2 follow this by a brief sketch of the proof of Milnor's fibration theorems. In § 7.4 we recount the known connection between Milnor fibrations and harmonic morphisms. Following this, in § 7.3 we show the interesting connection between simple Ł maps and HWC maps and under what circumstances a Ł analytic map (and therefore an HWC map) guarantees the existence of a Milnor fibration. In § 7.5 we prove a proposition useful to our final result discussed in § 7.6 where we return to the problem presented in § 7.1.

7.1 The Problem

Let $G : \mathbb{R}^m \to \mathbb{R}^n$ with $m \geq n \geq 2$, be a map with isolated critical point at the origin. Let $V_G := G^{-1}(0)$ be the variety defined by the preimage of the zero set of G. Let $K_\epsilon := V_G \cap S_\epsilon$, the intersection between the variety defined by the preimage of the zero set of G with a sphere S_ϵ^{m-1} of radius ϵ, where ϵ is chosen such that the only critical point of G contained in S_ϵ^{m-1} is the origin. Suppose G is such that the following diagram commutes with φ a submersion and $\Psi|$ a fibration.

$$\begin{array}{ccc} \mathbb{R}^m \setminus V_G & \xrightarrow{\ G\ } & \mathbb{R}^n \setminus \{0\} \\ \varphi \downarrow & \overset{\Psi}{\searrow} & \downarrow \pi \\ S_\epsilon^{m-1} \setminus K_\epsilon & \xrightarrow[\Psi|]{} & S^{n-1} \end{array}$$

The fibration $\Psi|$ is often called a Milnor fibration map.

The question we originally investigated was as follows: Under what assumption could one assert that if G is a harmonic morphism, then $\Psi|$ is also a harmonic morphism.

Below in figure 7.1 is shown a very simplified picture of how the commutative diagram in figure 7.1 requires a certain compatibility between the maps G, φ, and π. G is shown as some homogeneous map (mapping a submanifold of $\mathbb{R}^m$ to a submanifold of $\mathbb{R}^n$), and φ and π as radial projection down to their respective spheres (S^{m-1} and S^{n-1}).

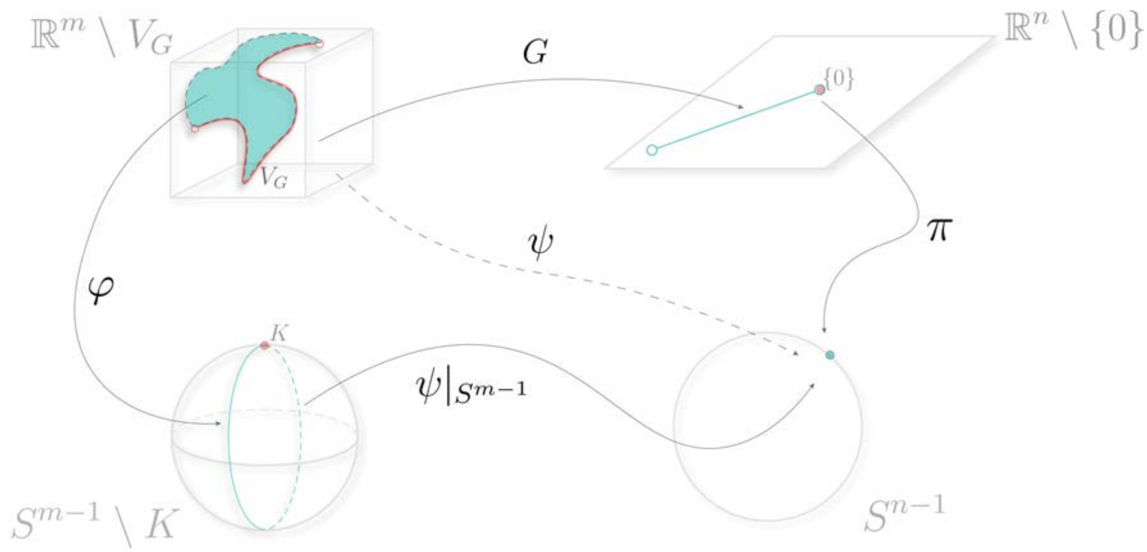

Figure 7.1: **Simplified Example of Commutative Diagram**:

7.2 A Sketch of the Proof of the Milnor Fibration Theorems

In this section we provide a simple sketch of Milnor's fibration theorems, following closely the presentation given in [12] and in [10]. The primary utility in going over the basic proof of Milnor's fibration theorems is to show the existence of a Milnor vector field X whose integral curves are the fibres of a important map we examine in the following section.

Remark 7.2.1. Milnor's original presentation of his now famous fibration theorems is restricted to complex analytic maps from $\mathbb{C}^{n+1}$ to $\mathbb{C}$. With a considerable amount of work, these theorems can be generalized to apply for real analytic maps between Euclidean domains of arbitrary dimension.

Milnor fibrations arose primarily as a tool for finding so called "exotic spheres" which are also known as a homotopy spheres. In short, an exotic sphere is an object that is homeomorphic to a standard sphere but not diffeomorphic to a standard sphere. Milnor showed that an exotic sphere occurs as the intersection of a complex hypersurface singularity with a $(2n+1)$-dimensional sphere S_ϵ^{2n+1} of radius ϵ. This intersection is called the link, K_ϵ. The Milnor fibration describes a smooth fibration between $S_\epsilon^{2n+1} \setminus K_\epsilon$ and the 1-sphere S^1.

Theorem 7.1. *(Milnor Fibration, 1st version) Let U be an open neighborhood of the origin $\tilde{0} \in \mathbb{C}^{n+1}$ and let $f : (\mathbb{C}^{n+1}, \tilde{0}) \to (\mathbb{C}, 0)$ be a germ of a complex analytic function with an isolated singularity at $\tilde{0}$. Let $V := f^{-1}(0)$ and $K_\epsilon := V \cap S_\epsilon^{2n+1}$, where S_ϵ^{2n+1} is a sufficiently small sphere about $\tilde{0}$. Then,*

$$\psi := \frac{f}{|f|} : S_\epsilon^{2n+1} \setminus K_\epsilon \to S^1 \tag{7.2.1}$$

is a smooth fibre bundle.

Milnor followed up the above theorem with a closely related theorem whose result is employed in the proof of Theorem 7.1.

Theorem 7.2. *(Milnor Fibration, 2nd version) Let $\delta > 0$ be sufficiently small relative to ϵ such that for every $t \in \mathbb{C}$ with $|t| \leq \delta$ the fibre $f^{-1}(t)$ meets the sphere S_ϵ^{2n+1} transversally. Let D_δ be the disk in $\mathbb{C}$ of radius δ and centered at 0. Let $\partial D_\delta \cong S^1$ be its boundary and let $N(\epsilon, \delta) := f^{-1}(\partial D_\delta) \cap B_\epsilon$ be the Milnor tube where B_ϵ is the open ball of radius ϵ in $\mathbb{C}^{n+1}$ bounded by S_ϵ^{2n+1}. Then,*

$$f|_{N(\epsilon,\delta)} : N(\epsilon, \delta) \to \partial D_\delta \tag{7.2.2}$$

is a smooth fibre bundle.

The proofs of Theorems 7.1 and 7.2 start by choosing an open ball of sufficiently small radius B_r so that $\tilde{0} \in \mathbb{C}^{n+1}$ is the only critical point contained. B_r is then equipped with a Thom stratification such that V is a union of strata and $\tilde{0}$ is its own stratum. Let $S_\epsilon^{2n+1} \subset B_r$ such that every sphere of radius $\leq r$ intersects each stratum of V transversally. Thus for each t where $|t| \leq \delta$, $f^{-1}(t)$ intersects S_ϵ^{2n+1} transversally, which implies each fibre is a smooth submanifold. Next we lift via $\mathrm{d}f$ vector fields on $\mathbb{C}$ to vector fields on B_ϵ such that the fields are normal to the fibre and tangent to S_ϵ^{2n+1}. We next need to map the tube to the sphere S_ϵ^{2n+1}, translating the fibration in the tube to a fibration on the sphere. This depends on the following lemma.

Lemma 7.3. *There exists an integrable vector field X on $B_\epsilon \setminus V$ such that the following are true:*

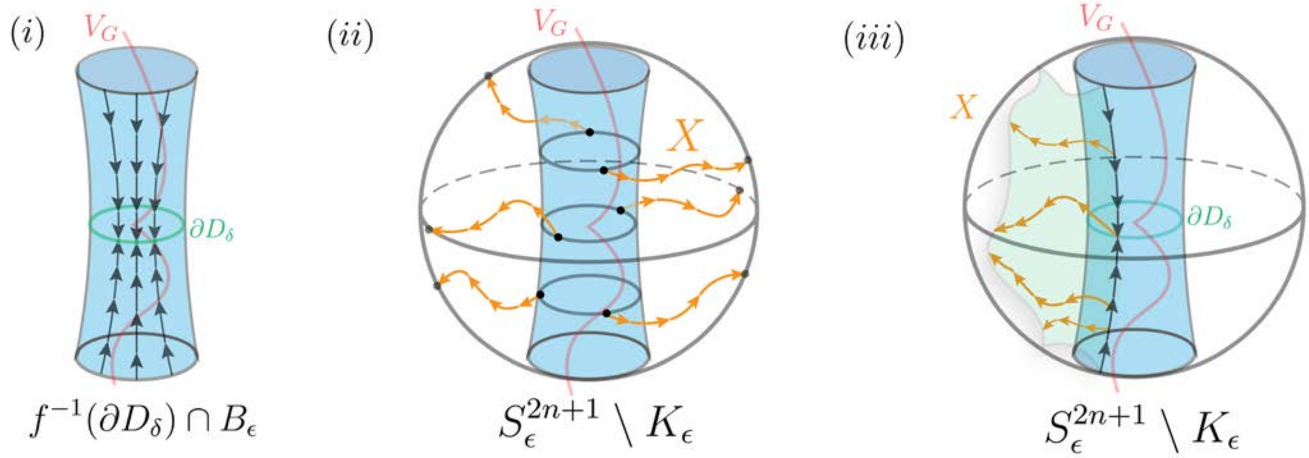

Figure 7.2: **Milnor Fibration** (i) Fibration of Milnor tube over ∂D, (ii) vector field X whose integral curves define a homeomorphism h between the Milnor tube and S_ϵ^{2n+1}, and (iii) the composition of $f| \circ h$ defines a fibration of S_ϵ^{2n+1} over ∂D. The fibre is shown in green.

(i) its integral lines are transversal to all Milnor tubes $f^{-1}(S_\delta)$ *.*

(ii) its integral lines are transversal to all spheres S_ϵ^{2n+1} *centered at* $\tilde{0}$.

(iii) its integral lines travel along paths where $\arg f$ *is constant, i.e. if* $x, y \in B_\epsilon \setminus V$ *lie on the same integral line, then* $f(x)/|f(x)| = f(y)/|f(y)|$.

With such a vector field X, we can define a diffeomorphism between $N(\epsilon, \delta)$ and S_ϵ^{2n+1} as follows: let $\gamma(t)$ be an integral curve of X such that for $z \in N(\epsilon, \delta)$ there exists a t_0 with $\gamma(t_0) = z$ and $\gamma'(t)|_{t_0} = X(z)$. Then since the integral lines of X are transversal to S_ϵ^{2n+1} implies there exists a later t^* such that $\gamma(t^*) = z^*$ for some $z^* \in S_\epsilon^{2n+1}$. So, for each $z \in N(\epsilon, \delta)$ there exists a unique integral curve γ_z of X that passes through a unique $z^* \in S_\epsilon^{2n+1}$. Thus, we may define a diffeomorphism $h : N(\epsilon, \delta) \to S_\epsilon^{2n+1}$ establishes an equivalence relation between each z and z^*. Furthermore, the vector field X on B_ϵ has been chosen such that for each $x \in h^{-1}(z)$, $\arg f(x)$ is constant, thus giving us the final result that $f \circ h^{-1} = f/|f|$. Hence, Theorem 7.1 is shown.

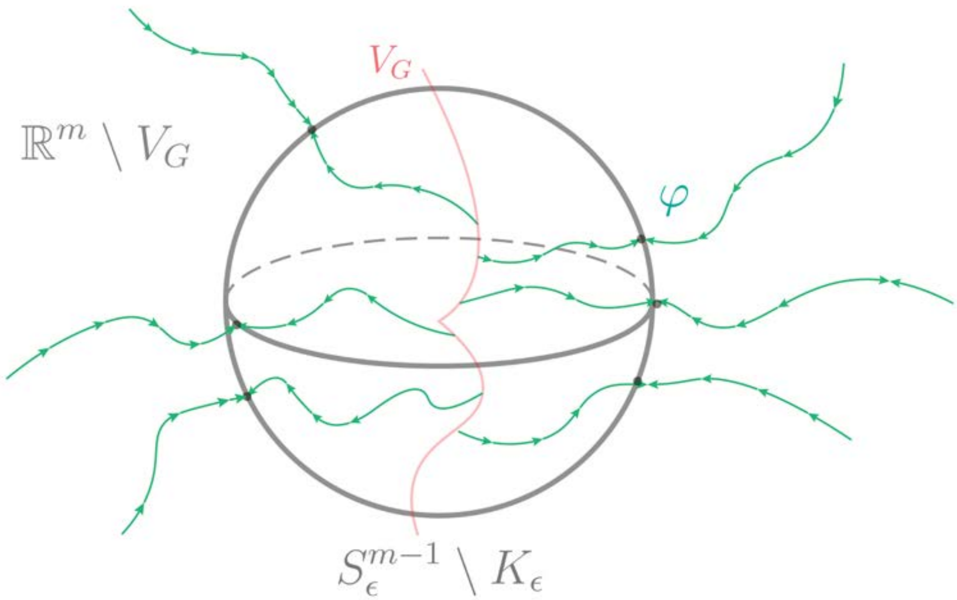

Figure 7.3: **Integral Curves Define a Submersion φ**: Provided the Milnor vector field exists in the real case (i.e. φ is what is called ρ-regular: see [11]), The Milnor vector field X which Milnor utilized in the proof of his fibration theorems can also be used to define a submersion $\varphi : U \subseteq \mathbb{R}^m \setminus V_G \to S_\epsilon^{m-1} \setminus K_\epsilon$.

$$\begin{array}{ccc} N(\epsilon,\delta) & \xrightarrow{f|_{N(\epsilon,\delta)}} & \partial D_\delta \cong S^1 \\ {\scriptstyle h}\downarrow & \nearrow_{f|\circ h^{-1}} & \\ S_\epsilon^{2n+1} \setminus K_\epsilon & & \end{array}$$

Remark 7.2.2. Since integral curves of the vector field X are transversal to all Milnor tubes $f^{-1}(\partial D_\delta) \cap B_\epsilon$ implies we can define a map $\varphi : B_\epsilon \setminus K_\epsilon \to S_\epsilon^{2n+1}$. Locally, The fibres of φ coincide with the integral curves of the Milnor vector field X, which allows us to assert (insofar as S_ϵ^{2n+1} can be viewed as an embedding) that for each $z \in S_\epsilon^{2n+1} \setminus K_\epsilon \subset \mathbb{C}^{n+1} \setminus V_f$ that $\varphi(z) = z$, or in other words: the map φ preserves S_ϵ^{2n+1}.

7.3 Simple Ł-maps and HWC maps

The first hint that harmonic morphisms and Milnor fibrations had any intersection was from the interesting similarities between simple Ł maps and horizontally weakly conformal maps.

In [7] it is shown that the existence of Milnor fibrations inside a ball of small enough radius can be guaranteed when a map is Ł-analytic. Ł-analytic maps are maps which satisfy the strong Łojasiewics inequality.

Definition 7.1. (Strong Lojasiewicz inequality) Let $F = (f_1, \ldots, f_m) : \mathcal{U} \subseteq \mathbb{R}^n \to \mathbb{R}^m$ where each $f_i : \mathcal{U} \to \mathbb{R}$ is a real analytic function. We say F is Ł-analytic at 0 with $F(0) = 0$ if and only if there exists an open neighborhood $\mathcal{W}$ of 0 in $\mathcal{U}$ and there exists $c, \theta \in \mathbb{R}$ such that $c > 0$, $0 < \theta < 1$, and, for all x $\in \mathcal{W}$

$$|F(x)|^{\theta} \leq c \cdot \min_{|(a_1,\ldots,a_m)|=1} \left| \sum_{i=1}^{m} a_i \nabla f_i(x) \right| \tag{7.3.1}$$

A special case of such a map is a so-called simple Ł map.

Definition 7.2. A simple Ł map is a map $F = (f_1, \ldots, f_m)$ whose component functions have the property that their gradients are mutually orthogonal and of equal length at each point p (i.e. $|\nabla f_i|^2 = |\nabla f_j|^2$ for $i, j \leq m, i \neq j$ and $\nabla f_i \cdot \nabla f_j = 0$). In the case that F is a simple Ł map, the strong Lojasiewicz inequality simplifies:

$$|F(x)|^{\theta} \leq c\, |\nabla f_1(x)| \tag{7.3.2}$$

Remark 7.3.1. Notice that the definition of a simple Ł map directly corresponds to the characterization of HWC maps between Euclidean spaces (see Theorem 2.1).

The next theorem attributed to Massey in [7] connects Ł anayltic maps to Milnor fibrations:

Theorem 7.4. *Let a map F be Ł analytic at* 0*. Then Milnor fibrations (from sphere to sphere or inside a ball) for F centered at* 0 *exist.*

Hence, given the commonality between simple Ł maps and HWC maps, it stands to reason that we might substitute harmonic morphisms for simple Ł maps while maintaining the guarantee granted by Theorem 7.4 of the existence of Milnor fibrations.

7.4 Milnor Fibrations and Homogeneous Polynomial Harmonic Morphisms

The following is already known vis-a-vis the relation between harmonic morphism and Milnor fibrations [9]:

Let $G : \mathbb{R}^m \to \mathbb{R}^n$ be a polynomial map which takes the origin to the origin and where the origin is an isolated critical point. Let S_ϵ^{m-1} be a sphere about the origin with radius ϵ, where ϵ is chosen such that the origin is the only critical point contained within the sphere. Milnor has shown how the complement $S_\epsilon^{m-1} \setminus G^{-1}(0)$ fibres over the $(n-1)$-sphere S^{n-1}.

For a *homogeneous* polynomial map G, the Milnor fibration is given as

$$x \mapsto \frac{G(x)}{|G(x)|} \tag{7.4.1}$$

Theorem 7.5. *Let $G : \mathbb{R}^m \to \mathbb{R}^n$ be a harmonic morphism defined by homogeneous polynomials of the same degree p. Then G retracts to a full submersive harmonic morphism (The Milnor Fibration)*

$$\psi| : S^{m-1} \setminus K_\epsilon \to S^{n-1} \tag{7.4.2}$$

whose component functions are the restrictions of irrational functions homogeneous of degree 0.

7.5 Main Result

Recall that our initial aim was to relax the conditions placed the harmonic morphism G in Theorem 7.5. There it is shown that if G is harmonic morphism defined by homogeneous polynomials of the same degree p, then G retracts to a submersive harmonic morphism on $S_\epsilon^{m-1} \setminus K_\epsilon$ that defines a Milnor fibration map to S^{n-1}.

Now, in Remark 7.2.2, it was stated that a map $\varphi : \mathbb{C}^{n+1} \setminus V_G \to S_\epsilon^{2n+1}$ could be defined whose fibres are at least locally the integral curves of a Milnor vector field $X \in \Gamma(\mathbb{C}^{n+1} \setminus V_G)$ with the properties outlined in Lemma 7.3. This construction can be extended to a real analytic φ, provided φ is ρ-regular which guarantees the existence of a Milnor vector field (see [11]). We then, as stated in Remark 7.2.2, require that for each $z \in S_\epsilon^{2n+1}$ then $\varphi(z) = z$. So, φ preserves the sphere S_ϵ^{2n+1}.

Using the fact that the submersion φ needs to preserve the sphere, we prove the following proposition:

Proposition 7.5.1. Let $\varphi : \mathbb{R}^m \setminus V_G \to S^{m-1}$ be harmonic morphism. Then if for each $z \in S_\epsilon^{m-1} \subset \mathbb{R}^m \setminus \{0\}$ we have $\varphi(z) = z$, then φ is radial projection.

Proof. Notice, since φ preserves S_ϵ^{m-1}, that the dilation is constant on the sphere. Now, observe that the fibre of $\varphi|_{S_\epsilon^{m-1}}$ is a single point and hence S_ϵ^{m-1} must correspond to a horizontal submanifold of the horizontal distribution $\mathcal{H}$ induced by φ since the tangent space of a point is $\{0\}$. Hence, we know φ is horizontally homothetic on S_ϵ^{m-1}. It follows that φ locally may either be of Killing type or warped-product type. From Theorem 6.5.1, since φ is locally of either Killing or warped product type, means it is globally either Killing or warped product type.

There are now two distinct possibilities which follow directly from Theorem 6.1. 1) λ is constant on all of $\mathbb{R}^m \setminus V_G$. Then φ is Killing and furthermore is a Riemannian submersion and thus is, up to homothety, orthogonal projection. 2) The second option is λ is non-constant on $\mathbb{R}^m \setminus V_G$. Then since S_ϵ^{m-1} is a level-set of λ and $\vec{\nabla}\lambda$ is non-zero on the sphere shows by Proposition 6.3.3 that φ is of warped product type, meaning ultimately by Proposition 6.3.4 that φ is radial projection since the horizontal submanifolds must constitute a parallel family of umbilic hypersurfaces. So either φ is orthogonal projection or radial projection. But note that there is no way to define orthogonal projection from $\mathbb{R}^m \setminus V_G$ to a sphere. Hence φ must be radial projection. □

Proof. (Alternate) The above result can be seen even more immediately once we identify φ is horizontally homothetic. From Theorem 6.3, we can say φ is either Killing or warped product type as a map from a space form. In either case, φ is horizontally homothetic. Then Theorem 6.4 shows φ must be radial projection. □

Remark 7.5.1. Since φ must be radial projection shows each consecutive sphere constitutes a submanifold of constant dilation equal to the reciprocal of the chosen radius. Since the dilation changes as we move from one sphere to another (in other words, as we move along the fibres of φ) shows clearly φ is of warped product type. The fibres of φ are straight lines (minimal) in $\mathbb{R}^m \setminus \{0\}$ projecting radially from the origin. The horizontal submanifolds of the horizontal distribution $\mathcal{H}$ induced by φ are a set of nested spheres, which constitute a isoparametric parallel family of umbilic hypersurfaces in $\mathbb{R}^n \setminus \{0\}$ (see Proposition 6.3.4). All of the above clearly demonstrates that φ is radial projection.

7.6 Implications for Harmonics Morphisms and Milnor Fibrations

Returning to the problem outlined in § 7.1. In order for G to retract to a harmonic morphism Milnor fibration map, we need φ to preserve the sphere $S_\epsilon^{m-1} \setminus K_\epsilon$. Since we require φ to be a harmonic morphism as well, means by Proposition 7.5.1 that φ must be radial projection. This means φ has totally geodesic fibres and hence maps radial lines in $\mathbb{R}^m \setminus V_f$ to points in $S_\epsilon^{m-1} \setminus K_\epsilon$. By assumption π is also radial projection, meaning it too maps radial lines in $\mathbb{R}^n \setminus \{0\}$ to points in S^n. So, in order for the diagram in § 7.1 to commute, we must have that the map G maps radial lines in $\mathbb{R}^m \setminus V_f$ to radial lines in $\mathbb{R}^n \setminus \{0\}$, meaning that G must be homogeneous. Thus, the class of harmonic morphisms that retract to a harmonic morphism Milnor fibration maps is strictly limited to homogeneous maps.

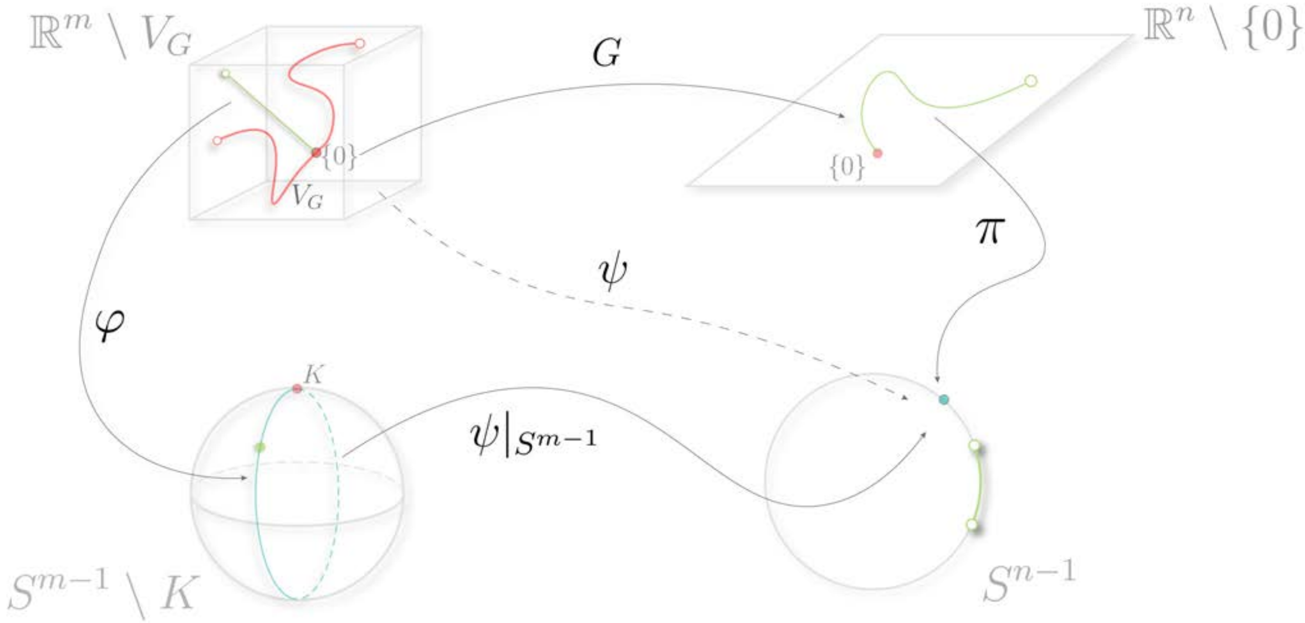

Figure 7.4: **What if G is not homogeneous?**: ($\rightarrow\downarrow$) Following the above diagram from the upper left, since G is assumed to be non-homogeneous, G maps at least one radial line (green) from $\mathbb{R}^m \setminus V_G$ to a non-radial curve in $\mathbb{R}^n \setminus \{0\}$ (green). Since π is radial projection, it maps a non-radial line in $\mathbb{R}^n \setminus \{0\}$ to some subset of S^{n-1}. ($\downarrow\rightarrow$) Since φ *must* be radial projection as well, means the radial line (green) is mapped to a point in a submanifold of $S^{m-1} \setminus K$ (blue) which is then mapped by the restriction of ψ to a single point (blue) in S^{n-1}. This shows a breakdown in the commutativity of the diagram outlined in §7.1

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