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Elliptic functions and iterative algorithms for π

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THESIS/DISSERTATION CERTIFICATE OF APPROVAL

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Contents

Abstract

Preliminary identities in the theory of basic hypergeometric series, or 'q-series', are proven. These include q-analogues of the exponential function, which lead to a fairly simple proof of Jacobi's celebrated triple product identity due to Andrews. The Dedekind η function is introduced and a few identities of it derived. Euler's pentagonal number theorem is shown as a special case of Ramanujan's theta function and Watson's quintuple product identity is proved in a manner given by Carlitz and Subbarao. The Jacobian theta functions are introduced as special kinds of basic hypergeometric series and various relations between them derived using the triple product identity, among other previously established results. A special quotient of theta functions is introduced as the modular λ function. The Eisenstein series are first defined through their Lambert series expansions and a series of differential equations due to Ramanujan are developed. Modular forms and functions and subsequently elliptic functions are introduced. The Weierstrass $\wp$ function is developed along other elliptic functions, those being defined as certain quotients of theta functions. The first few Eisenstein series are then shown to be expressible in terms of theta functions. Theta functions are shown to be related to Gauss' hypergeometric series ${}_2F_1(a, b; c; z)$ through the Jacobi inversion theorem. This is shown to have use in relating modular equations and hypergeometric series to π. The arithmetic-geometric mean iteration of Gauss is developed and used in conjunction with other results established in proofs of two iterative algorithms for π. Recent applications of π algorithms using and not using the techniques developed here are then discussed.

0 Introduction

The fascination with calculating $\pi = 3.14...$ to greater and greater accuracy using whatever mathematical tools available has a history going back to antiquity[5], with Archimedes' estimation $223/71 < \pi < 22/7$. The ancient methods were geometric in character and rather slow, and until the advent of calculus only a few dozen digits of π were known. With calculus and in particular the series expansion of the arctangent function, the known digits of π surpassed 500; with the aid of computers in the 20th century, this method using arctan gave π to over one million digits.

Soon after that million-digit mark however, the calculation methods of choice shifted to the application of 19th century theories regarding elliptic integrals and their inverses: elliptic functions. Related to elliptic functions are Jacobi's theta functions, and in the manipulation of these functions lies a large part of Ramanujan's work. In [20], Ramanujan investigates properties of almost-integers such as $e^{\pi\sqrt{58}} = 24591257751.99999982...$ and, more importantly for the purpose of calculating π, several rapidly convergent infinite series for $1/\pi$. The investigation of such series later in the 20th century, along with the rise in computing power and the reduction in computational complexity by use of the fast Fourier transform, led to several other rapid convergent series for π which are the main tool today for calculating π to extraordinary precision.

Tangentially to these infinite series for π, another 19th century tool related to elliptic integrals, the arithmetic-geometric mean, was appropriated by π computers in the 1970s ([7],[21]) to derive some of the most rapidly convergent iterative algorithms known today. It is along this line of thought we seek to compute π. Rather than use the theory of elliptic integrals to derive these algorithms, which is the usual way (for example, in [4]), we use the related theory of theta functions. This way is done for example in [11] and extends itself more easily to possible q-analogues of π algorithms.

1 Infinite Products

A q-analog of a theorem is a similar statement given in terms of a complex variable $|q| < 1$, such that a limiting case as $q \to 1$ reduces to a familiar statement. One example of a q-analog is the q-binomial theorem, a generalization of the more famous binomial theorem. Before proving that theorem some notation is introduced.

Definition 1.1 (q-Pochhammer Symbol): *Define $(a;q)_0 = 1$ and for any integer $n > 0$, let*

$$(a;q)_n = (1-a)(1-aq)(1-aq^2)\cdots(1-aq^{n-1}). \tag{1}$$

By L'Hôpital's rule,

$$\begin{aligned}
\lim_{q\to 1}\frac{(q^a;q)_n}{(1-q)^n} &= \lim_{q\to 1}\frac{1-q^a}{1-q}\cdot\frac{1-q^{a+1}}{1-q}\cdots\frac{1-q^{a+n-1}}{1-q} \\
&= \lim_{q\to 1}\frac{\frac{d}{dq}(1-q^a)}{\frac{d}{dq}(1-q)}\cdot\frac{\frac{d}{dq}(1-q^{a+1})}{\frac{d}{dq}(1-q)}\cdots\frac{\frac{d}{dq}(1-q^{a+n-1})}{\frac{d}{dq}(1-q)} \\
&= \lim_{q\to 1}\frac{-aq^{a-1}}{-1}\cdot\frac{-(a+1)q^a}{-1}\cdots\frac{-(a+n-1)q^{a+n}}{-1} \\
&= a(a+1)\cdots(a+n-1)
\end{aligned} \tag{2}$$

so that the q-Pochhammer symbols are indeed q-analogs of the ordinary Pochhammer symbols, defined as

$$(a)_n := a(a+1)\cdots(a+n-1). \tag{3}$$

One can extend the q-Pochhammer symbols to an infinite product by defining

$$(a;q)_\infty = \prod_{n\geq 0}(1-aq^n). \tag{4}$$

Now, it is known from analysis[19] that an infinite product $\prod_{n\geq 0}(1+a_n)$ converges and is not zero if and only if the sum $\sum_{n\geq 0}\log(1+a_n)$ converges and is not 0. A stronger result states that the infinite product $\prod_{n\geq 0}(1+a_n)$ is absolutely convergent if and only if $\sum_{n\geq 0}a_n$ is absolutely convergent. Since in the case of the infinite q-Pochhammer symbols $|q|<1$, by the ratio test we have $\lim_{n\to\infty}|(-aq^{n+1})/(-aq^n)| = |q| < 1$, and so the series and thus the product converges absolutely. The special case $(q;q)_\infty$ will sometimes be denoted $E(q)$, and

more generally let

$$E(q^k) := (q^k; q^k)_\infty. \tag{5}$$

For brevity, let

$$(a_1; q)_k (a_2; q)_k \cdots (a_n; q)_k := (a_1, a_2, ..., a_n; q)_k \tag{6}$$

and more generally

$$\begin{pmatrix} a_1, ..., a_r \\ & ; q \\ b_1, ..., b_s \end{pmatrix}_k := (a_1, ..., a_r; b_1, ..., b_s; q)_k := \frac{(a_1, ..., a_r; q)_k}{(b_1, ..., b_s; q)_k} \tag{7}$$

These notations will apply to the infinite extensions as well. A polynomial in the variable q can be viewed as a generating function of its coefficients. This take had its genesis in Euler's Pentagonal Number Theorem, and later found great use during the 19th and 20th centuries in combinatorial arguments relating to partitions[1]. More recently, q-analogs have become popular as one can develop a calculus without limits through them, and this has found some use in quantum mechanics[17].

Definition 1.2 (q-Hypergeometric Series): *For $z \neq 0$, define*[17]

$${}_r\phi_s \begin{bmatrix} a_1, ..., a_r \\ & ; q; z \\ b_1, ..., b_s \end{bmatrix} := \sum_{n \geq 0} \begin{pmatrix} a_1, ..., a_r \\ & ; q \\ q, b_1, ..., b_s \end{pmatrix}_n \left[(-1)^n q^{\binom{n}{2}} \right]^{1-r+s} z^n. \tag{8}$$

The name 'q-hypergeometric series' will henceforth be shortened to 'q-series'. The convergence of such an infinite series can be seen intuitively from that of the infinite q-Pochhammer symbols above where for $|q| < 1$ and large n, the q-Pochhammer terms approach multiplication by 1 and $q^{\binom{n}{2}}$ goes to 0 faster than z^n grows. Taking

the limit of ${}_2\phi_1[q^a, q^b; q^c; q; z]$ as $q \to 1$ results in

$$\lim_{q\to 1} {}_2\phi_1[q^a, q^b; q^c; q; z] = \lim_{q\to 1} \sum_{n\geq 0} \begin{pmatrix} q^a, q^b \\ & ; q \\ q, q^c \end{pmatrix}_n z^n = \sum_{n\geq 0} \lim_{q\to 1} \begin{pmatrix} q^a, q^b \\ & ; q \\ q, q^c \end{pmatrix}_n z^n$$

$$= \sum_{n\geq 0} \lim_{q\to 1} \begin{pmatrix} q^a, q^b \\ & ; q \\ q, q^c \end{pmatrix}_n \frac{(1-q)^{2n}}{(1-q)^{2n}} z^n = \sum_{n\geq 0} \frac{(a)_n (b)_n}{(1)_n (c)_n} z^n \tag{9}$$

$$= \sum_{n\geq 0} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}$$

which is the standard hypergeometric series ${}_2F_1(a, b; c; z)$.

Theorem 1.3 (q-Binomial Theorem): *With the notation above,*

$$\begin{pmatrix} at \\ & ; q \\ t \end{pmatrix}_\infty = {}_1\phi_0[a; -; q; t] \tag{10}$$

where '-' in a q-series denotes an absence of such terms.

Proof: First note that with $|q| < 1$ both sides do converge, so this expression makes sense in that respect. Now[18],

$$(1-t) \begin{pmatrix} at \\ & ; q \\ t \end{pmatrix}_\infty = (1-at) \begin{pmatrix} aqt \\ & ; q \\ qt \end{pmatrix}_\infty \tag{11}$$

since

$$\begin{pmatrix} at \\ & ; q \\ t \end{pmatrix}_\infty = \frac{(1-at)}{(1-t)} \begin{pmatrix} aqt \\ & ; q \\ qt \end{pmatrix}_\infty . \tag{12}$$

Then if

$$\begin{pmatrix} at \\ & ; q \\ t \end{pmatrix}_\infty = \sum_{k\geq 0} c_k t^k, \tag{13}$$

this implies

$$(1-t)\sum_{k\geq 0} c_k t^k = (1-at)\sum_{k\geq 0} c_k q^k t^k. \tag{14}$$

Comparing the coefficients of t^k results in

$$c_k - c_{k-1} = c_k q^k - a c_{k-1} q^{k-1}, \tag{15}$$

or by solving for c_k,

$$c_k = \frac{(1-aq^{k-1})}{(1-q^k)} c_{k-1}. \tag{16}$$

A quick inspection of $(at;t;q)_\infty$ shows $c_0 = 1$, and the result follows by induction. □

To justify the name of the theorem, let $a = q^a$. Then as $q \to 1$, the q-binomial theorem implies

$$\lim_{q\to 1} {}_1\phi_0[q^a;-;q;z] = \sum_{n\geq 0} \frac{(a)_n}{n!} z^n = \lim_{q\to 1}(zq^a;z;q)_\infty = \prod_{n=0}^{a-1} \frac{1}{1-z} = (1-z)^{-a}. \tag{17}$$

The expression

$$\sum_{n\geq 0} \frac{(a)_n}{n!} z^n = (1-z)^{-a} \tag{18}$$

is a form of the generalized binomial theorem, so the q-binomial theorem is a q-analog of it.

Corollary 1.4:

$$(-a;q)_\infty = \sum_{n\geq 0} \frac{a^n q^{\binom{n}{2}}}{(q;q)_n}. \tag{19}$$

Proof: Replace a by $-a/t$ in the q-binomial theorem to obtain

$$\left(\begin{matrix} -a \\ t \end{matrix};q\right)_\infty = \sum_{n\geq 0} \left(\begin{matrix} -a/t \\ q \end{matrix};q\right)_n t^n. \tag{20}$$

Note that[18]

$$t^n(1+a/t)(1+qa/t)\cdots(1+q^{n-1}a/t) = (t+a)(t+aq)\cdots(t+aq^{n-1}). \tag{21}$$

Letting $t \to 0$ then gives

$$(-a;q)_\infty = \sum_{n\geq 0} \frac{a(aq)(aq^2)\cdots(aq^{n-1})}{(q;q)_n} = \sum_{n\geq 0} \frac{a^n q^{1+2+\cdots+(n-1)}}{(q;q)_n} \tag{22}$$

and the result follows. $\square$

Corollary 1.5:

$$\frac{1}{(a;q)_\infty} = \sum_{n\geq 0} \frac{a^n}{(q;q)_n}. \tag{23}$$

Proof: Take $a = 0$ and $t = a$ in the q-binomial theorem so that

$$\left(\begin{matrix} 0 \\ & ;q \\ a \end{matrix}\right)_\infty = {}_1\phi_0[0;-;q;a] = \sum_{n\geq 0} \frac{a^n}{(q;q)_n}. \ \square \tag{24}$$

The corollaries above were known to Euler, and can be used to give a proof of a product-to-sum identity due to Jacobi. Upon taking the limit $q \to 1$, the above corollaries turn into the infinite series expansion for e^x; thus they can be seen as q-analogues of the exponential function[17].

Theorem 1.6 (Jacobi Triple Product): *For $x \neq 0$ and $|q| < 1$,*

$$\sum_{n=-\infty}^{\infty} q^{n^2} x^n = (-xq, -q/x, q^2; q^2)_\infty. \tag{25}$$

Proof: As in Andrews[1], we first prove the theorem for when $|q| < |x|$.

$$\begin{aligned}(-xq;q^2)_\infty &= \sum_{n\geq 0}\frac{(xq)^n(q^2)^{\binom{n}{2}}}{(q^2;q^2)_n} = \sum_{n\geq 0}\frac{x^n q^{n^2}}{(q^2;q^2)_n} = \sum_{n\geq 0} q^{n^2}x^n(q^2;q^2)_n^{-1} \\ &= \sum_{n\geq 0} q^{n^2}x^n(q^{2(n+1)};q^2;q^2)_\infty = (q^2;q^2)_\infty^{-1}\sum_{n\geq 0} q^{n^2}x^n(q^{2(n+1)};q^2)_\infty.\end{aligned} \tag{26}$$

Notice that if $n < 0$, the whole term x^n goes to zero because then the $(-n-1)$-th term of the product $(q^{2(n+1)};q^2)_\infty$ is

$$(1-q^{2n+2+2(-n-1)}) = (1-q^0) = 0. \tag{27}$$

So continuing on,

$$\begin{aligned}(-xq;q^2)_\infty &= (q^2;q^2)_\infty^{-1}\sum_{n=-\infty}^{\infty} q^{n^2}x^n(q^{2(n+1)};q^2)_\infty \\ &= (q^2;q^2)_\infty^{-1}\sum_{n=-\infty}^{\infty} q^{n^2}x^n\sum_{m\geq 0}\frac{(-1)^m q^{m^2+2mn+m}}{(q^2;q^2)_m} \\ &= (q^2;q^2)_\infty^{-1}\sum_{m\geq 0}\frac{(-1)^m q^m}{(q^2;q^2)_m}\sum_{n=-\infty}^{\infty} q^{(n+m)^2}x^n\cdot(x^m x^{-m}) \\ &= (q^2;q^2)_\infty^{-1}\sum_{m\geq 0}\frac{(-qx^{-1})^m}{(q^2;q^2)_m}\sum_{n=-\infty}^{\infty} q^{(n+m)^2}x^{n+m} \\ &= (q^2;q^2)_\infty^{-1}(-q/x;q^2)_\infty^{-1}\sum_{n=-\infty}^{\infty} q^{(n+m)^2}x^{n+m}.\end{aligned} \tag{28}$$

The theorem follows upon replacing $n+m$ with n and rearranging. Upon replacing x with x^{-1}, the result stays the same due to $q^{n^2}x^n = q^{n^2}x^{-n}$ in the sum and the symmetry between the $-xq$ and $-q/x$ terms in the product. However, the restriction becomes $|q| < |x^{-1}|$ in this case. Since $|q| < 1$, either $|q| < |x|$ or $|q| < |x^{-1}|$ so we can take simply $x \neq 0$. $\square$

The Jacobi triple product can give useful corollaries upon changing the values of q

and x. For example, setting q to $q^{1/2}$ and x to $-xq^{-1/2}$ results in

$$\sum_{n=-\infty}^{\infty} (q^{1/2})^{n^2}(-xq^{-1/2})^n = (-(-xq^{-1/2})q^{1/2}, -q^{1/2}/(-xq^{-1/2}), q; q)_\infty, \tag{29}$$

which simplifies to

Corollary 1.7:

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{\binom{n}{2}} x^n = (x, q/x, q; q)_\infty. \tag{30}$$

The product form of this identity is simpler than that of Theorem 1.6, and the symmetry between x and q/x a bit more obvious. We now give a couple of definitions to set some more notation.

Definition 1.8 (Dedekind η Function): *Let $q^* := e^{2\pi i\tau}$, where τ is a complex number with positive imaginary part so that $|q^*| < 1$. The Dedekind eta function $\eta(\tau)$ is defined as:*

$$\eta(\tau) = (q^*)^{1/24}E(q^*) = (q^*)^{1/24}(q^*; q^*)_\infty. \tag{31}$$

When there is no confusion, q^* will be denoted q. Observe that $\eta(\tau)$ is defined in terms of τ while $E(q)$ is in terms of q, so that

$$\eta(k\tau) = q^{k/24}E(q^k). \tag{32}$$

Since $\eta(\tau)$ and $E(q)$ are so closely related, proving an identity for either one essentially proves it for both. Thus in what follows results proven in terms of $E(q^n)$ will not separately be proven in terms of $\eta(n\tau)$, and vice versa.

Proposition 1.9[15]:

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} = \frac{\eta^2(\tau)}{\eta(2\tau)}. \tag{33}$$

Proof: By the triple product with $x=-1$,

$$\begin{aligned}\sum_{n=-\infty}^{\infty}(-1)^n q^{n^2} &= (q,q,q^2;q^2)_\infty = (q;q)_\infty(q;q^2)_\infty \\ &= \frac{(q;q)_\infty^2}{(q^2;q^2)_\infty} = \frac{E^2(q)}{E(q^2)}\cdot\frac{q^{1/12}}{q^{1/12}} = \frac{\eta^2(\tau)}{\eta(2\tau)}. \ \square\end{aligned} \tag{34}$$

Definition 1.10 (Ramanujan Theta Function): *Let* $|ab|\neq 0$*. Define*[3]

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{\binom{n+1}{2}} b^{\binom{n}{2}}. \tag{35}$$

The explanation for the name will come later, but observe that many infinite series can be represented in this form. Three in particular were important enough for Ramanujan to distinguish:

$$\phi(q) := f(q,q) = \sum_{n=-\infty}^{\infty} q^{\binom{n+1}{2}} q^{\binom{n}{2}} = \sum_{n=-\infty}^{\infty} q^{n^2} \tag{36}$$

$$\psi(q) := f(q,q^3) = \sum_{n=-\infty}^{\infty} q^{n(2n-1)} = \sum_{n\geq 0} q^{\binom{n+1}{2}} \tag{37}$$

$$f(-q) := f(-q,-q^2) = \sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2} \tag{38}$$

The triple product identity can be used to give $f(a,b)$ as an infinite product. Let $a=x$ and $b=q/x$. Then $q=ab$ and Corollary 1.7 states

$$f(a,b) = (-a,-b,ab;ab)_\infty \tag{39}$$

This implies the functions $\phi(q)$, $\psi(q)$ and $f(-q)$ defined above each have infinite product representations. They are:

$$\phi(q) = (-q, -q, q^2; q^2)_\infty = (-q; q^2)_\infty^2 (q^2; q^2)_\infty \tag{40}$$

$$\psi(q) = (-q, -q^3, q^4; q^4)_\infty \tag{41}$$

$$f(-q) = (q, q^2, q^3; q^3)_\infty = (q; q)_\infty \tag{42}$$

This last equation is Euler's pentagonal number theorem, restated here:

Theorem 1.11 (Pentagonal Number Theorem):

$$E(q) = (q; q)_\infty = \prod_{n=1}^{\infty}(1 - q^n) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n-1)/2} = f(-q). \tag{43}$$

The products of $\phi(q)$ and $\psi(q)$ can be simplified further through elementary manipulations to quotients of $E(q)$ and $\eta(\tau)$.

Proposition 1.12:

$$\phi(q) = \frac{\eta^5(2\tau)}{\eta^2(\tau)\eta^2(4\tau)}. \tag{44}$$

Proof:

$$\begin{aligned} \phi(q) &= (-q; q^2)_\infty^2 (q^2; q^2)_\infty = \frac{(q^2; q^4)_\infty^2}{(q; q^2)_\infty^2}(q^2; q^2)_\infty = \frac{(q^2; q^4)_\infty^2 (q^2; q^2)_\infty^3}{(q; q)_\infty^2} \\ &= \frac{(q^2; q^2)_\infty^5}{(q; q)_\infty^2 (q^4; q^4)_\infty^2} = \frac{E^5(q^2)}{E^2(q)E^2(q^4)} \cdot \frac{q^{5/12}}{q^{5/12}} = \frac{\eta^5(2\tau)}{\eta^2(\tau)\eta^2(4\tau)}. \ \square \end{aligned} \tag{45}$$

If the relation $(q^2; q^4)_\infty = E(q^2)/E(q^4)$ is not obvious, observe that the powers of q in $E(q^2)$ are the even numbers, which are simply all the numbers $n \equiv 0, 2 \mod 4$.

Proposition 1.13:

$$\psi(q) = \frac{E^2(q^2)}{E(q)}. \tag{46}$$

Proof:

$$\begin{aligned}
\psi(q) &= (-q,-q^3,q^4;q^4)_\infty = (-q,-q^3;q^4)_\infty(-q^2,q^2;q^2)_\infty \\
&= (-q,-q^2,q^2;q^2)_\infty = (-q;q)_\infty(q^2;q^2)_\infty = \frac{(q^2;q^2)_\infty}{(q;q)_\infty} \qquad (47)\\
&= \frac{E^2(q^2)}{E(q)}. \ \square
\end{aligned}$$

Similar to the Jacobi triple product, there is another useful product-to-sum formula called the quintuple product identity.

Theorem 1.14 (Quintuple Product Identity): *For $x \neq 0$,*

$$\sum_{n=-\infty}^{\infty} q^{n(3n+1)/2}(x^{3n} - x^{-3n-1}) = (q, xq, 1/x; q)_\infty(x^2q, q/x^2; q^2)_\infty. \qquad (48)$$

Proof: Following Carlitz and Subbarao[9], first note that by the triple product identity with $x = -x$ gives

$$\sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} x^n = (xq, q/x, q^2; q^2)_\infty. \qquad (49)$$

Now taking $q = q^2$, the statement of the quintuple product identity becomes

$$\sum_{n=-\infty}^{\infty} q^{n(3n+1)}(x^{3n} - x^{-3n-1}) = (q^2, xq^2, 1/x; q^2)_\infty(x^2q^2, q^2/x^2; q^4)_\infty. \qquad (50)$$

Let $A(q,x)$ denote the product

$$A(q,x) = (q^2, xq^2, 1/x; q^2)_\infty(x^2q^2, q^2/x^2, q^4; q^4)_\infty. \qquad (51)$$

By the triple product applied twice this becomes

$$\begin{aligned} A(q,x) &= (q^2, xq^2, 1/x; q^2)_\infty (x^2q^2, q^2/x^2, q^4; q^4)_\infty \\ &= \sum_{j=-\infty}^{\infty} (-1)^j q^{j^2+j} x^j \sum_{k=-\infty}^{\infty} (-1)^k q^{2k^2} x^{2k} \\ &= \sum_{j,k=-\infty}^{\infty} (-1)^{j+k} q^{j^2+2k^2+j} x^{j+2k}. \end{aligned} \tag{52}$$

Writing $n = 2k + j$, this becomes

$$\begin{aligned} A(q,x) &= \sum_{n=-\infty}^{\infty} (-1)^n x^n \sum_{k=-\infty}^{\infty} (-1)^{-k} q^{(n-2k)^2+2k^2+(n-2k)} \\ &= \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2+n} x^n \sum_{k=-\infty}^{\infty} (-1)^k q^{6k^2-4kn-2k}. \end{aligned} \tag{53}$$

Now consider the sum

$$\sum_{k=-\infty}^{\infty} (-1)^k q^{6k^2+6k(2p+1)} \tag{54}$$

where p is some integer. Re-indexing k with $-k-2p-1$ gives

$$\begin{aligned} \sum_{k=-\infty}^{\infty} (-1)^k q^{6k^2+6k(2p+1)} &= \sum_{k=-\infty}^{\infty} (-1)^{-k-2p-1} q^{6(-k-2p-1)^2+6(-k-2p-1)(2p+1)} \\ &= \sum_{k=-\infty}^{\infty} (-1)^{k+1} q^{6k^2+12kp+6k} = -\sum_{k=-\infty}^{\infty} (-1)^k q^{6k^2+6k(2p+1)} \end{aligned} \tag{55}$$

so the entire sum is 0 for any p. But this implies that the inner sum of $A(q,x)$, and thus $A(q,x)$, is 0 except when $2n+1 \not\equiv 0 \mod 3$. Since $2(3n+1)+1 = 6n+3 = 3(2n+1)$, while $2(3n)+1 = 6n+1 = 3(2n+1/3)$ and $2(3n+2)+1 = 6n+5 = 3(2n+5/3)$, we may assume $n \equiv 0, 2 \mod 3$. Let $A_0(q,x)$ denote that part of $A(q,x)$ with $n \equiv 0 \mod 3$ and $A_2(q,x)$ denote the part with

$n \equiv 2 \mod 3$. Clearly $A(q,x) = A_0(q,x) + A_2(q,x)$. For $A_0(q,x)$,

$$\begin{aligned} A_0(q,x) &= \sum_{n=-\infty}^{\infty} (-1)^{3n} q^{(3n)^2+3n} x^{3n} \sum_{k=-\infty}^{\infty} (-1)^k q^{6k^2-4k(3n)-2k} \\ &= \sum_{n=-\infty}^{\infty} (-1)^n q^{9n^2+3n} x^{3n} \sum_{k=-\infty}^{\infty} (-1)^k q^{6k^2-12kn-2k}. \end{aligned} \tag{56}$$

Setting $m = k - n$, this becomes

$$\begin{aligned} A_0(q,x) &= \sum_{n=-\infty}^{\infty} (-1)^n q^{9n^2+3n} x^{3n} \sum_{m=-\infty}^{\infty} (-1)^{m+n} q^{6(m+n)^2-12(m+n)n-2(m+n)} \\ &= \sum_{n=-\infty}^{\infty} q^{3n^2+n} x^{3n} \sum_{m=-\infty}^{\infty} (-1)^m q^{6m^2-2m}. \end{aligned} \tag{57}$$

By the pentagonal number theorem, the inner sum is equal to $E(q^4)$ and so

$$A_0(q,x) = E(q^4) \sum_{n=-\infty}^{\infty} q^{n(3n+1)} x^{3n}. \tag{58}$$

Similarly for $A_2(q,x)$,

$$\begin{aligned} A_2(q,x) &= \sum_{n=-\infty}^{\infty} (-1)^{3n-1} q^{(3n-1)^2+(3n-1)} x^{3n-1} \sum_{k=-\infty}^{\infty} (-1)^k q^{6k^2-4k(3n-1)-2k} \\ &= \sum_{n=-\infty}^{\infty} (-1)^{n-1} q^{9n^2-3n} x^{3n-1} \sum_{k=-\infty}^{\infty} (-1)^k q^{6k^2-12kn+2k}. \end{aligned} \tag{59}$$

Setting $m = k - n$ gives

$$\begin{aligned} A_2(q,x) &= \sum_{n=-\infty}^{\infty} (-1)^{n-1} q^{9n^2-3n} x^{3n-1} \sum_{k=-\infty}^{\infty} (-1)^{m+n} q^{6(m+n)^2-12(m+n)n+2(m+n)} \\ &= \sum_{n=-\infty}^{\infty} (-1)^{2n-1} q^{3n^2-n} x^{3n-1} \sum_{k=-\infty}^{\infty} (-1)^m q^{6m^2+2m} \\ &= -\sum_{n=-\infty}^{\infty} q^{3n^2-n} x^{3n-1} \sum_{k=-\infty}^{\infty} (-1)^m q^{6m^2+2m}. \end{aligned} \tag{60}$$

Upon replacing n and m with $-n$ and $-m$ respectively,

$$A_2(q,x) = -\sum_{n=-\infty}^{\infty} q^{3n^2+n}x^{-3n-1}\sum_{k=-\infty}^{\infty}(-1)^m q^{6m^2-2m} = -E(q^4)\sum_{n=-\infty}^{\infty} q^{n(3n+1)}x^{-3n-1} \tag{61}$$

and so

$$A(q,x) = A_0(q,x) + A_2(q,x) = E(q^4)\sum_{n=-\infty}^{\infty} q^{n(3n+1)}(x^{3n} - x^{-3n-1}).\ \square \tag{62}$$

2 Theta Functions

Definition 2.1 (Jacobi Theta Functions): *Let $q^\dagger := \sqrt{q^*} = e^{\pi i\tau}$ with τ a complex number in the upper half-plane so that $|q^\dagger| < 1$. For any $x \in \mathbb{C}$, let*[22]

$$\theta_1(q^\dagger, x) = 2\sum_{n\geq 0}(-1)^n (q^\dagger)^{(n+1/2)^2}\sin(2n+1)x \tag{63}$$

$$\theta_2(q^\dagger, x) = 2\sum_{n\geq 0}(q^\dagger)^{(n+1/2)^2}\cos(2n+1)x \tag{64}$$

$$\theta_3(q^\dagger, x) = 1 + 2\sum_{n\geq 1}(q^\dagger)^{n^2}\cos 2nx \tag{65}$$

$$\theta_4(q^\dagger, x) = 1 + 2\sum_{n\geq 1}(-1)^n(q^\dagger)^{n^2}\cos 2nx \tag{66}$$

The reasons why the Jacobi theta functions are in terms of $q^\dagger = e^{\pi i\tau}$ while the Dedekind eta function is in terms of $q^* = e^{2\pi i\tau}$ are mainly historical. Since they can both be taken as arbitrary complex numbers with positive imaginary part, when there is no confusion $q^\dagger$, like q^*, will be denoted simply as q. Notice that $\theta_1(q,x)$ is an odd function, while the others are all even functions. By Euler's formula ($e^{i\theta} = \cos\theta + i\sin\theta$), the theta functions can be written

$$\theta_1(q,x) = -i\sum_{n=-\infty}^{\infty}(-1)^n q^{(n+1/2)^2}e^{(2n+1)ix} \tag{67}$$

$$\theta_2(q,x) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} e^{(2n+1)ix} \tag{68}$$

$$\theta_3(q,x) = \sum_{n=-\infty}^{\infty} q^{n^2} e^{2nix} \tag{69}$$

$$\theta_4(q,x) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} e^{2nix} \tag{70}$$

When $x = 0$ we write $\theta_i(q,0)$ as $\theta_i(q)$. If the precise value of q is unimportant we write $\theta_i(q,x)$ as $\theta_i(x)$. Now let $z := e^{2ix}$. Then the theta functions can again be rewritten, this time as

$$\theta_1(q,z) = -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+1/2)^2} z^{n+1/2} \tag{71}$$

$$\theta_2(q,z) = \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} z^{n+1/2} \tag{72}$$

$$\theta_3(q,z) = \sum_{n=-\infty}^{\infty} q^{n^2} z^n \tag{73}$$

$$\theta_4(q,z) = \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} z^n \tag{74}$$

The naming of the Ramanujan theta function can now be explained as a generalization of the Jacobi theta functions. A quick check shows that with $q = q^{\dagger}$ and z defined as above, the Jacobi theta functions can be written

$$\theta_1(q,z) = -iq^{1/4} z^{1/2} f(-q^2 z, -1/z) \tag{75}$$

$$\theta_2(q,z) = q^{1/4} z^{1/2} f(q^2 z, 1/z) \tag{76}$$

$$\theta_3(q,z) = f(qz, q/z) \tag{77}$$

$$\theta_4(q,z) = f(-qz, -q/z) \tag{78}$$

By the triple product identity, the theta functions can be presented as infinite products. The cases $\theta_3(q,x)$ and $\theta_4(q,x)$ are clear from their definitions. For $\theta_1(q,x)$

and $\theta_2(q,x)$, one simply observes that

$$(\pm 1/z;q^2)_\infty = (1 \mp 1/z)(\pm q^2/z;q^2)_\infty. \tag{79}$$

The four theta functions of Jacobi then become

$$\theta_1(q,z) = 2q^{1/4}\sin(x)(q^2z, q^2/z, q^2;q^2)_\infty \tag{80}$$

$$\theta_2(q,z) = 2q^{1/4}\cos(x)(-q^2z, -q^2/z, q^2;q^2)_\infty \tag{81}$$

$$\theta_3(q,z) = (-qz, -q/z, q^2;q^2)_\infty \tag{82}$$

$$\theta_4(q,z) = (qz, q/z, q^2;q^2)_\infty \tag{83}$$

Some very nice identities can be proved using these product forms of the theta functions, such as the following:

Proposition 2.2[12]:

$$\eta(2\tau)\frac{\theta_1(q,2x)}{\theta_1(q,x)} = 2\sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2/12}\cos(6n+1)x. \tag{84}$$

Proof: Recalling the basic identity $\sin 2x = 2\sin x\cos x$, observe that

$$\begin{aligned}
\frac{\theta_1(q,2x)}{\theta_1(q,x)} &= \frac{\sin 2x}{\sin x}\cdot(q^2z^2, q^2/z^2; q^2z, q^2/z; q^2)_\infty = 2\cos x\cdot\frac{(qz,-qz,q/z,-q/z;q)_\infty}{(q^2z^2,q^2/z^2;q^2)_\infty} \\
&= 2\cos x\cdot(qz,q/z;q^2)_\infty(-qz,-q/z;q)_\infty \\
&= 2\cos x\cdot(qz,q/z;q^2)_\infty(-qz,-q/z;q)_\infty\cdot\frac{(-qz,-q/z;-qz,-q/z;q^2)_\infty}{(-qz,-q/z;q^2)_\infty} \\
&= 2\cos x\cdot(q^2z^2,q^2/z^2;q^4)_\infty\cdot\frac{(-qz,-q/z;q)_\infty}{(-qz,-q/z;q^2)_\infty} \\
&= 2\cos x\cdot(q^2z^2,q^2/z^2;q^4)_\infty(-q^2z,-q^2/z;q^2)_\infty.
\end{aligned} \tag{85}$$

Then by the quintuple product identity and the definition of $E(q)$ (eq. 5),

$$\begin{aligned}
E(q^2)\frac{\theta_1(q,2x)}{\theta_1(q,x)} &= 2\cos x \cdot (q^2z^2, q^2/z^2; q^4)_\infty(-q^2z, -q^2/z, q^2; q^2)_\infty \\
&= z^{1/2}(q^2z^2, q^2/z^2; q^4)_\infty(-q^2z, -1/z, q^2; q^2)_\infty \\
&= z^{1/2}\sum_{n=-\infty}^{\infty} q^{n(3n+1)}((-z)^{3n} - (-z)^{-3n-1}) \\
&= z^{1/2}\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)}(z^{3n} + z^{-3n-1}).
\end{aligned} \tag{86}$$

Recalling the definition of z,

$$\begin{aligned}
E(q^2)\frac{\theta_1(q,2x)}{\theta_1(q,x)} &= e^{ix}\sum_{n=-\infty}^{\infty} (-1)^n q^{n(3n+1)}(e^{6nix} + e^{-6nix-2ix}) \\
&= \sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2+n}(e^{(6n+1)ix} + e^{-(6n+1)ix}) \\
&= 2\sum_{n=-\infty}^{\infty} (-1)^n q^{3n^2+n}\cos(6n+1)x.
\end{aligned} \tag{87}$$

Multiplying both sides by $q^{1/12}$ finishes the proof. $\square$

Now assume τ, and thus q, is fixed so that the theta functions depend only on x. As functions of x, we would like to take their derivative with respect to x; the case $\theta_1'(x)$ will be particularly important. There are two ways two view $\theta_1(x)$ of course, as either an infinite product or an infinite series. Thus we will need to consider if and when differentiating an infinite sum or product makes sense.

It is a known result in analysis that a series of functions $\sum_n f_n$ can be differentiated term-wise, so $(\sum_n f_n)' = \sum_n f_n'$, only when $\sum_n f_n'$ converges uniformly. It is also known[19] that an infinite product of functions $\prod_n f_n$ can be differentiated precisely when both the product and its logarithmic derivative $\sum_n f_n'/f_n$ converge locally

uniformly. To recall, a sequence $\{a_n\}$ converges *locally uniformly* in a metric space X if for any $x \in X$, there exists a non-empty open ball $B \subset X$ such that the sequence converges uniformly in B[16]. Clearly, uniform convergence implies locally uniform convergence. Now, since

$$\lim_{n\to\infty}\left|\frac{f_n'}{f_n}\right| = \lim_{n\to\infty}\left|\frac{q^n}{(1-aq^n)}\right| = 0 \tag{88}$$

when $|q| < 1$, the sum $\sum_n f_n'/f_n$ converges locally uniformly in $B(0,1) \subset \mathbb{C}$ with $f_n = (1-aq^n)$, and so the infinite product $\prod_{n\geq 0}(1-aq^n)$ is differentiable with respect to a. Since the theta functions, and in particular $\theta_1(x)$, are a product of convergent infinite products, the theta functions are also differentiable with respect to x. For the derivative of $\theta_1(x)$ as a series to exist, we need $\sum_n f_n'$ to converge uniformly. But

$$f_n' = 2(-1)^n q^{(n+1/2)^2}(2n+1)\cos(2n+1)x \tag{89}$$

and, upon observing $|\cos(2n+1)x| \leq 1$, this series converges uniformly when $|q| < 1$ and so its derivative exists. Looking at the derivative of $\theta_1(x)$ from both views, using either the product rule or linearity as appropriate, leads to an interesting identity.

Proposition 2.3:

$$E^3(q) = \sum_{n\geq 0}(-1)^n(2n+1)q^{\binom{n+1}{2}}. \tag{90}$$

Proof: Begin with the derivative of the product representation of $\theta_1(x)$. Let

$$T(x) = (q^2z, q^2/z, q^2; q^2)_\infty \tag{91}$$

By the product rule from calculus,

$$\theta_1'(x) = 2q^{1/4}\cos(x)T(x) + 2q^{1/4}\sin(x)T'(x). \tag{92}$$

By the discussion above, the differentiability of the three infinite products is allowed. The differentiability of $\sin x$ is not of concern and the product of differentiable functions is differentiable so the derivative here of θ_1 with respect to x is valid. Additionally, by the linearity of the derivative,

$$\theta_1'(x) = 2q^{1/4}\sum_{n\geq 0}(-1)^n q^{n(n+1)}(2n+1)\cos(2n+1)x. \tag{93}$$

Here, since the convergence of $f_n' = (-1)^n q^{(n+1/2)^2}(2n+1)\cos(2n+1)x$ is dominated by the behavior of $q^{(n+1/2)^2}$, for any $\epsilon > 0$ there is a natural number N such that for all $n \geq N$, $|f_n| < \epsilon$. Thus uniform convergence of the series is established, and the term-by-term differentiation done above is valid. Setting the above two expressions equal to one another by uniqueness of the derivative leaves

$$\cos(x)T(x) + \sin(x)T'(x) = \sum_{n\geq 0}(-1)^n q^{n(n+1)}(2n+1)\cos(2n+1)x. \tag{94}$$

Now evaluating at $x = 0$:

$$T(0) = \sum_{n\geq 0}(-1)^n(2n+1)q^{n(n+1)}. \tag{95}$$

But by the definition of $T(x)$,

$$T(0) = (q^2, q^2, q^2; q^2)_\infty = E^3(q^2). \tag{96}$$

Replacing q with $q^{1/2}$ completes the proof. $\square$

We now prove a series of identities relating the $\theta_i(q)$ to one another in ways that will become useful when dealing with π.

Theorem 2.4:

$$\theta_4^2(q^2) = \theta_3(q)\theta_4(q). \tag{97}$$

Proof:

$$\begin{aligned} \theta_3(q)\theta_4(q) &= (q, q, -q, -q, q^2, q^2; q^2)_\infty = (q^2, q^2; q^2)_\infty (q^2, q^2; q^4)_\infty \\ &= (q^2, q^4; q^4)_\infty (q^2, q^4; q^4)_\infty (q^2, q^2; q^4)_\infty \\ &= (q^2, q^2, q^4; q^4)_\infty^2 = \theta_4^2(q^2).\ \square \end{aligned} \tag{98}$$

Proposition 2.5:

$$\theta_1'(q) = \theta_2(q)\theta_3(q)\theta_4(q), \tag{99}$$

where the derivative is taken with respect to x.

Proof: First off, with $x = 0$,

$$\begin{aligned} \theta_2(q)\theta_3(q)\theta_4(q) &= 2q^{1/4}(-q^2, -q^2, -q, -q, q, q, q^2, q^2, q^2; q^2)_\infty \\ &= 2q^{1/4}(q^2, q^2, q^4, q^4; q^4)_\infty (q^2; q^2)_\infty \\ &= 2q^{1/4}(q^2, q^2, q^2; q^2)_\infty \\ &= 2q^{1/4}E^3(q^2) = 2\eta^3(2\tau). \end{aligned} \tag{100}$$

Now by Proposition 2.3,

$$\theta_1'(q, x) = 2q^{1/4}\cos(x)(q^2 z, q^2/z, q^2; q^2)_\infty + 2q^{1/4}\sin(x)\big[(q^2 z, q^2/z, q^2; q^2)_\infty\big]'. \tag{101}$$

Then when $x = 0$,

$$\theta_1'(q) = 2q^{1/4}(q^2, q^2, q^2; q^2)_\infty = 2\eta^3(2\tau) \tag{102}$$

and the equality holds. □

Proposition 2.6:

$$\theta_3^2(q) + \theta_4^2(q) = 2\theta_3^2(q^2). \tag{103}$$

Proof: First note that[4]

$$\theta_3(q)+\theta_4(q) = \sum_{n=-\infty}^{\infty} q^{n^2} + \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2} = 2\sum_{2|n} q^{n^2} = 2\sum_{n=-\infty}^{\infty} q^{(2n)^2} = 2\theta_3(q^4) \tag{104}$$

Now let $S(n)$ denote the number of distinct ways to express n as a sum of two squares. Explicitly,

$$S(n) = \#(\{(x, y) \in \mathbb{Z}^2 : x^2 + y^2 = n\}) \tag{105}$$

where $\#(X)$ denotes the cardinality of the set X. Then by definition:

$$\theta_3^2(q) = \sum_{n\geq 0} S(n)q^n \tag{106}$$

$$\theta_4^2(q) = \sum_{n\geq 0} S(n)(-1)^n q^n \tag{107}$$

Lemma 2.7:

$$S(n) = S(2n). \tag{108}$$

Proof: If $n = a^2 + b^2$, then

$$2n = 2a^2 + 2b^2 = (a + b)^2 + (a - b)^2 \tag{109}$$

so $S(n) \leq S(2n)$. But if $x^2 + y^2 = 2n$, this implies x and y have the same parity and so their median $a := (x+y)/2$ and distance from the median $b := a - x = y - a$ are defined and are positive integers. But by (eq. 109) above this implies $n = a^2 + b^2$ and so $S(2n) \leq S(n)$, implying their equality.

The result then follows by the lemma:

$$\begin{aligned} \theta_3^2(q) + \theta_4^2(q) &= \sum_{n \geq 0} S(n) q^n + \sum_{n \geq 0} S(n)(-1)^n q^n \\ &= 2 \sum_{2|n \geq 0} S(n) q^n = 2 \sum_{n \geq 0} S(2n) q^{2n} = 2\theta_3^2(q^2). \ \square \end{aligned} \tag{110}$$

Proposition 2.8:

$$\theta_2^2(q^4) + \theta_3^2(q^4) = \theta_3^2(q^2). \tag{111}$$

Proof: By the last lemma with $S(n)$ as defined in (eq. 105) we have:

$$\theta_3^2(q) - \theta_3^2(q^2) = \sum_{n \geq 0} S(n) q^n - \sum_{n \geq 0} S(2n) q^{2n} = \sum_{n \geq 0} S(2n+1) q^{2n+1} = \sum_{2 \nmid k+m} q^{k^2+m^2}. \tag{112}$$

Setting $k = i - j$ and $m = i + j + 1$, $k + m = 2i + 1$ is odd and so:

$$\sum_{i,j=-\infty}^{\infty} q^{(i-j)^2+(i+j+1)^2} = \sum_{i,j=-\infty}^{\infty} q^{2i^2+2j^2+2i+2j+1} = \sum_{i,j=-\infty}^{\infty} (q^2)^{(i+1/2)^2+(j+1/2)^2} = \theta_2^2(q^2). \tag{113}$$

Adding $\theta_3^2(q^2)$ to both sides and replacing q by q^2 gives the desired identity. $\square$

Corollary 2.9:

$$\theta_3^2(q^4) - \theta_2^2(q^4) = \theta_4^2(q^2). \tag{114}$$

Proof: By Proposition 2.8, $\theta_2^2(q^2) + \theta_3^2(q^2) = \theta_3^2(q)$. Plugging that value for $\theta_3^2(q)$ into the result of Proposition 2.6 gives

$$\theta_3^2(q^2) + \theta_2^2(q^2) + \theta_4^2(q) = 2\theta_3^2(q^2) \tag{115}$$

and the result follows upon setting q to q^2 and rearranging. $\square$

Theorem 2.10:

$$\theta_2^4(q) + \theta_4^4(q) = \theta_3^4(q). \tag{116}$$

Proof: Squaring Theorem 2.4 results in:

$$\theta_3^2(q)\theta_4^2(q) = \theta_4^4(q^2). \tag{117}$$

Plugging in the results for $\theta_3^2(q)$ and $\theta_4^2(q)$ from Proposition 2.8 and Corollary 2.9 respectively,

$$\theta_4^4(q^2) = [\theta_3^2(q^2) + \theta_2^2(q^2)] \cdot [\theta_3^2(q^2) - \theta_2^2(q^2)] = \theta_3^4(q^2) - \theta_2^4(q^2). \tag{118}$$

The theorem is complete after rearranging and letting q^2 be q. $\square$

Definition 2.11 (Modular Lambda Function): *Let $\lambda(q)$ be defined as:*

$$\lambda(q) = \frac{\theta_2^4(q)}{\theta_3^4(q)}. \tag{119}$$

Theorem 2.12

$$\lambda(q^2) = \left(\frac{1 - \sqrt{1 - \lambda(q)}}{1 + \sqrt{1 - \lambda(q)}}\right)^2. \tag{120}$$

Proof: Divide Corollary 2.9 by Proposition 2.8[12]:

$$\frac{\theta_3^2(q^2)-\theta_2^2(q^2)}{\theta_3^2(q^2)+\theta_2^2(q^2)}=\frac{\theta_4^2(q)}{\theta_3^2(q)} \tag{121}$$

Dividing the left side by $1=\theta_3^2(q^2)/\theta_3^2(q^2)$ and squaring both sides and simplifying using the identities above imply that

$$\left(\frac{1-\sqrt{\lambda(q^2)}}{1+\sqrt{\lambda(q^2)}}\right)^2=\frac{\theta_4^4(q)}{\theta_3^4(q)}=\frac{\theta_3^4(q)-\theta_2^4(q)}{\theta_3^4(q)}=1-\lambda(q). \tag{122}$$

Taking the square root of both sides and solving for $\lambda(q^2)$ finishes the proof. $\square$

An identity for $\lambda(q^n)$ in terms of $\lambda(q)$ such as above is called a modular equation, and such equations play a role in the theory of hypergeometric functions.

3 Eisenstein Series

Recall the Taylor series expansion of e^x:

$$e^x=\sum_{n\geq 0}\frac{x^n}{n!}=1+\sum_{n\geq 1}\frac{x^n}{n!} \tag{123}$$

This implies the Taylor expansion of e^x-1 is $\sum_{n\geq 1}\frac{x^n}{n!}$. Manipulating this further by dividing both sides by x, we have:

$$\frac{e^x-1}{x}=\sum_{n\geq 1}\frac{x^{n-1}}{n!}=\sum_{n\geq 0}\frac{x^n}{(n+1)!} \tag{124}$$

Now however, since there is an x in the denominator, so we exclude $x=0$ from the radius of convergence. From analysis we know that the reciprocal of a function with

a power series expansion also has a power series expansion; that is,

$$\frac{x}{e^x - 1} = \sum_{n \geq 0} a_n x^n. \tag{125}$$

The coefficients of this power series turn out to be related to the Bernoulli numbers, and often the Bernoulli numbers are defined to be the coefficients of this power series, so that it becomes a generating function for the Bernoulli numbers.

Definition 3.1 (Bernoulli Numbers): *The Bernoulli numbers B_n are given by the generating function:*

$$\frac{x}{e^x - 1} = \sum_{n \geq 0} B_n \frac{x^n}{n!}. \tag{126}$$

The Bernoulli numbers can be computed from the power series for $e^x - 1$ and comparing coefficients. In particular, the powers series expansions derived above imply

$$(e^x - 1) \cdot \left(\frac{x}{e^x - 1} \right) = \left(\sum_{n \geq 1} \frac{x^n}{n!} \right) \left(\sum_{m \geq 0} \frac{B_m}{m!} x^m \right) = x. \tag{127}$$

It was shown by Mertens[8] that if two infinite series converge to A and B, respectively, with at least one series converging absolutely, then their Cauchy product converges to AB. The Cauchy product of two series is simply the product taken where like powers of x are summed together, exactly like a multiplication of two polynomials. By the absolute convergence of e^x, the power series of $e^x - 1$ converges absolutely and so we are valid in assuming the Cauchy product of the two Taylor series above converges to x.

To illustrate, comparing the coefficients of x gives $xB_0 = x$, or $B_0 = 1$. Continuing further, $1/2 \cdot B_0 + B_1 = 0$ and so $B_1 = -1/2$. One also gets $B_0/6 + B_1/2 + B_2/2 = 0$

resulting in $B_2 = 1/6$. Note however that for B_3,

$$\frac{B_0}{4!} + \frac{B_1}{3!} + \frac{B_2}{2!2!} + \frac{B_3}{3!} = \frac{1}{24} - \frac{1}{12} + \frac{1}{24} + \frac{B_3}{6} = \frac{B_3}{6} = 0 \tag{128}$$

so $B_3 = 0$. In fact, for odd $n > 1$, $B_n = 0$. To see this, we can use the power series expansion of $\cot x$ about 0 with a disk of radius $< \pi$. Note that

$$\begin{aligned} \cot x = \frac{\cos x}{\sin x} = \frac{e^{ix} + e^{-ix}}{2} \frac{2i}{e^{ix} - e^{-ix}} = i\left(\frac{e^{ix} + e^{-ix}}{e^{ix} - e^{-ix}}\right) \\ = i\left(\frac{e^{2ix} + 1}{e^{2ix} - 1}\right) = i\left(1 + \frac{2}{e^{2ix} - 1}\right). \end{aligned} \tag{129}$$

By the generating function for the Bernoulli numbers (Definition 3.1) it follows that

$$\cot x = i + \frac{1}{x} \sum_{n \geq 0} B_n \frac{(2i)^n}{n!} x^n = i + \frac{1}{x} - i + \frac{1}{x} \sum_{n \geq 2} B_n \frac{(2i)^n}{n!} x^n = \frac{1}{x} + \frac{1}{x} \sum_{n \geq 2} B_n \frac{(2i)^n}{n!} x^n. \tag{130}$$

Clearly $\cot x$ has a simple pole at $x = 0$ with $a_{-1} = 1$ in its Laurent series, and so

$$x \cot x = 1 + \sum_{n \geq 2} B_n \frac{(2i)^n}{n!} x^n \tag{131}$$

is analytic within the annulus about zero of radius given above for $\cot x$, and so the function and its power series agree. Since $x \cot x$ is an even function, B_n is indeed 0 for odd $n \geq 3$. With this in mind, replace n by $2n$ to get:

$$\cot x = \frac{1}{x} + \frac{1}{x} \sum_{n \geq 1} B_{2n} \frac{(2i)^{2n}}{(2n)!} x^{2n} = \frac{1}{x} + \sum_{n \geq 1} (-1)^n 2^{2n} \frac{B_{2n}}{(2n)!} x^{2n-1} = \sum_{n \geq 0} \frac{(-4)^n}{(2n)!} B_{2n} x^{2n-1}. \tag{132}$$

Definition 3.2 (Eisenstein Series): *With $q = q^*$ and $k > 0$ an integer, the Eisenstein series G_{2k} are defined as:*

$$G_{2k} = 2\zeta(2k)\left(1 - \frac{4k}{B_{2k}} \sum_{n \geq 1} \frac{n^{2k-1} q^n}{1 - q^n}\right). \tag{133}$$

It is often helpful to work with what are called the normalized Eisenstein series,

$$E_{2k} = \frac{G_{2k}}{2\zeta(2k)} = 1 - \frac{4k}{B_{2k}} \sum_{n \geq 1} \frac{n^{2k-1} q^n}{1 - q^n}. \tag{134}$$

Recalling the Riemann zeta function,

$$\zeta(s) = \sum_{n \geq 1} \frac{1}{n^s}, \tag{135}$$

where $\Re(s) > 1$, it is clear $\zeta(2k) > 0$ for any positive integer k so this division is valid. Additionally there is an expression relating the Bernoulli numbers B_{2k} to $\zeta(2k)$ stated here as:

$$B_{2k} = \frac{(-1)^{k+1} 2(2k)!}{(2\pi)^{2k}} \zeta(2k). \tag{136}$$

By this it is clear also that division by B_{2k} is permitted.

Theorem 3.3 (The ζ Function at Even Positive Integers): *For all positive integers k,*

$$B_{2k} = \frac{(-1)^{k+1} 2(2k)!}{(2\pi)^{2k}} \zeta(2k). \tag{137}$$

Proof: Observe from (eq. 132) the expansion of $\pi x \cot(\pi x)$ about 0 for $|x| < 1$ is:

$$\pi x \cot \pi x = \sum_{n \geq 0} \frac{(-4)^n}{(2n)!} B_{2n} \pi^{2n} x^{2n}. \tag{138}$$

Alternatively, since $\cot(\pi x) = \cos(\pi x)/\sin(\pi x)$, it is clear that since $\sin(\pi x) = 0$ only when x is an integer, $\cot(\pi x)$ is meromorphic and so can be written as a partial fraction expansion

$$\cot(\pi x) = \frac{a}{x} + \sum_{n\geq 1}\left(\frac{b_n}{x-n} + \frac{c_n}{x+n}\right) \tag{139}$$

where a, b_n, c_n are undetermined coefficients. Multiplying both sides by x and then letting $x = 1/4$ so that $\cot(\pi x) = 1$, we see:

$$1/4 = a - b_1/3 + c_1/5 - b_2/7 + c_2/9 - \cdots \tag{140}$$

Comparing this with the well-known formula $\pi/4 = 1 - 1/3 + 1/5 - 1/7 + \cdots$, it becomes obvious that $a = b_n = c_n = 1/\pi$. Then, upon multiplying both sides by π,

$$\pi\cot(\pi x) = \frac{1}{x} + 2\sum_{n\geq 1}\frac{x}{x^2-n^2}. \tag{141}$$

Take $|x| < 1$. Now by factoring out $-1/n^2$ from the terms in the summation,

$$\pi\cot(\pi x) = \frac{1}{x} - 2\sum_{n\geq 1}\left(\frac{x}{n^2}\right)\left(\frac{1}{1-(x^2/n^2)}\right) = \frac{1}{x} - 2\sum_{n\geq 1}\left(\frac{x}{n^2}\right)\sum_{k\geq 1}\left(\frac{x^2}{n^2}\right)^{k-1} \tag{142}$$

where for the last equality we used a geometric series, which since $|x| < 1$ does not affect the convergence. Combining this all into a double sum,

$$\pi\cot(\pi x) = \frac{1}{x} - 2\sum_{n\geq 1}\sum_{k\geq 1}\left(\frac{1}{n^{2k}}\right)x^{2k-1}. \tag{143}$$

Since $n \geq 1$, $k \geq 1$ and $|x| < 1$, we have absolute convergence and thus can switch the order of summation. By the definition of $\zeta(s)$ (eq. 135),

$$\pi\cot(\pi x) = \frac{1}{x} - 2\sum_{n\geq 1}\zeta(2n)x^{2n-1}. \tag{144}$$

This function has a simple pole at $x = 0$ and so $\pi x \cot(\pi x)$ is analytic and thus agrees with its power series expansion about $x = 0$ in the annulus $0 < |x| < 1$, which is:

$$\pi x \cot(\pi x) = 1 - 2 \sum_{n \geq 1} \zeta(2n) x^{2n}. \tag{145}$$

But we now have two power series expansions for the same function in the same radius of convergence, and thus they must be identical. So like terms must agree, and this implies:

$$-2\zeta(2n) = \frac{(-4)^n}{(2n)!} B_{2n} \pi^{2n} \tag{146}$$

$$(-1)2\zeta(2n) = \frac{(-1)^n (2\pi)^{2n}}{(2n)!} B_{2n} \tag{147}$$

$$\frac{2(2n)!}{(-1)^{n-1}(2\pi)^{2n}} \zeta(2n) = B_{2n} \tag{148}$$

which is equivalent to the desired result. □

Proposition 3.4:

$$\frac{\theta_1'(x)}{\theta_1(x)} = \sum_{n \geq 0} \frac{(-4)^n}{(2n)!} B_{2n} E_{2n} x^{2n-1}. \tag{149}$$

Proof: In Proposition 2.3 the validity of differentiation here ofr $\theta_1(x)$ was established. Let $T(x)$ be as in Proposition 2.3 so that[12]

$$\theta_1(x) = 2q^{1/4} \sin(x) T(x). \tag{150}$$

By the same proposition, it follows that the logarithmic derivative of $\theta_1(x)$ is

$$\frac{\theta_1'(x)}{\theta_1(x)} = \frac{2q^{1/4}\cos(x)T(x) + 2q^{1/4}\sin(x)T'(x)}{2q^{1/4}\sin(x)T(x)} = \cot x + \frac{T'(x)}{T(x)}. \tag{151}$$

By use of the product rule, $T'(x)/T(x)$ is

$$\begin{aligned}\frac{T'(x)}{T(x)} &= \frac{-2izq^2}{(1-zq^2)} + \frac{2iz^{-1}q^2}{(1-z^{-1}q^2)} + \frac{-2izq^4}{(1-zq^4)} + \frac{2iz^{-1}q^4}{(1-z^{-1}q^4)} + \cdots \\ &= 2i\left(\sum_{n\geq 1}\frac{z^{-1}q^{2n}}{1-z^{-1}q^{2n}} - \sum_{n\geq 1}\frac{zq^{2n}}{1-zq^{2n}}\right).\end{aligned} \tag{152}$$

The differentiability of $T(x)$ is a result of the differentiability of the infinite products involved, which was established in the previous section. By the summation of a geometric series,

$$\frac{T'(x)}{T(x)} = 2i\sum_{n\geq 1}\sum_{m\geq 1} q^{2nm}(z^{-m} - z^m) = 4\sum_{m\geq 1}\frac{q^{2m}}{1-q^{2m}}\sin 2mx. \tag{153}$$

Using the series expansion of $\sin x$,

$$\begin{aligned}\frac{\theta_1'(x)}{\theta_1(x)} &= \sum_{n\geq 0}\frac{(-4)^n}{(2n)!}B_{2n}x^{2n-1} + 4\sum_{m\geq 1}\frac{q^{2m}}{1-q^{2m}}\left(\sum_{n\geq 0}\frac{(-1)^n}{(2n+1)!}(2mx)^{2n+1}\right) \\ &= \frac{1}{x} + \sum_{n\geq 1}\frac{(-4)^n}{(2n)!}B_{2n}x^{2n-1} + 4\sum_{n\geq 0}\left(\sum_{m\geq 1}\frac{q^{2m}}{1-q^{2m}}\right)\frac{(-1)^n}{(2n+1)!}(2mx)^{2n+1} \\ &= \frac{1}{x} + \sum_{n\geq 1}\frac{(-4)^n}{(2n)!}B_{2n}x^{2n-1} + 4\sum_{n\geq 1}\left(\sum_{m\geq 1}\frac{m^{2n-1}q^{2m}}{1-q^{2m}}\right)\frac{(-1)^{n-1}}{(2n-1)!}2^{2n-1}x^{2n-1} \\ &= \frac{1}{x} + \sum_{n\geq 1}\frac{(-4)^n}{(2n)!}B_{2n}x^{2n-1} + 4\sum_{n\geq 1}\frac{B_{2n}}{4n}(1-E_{2n})\frac{(-1)^{n-1}}{(2n-1)!}2^{2n-1}x^{2n-1} \\ &= \frac{1}{x} + \sum_{n\geq 1}\frac{(-4)^n}{(2n)!}B_{2n}x^{2n-1} - \sum_{n\geq 1}B_{2n}(1-E_{2n})\frac{(-1)^n 2^{2n}}{(2n)(2n-1)!}x^{2n-1} \\ &= \frac{1}{x} + \sum_{n\geq 1}\frac{(-4)^n}{(2n)!}B_{2n}x^{2n-1} - \sum_{n\geq 1}\frac{(-4)^n}{(2n)!}B_{2n}(1-E_{2n})x^{2n-1} \\ &= \frac{1}{x} + \sum_{n\geq 1}\frac{(-4)^n}{(2n)!}B_{2n}E_{2n}x^{2n-1} = \sum_{n\geq 0}\frac{(-4)^n}{(2n)!}B_{2n}E_{2n}x^{2n-1}. \ \square\end{aligned} \tag{154}$$

We are now in a position to prove a series of important differential equations attributed to Ramanujan involving the first few Eisenstein series.

Theorem 3.5 (Ramanujan's Differential Equations):

$$q\frac{dE_2}{dq} = \frac{E_2^2 - E_4}{12} \tag{155}$$
$$q\frac{dE_4}{dq} = \frac{E_2E_4 - E_6}{3} \tag{156}$$
$$q\frac{dE_6}{dq} = \frac{E_2E_6 - E_4^2}{2} \tag{157}$$

Proof: First note that term-by-term differentiation of E_{2k} with respect to q is allowed, since the terms of such a derivative

$$f_n' = \frac{n^{2k}q^{n-1}}{(q^n-1)^2} \to 0 \tag{158}$$

as $n \to \infty$ and converge uniformly where $|q| < 1$. Since the q in E_{2n} is q^* while the q in $\theta_1(q,x)$ is $q^\dagger$, in accordance with Chan[10] we now give $\theta_1(q,x)$ with $q = q^*$ so that the two expressions of q are compatible. This results in

$$\theta_1(q,x) = 2\sum_{n\geq 0}(-1)^n q^{(n+1/2)^2/2}\sin(2n+1)x. \tag{159}$$

Noticing that $(n+1/2)^2/2 = (2n+1)^2/8$, and expanding sine as a power series results in:

$$\theta_1(q,x) = 2\sum_{n\geq 0}(-1)^n q^{(2n+1)^2/8}\left(\sum_{m\geq 0}\frac{(-1)^m}{(2m+1)!}(2m+1)^{2m+1}x^{2m+1}\right). \tag{160}$$

Letting

$$S_{2n+1} = 2\sum_{k\geq 0}(-1)^k(2k+1)^{2n+1}q^{(2k+1)^2/8}, \tag{161}$$

$\theta_1(q,x)$ becomes

$$\theta_1(q,x) = \sum_{n\geq 0}\frac{(-1)^n}{(2n+1)!}S_{2n+1}x^{2n+1}. \tag{162}$$

Using this and Proposition 3.4,

$$\begin{aligned}\theta_1'(x) &= \sum_{n\geq 0}\frac{(-1)^n(2n+1)}{(2n+1)!}S_{2n+1}x^{2n} = \sum_{n\geq 0}\frac{(-1)^n}{(2n)!}S_{2n+1}x^{2n} \\ &= \left(\sum_{n\geq 0}\frac{(-1)^n}{(2n+1)!}S_{2n+1}x^{2n+1}\right)\left(\sum_{n\geq 0}\frac{(-4)^n}{(2n)!}B_{2n}E_{2n}x^{2n-1}\right).\end{aligned} \tag{163}$$

Comparing the coefficients of x^{2n} gives

$$\frac{(-1)^n}{(2n)!}S_{2n+1} = S_1\frac{(-4)^n}{(2n)!}B_{2n}E_{2n} - \frac{1}{3!}S_3\frac{(-4)^{n-1}}{(2n-2)!}B_{2n-2}E_{2n-2} + \cdots + \frac{(-1)^n}{(2n+1)!}S_{2n+1}. \tag{164}$$

Dividing both sides by $(-1)^n/(2n)!$ leads to a recurrence relation for S_{2n+1}:

$$S_{2n+1} = 4^nS_1B_{2n}E_{2n} + \frac{4^{n-1}(2n)!}{3!(2n-2)!}S_3B_{2n-2}E_{2n-2} + \cdots + \frac{(2n)!}{(2n+1)!}S_{2n+1} \tag{165}$$

and so

$$S_{2n+1}\left(1-\frac{1}{2n+1}\right) = 4^nS_1B_{2n}E_{2n} + \frac{4^{n-1}(2n)!}{3!(2n-2)!}S_3B_{2n-2}E_{2n-2} + \cdots + \frac{4(2n)!}{(2n-1)!}S_{2n-1}, \tag{166}$$

resulting in

$$S_{2n+1} = \left(\frac{2n+1}{2n}\right)\sum_{k=0}^{n-1}\frac{4^{n-k}}{2k+1}\binom{2n}{2(n-k)}S_{2k+1}B_{2(n-k)}E_{2(n-k)}. \tag{167}$$

For $n = 1, 2, 3, 4$ we have, in terms of S_1:

$$S_3 = 6S_1B_2E_2 = S_1E_2 \tag{168}$$

$$S_5 = (5/4)[16S_1B_4E_4 + 8S_3B_2E_2] = S_1\frac{(5E_2^2 - 2E_4)}{3} \tag{169}$$

$$S_7 = (7/6)[(32/21)S_1E_6 - (8/3)S_3E_4 + 2S_5E_2] = S_1\frac{(35E_2^3 - 42E_2E_4 + 16E_6)}{9} \tag{170}$$

$$S_9 = S_1[(-48/5)E_8 + (64/3)E_2E_6 + (28/5)E_4^2 - 28E_2^2E_4 + (35/3)E_2^4] \tag{171}$$

Observe by the definition of S_{2n+1},

$$\begin{aligned} 8q\frac{dS_{2n+1}}{dq} &= 8q \cdot 2\sum_{k=0}^{\infty}(-1)^k(2k+1)^{2n+1}\frac{(2k+1)^2}{8}q^{\frac{(2k+1)^2}{8}-1} \\ &= 2\sum_{k=0}^{\infty}(-1)^k(2k+1)^{2n+3}q^{(2k+1)^2/8} = S_{2n+3}. \end{aligned} \tag{172}$$

Applying $8q\frac{d}{dq}$ to S_3 gives

$$8q\frac{dS_3}{dq} = S_5 = 8q\frac{dE_2}{dq}S_1 + E_2S_3. \tag{173}$$

Comparing that with $S_5 = S_1(5E_2^2 - 2E_4)/3$ and $S_3 = S_1E_2$,

$$S_1\frac{(5E_2^2 - 2E_4)}{3} = 8q\frac{dE_2}{dq}S_1 + E_2^2S_1, \tag{174}$$

or equivalently

$$q\frac{dE_2}{dq} = \frac{2E_2^2 - 2E_4}{24} = \frac{E_2^2 - E_4}{12}, \tag{175}$$

proving the first equation. In the same way,

$$\begin{aligned} 8q\frac{dS_5}{dq} &= S_7 = S_3(5E_2^2 - 2E_4)/3 + S_1\frac{8q}{3}\frac{d(5E_2^2 - 2E_4)}{dq} \\ &= S_1E_2(5E_2^2 - 2E_4)/3 + S_1\frac{40q}{3}\frac{d(E_2^2)}{dq} - S_1\frac{16q}{3}\frac{dE_4}{dq}. \end{aligned} \tag{176}$$

Since

$$\frac{40q}{3}\frac{d(E_2^2)}{dq} = \frac{80E_2}{3}\frac{E_2^2 - E_4}{12}, \tag{177}$$

it follows that

$$\frac{S_1(35E_2^3 - 42E_2E_4 + 16E_6)}{9} = \frac{S_1E_2(5E_2^2 - 2E_4)}{3} + S_1\frac{40q}{3}\frac{d(E_2^2)}{dq} - S_1\frac{16q}{3}\frac{dE_4}{dq} \tag{178}$$

$$\frac{35}{9}E_2^3 - \frac{14}{3}E_2E_4 + \frac{16}{9}E_6 = \frac{5}{3}E_2^3 - \frac{2}{3}E_2E_4 + \frac{20}{9}E_2^3 - \frac{20}{9}E_2E_4 - \frac{16}{3}q\frac{dE_4}{dq} \tag{179}$$

$$\frac{16}{9}E_6 - \frac{16}{9}E_2E_4 = -\frac{16}{3}q\frac{dE_4}{dq} \quad (180)$$
$$\frac{E_2E_4 - E_6}{3} = q\frac{dE_4}{dq} \quad (181)$$

so the second equation is proved as well. Moving on,

$$\begin{aligned} 8q\frac{dS_7}{dq} &= S_9 = S_1[(-48/5)E_8 + (64/3)E_2E_6 + (28/5)E_4^2 - 28E_2^2E_4 + (35/3)E_2^4] \\ &= 8q\frac{d}{dq}\left(S_1\frac{35E_2^3 - 42E_2E_4 + 16E_6}{9}\right) \\ &= S_1\Big[(35/9)E_2^4 - (42/9)E_2^2E_4 + (16/9)E_2E_6 + (280/9)q\frac{d}{dq}(E_2^3) \\ &\quad - (336/9)q\frac{d}{dq}(E_2E_4) + (128/9)q\frac{dE_6}{dq}\Big] \end{aligned} \quad (182)$$

Using the product rule and plugging in the previous two results yields, after some lengthy simplification,

$$40q\frac{dE_6}{dq} + 27E_8 - 20E_2E_6 - 7E_4^2 = 0. \quad (183)$$

In order to get the desired result, another recurrence relation is needed. With the series expansion of cosine, Proposition 2.2 with $q = q^*$ becomes

$$\eta(2\tau)\frac{\theta_1(2x)}{\theta_1(x)} = 2\sum_{n=-\infty}^{\infty}(-1)^n q^{(6n+1)^2/24}\sum_{m\geq 0}\frac{(-1)^m}{(2m)!}(6n+1)^{2m}x^{2m} \quad (184)$$

and with

$$T_{2m} = 2\sum_{n=-\infty}^{\infty}(-1)^n(6n+1)^{2m}q^{(6n+1)^2/24}, \quad (185)$$

we get

$$\eta(2\tau)\frac{\theta_1(2x)}{\theta_1(x)} = \sum_{m\geq 0}(-1)^m\frac{T_{2m}}{(2m)!}x^{2m}. \quad (186)$$

Logarithmically differentiating both sides with respect to x gives

$$2 \cdot \frac{\theta_1'(2x)}{\theta_1(2x)} - \frac{\theta_1'(x)}{\theta_1(x)} = \left(\frac{\sum_{m\geq 0} (-1)^m \frac{T_{2m}}{(2m-1)!} x^{2m-1}}{\sum_{m\geq 0} (-1)^m \frac{T_{2m}}{(2m)!} x^{2m}} \right) \tag{187}$$

which, after using Proposition 3.4 and re-indexing m with $j+1$ in the numerator of the above equation, simplifies to:

$$\left(\sum_{n\geq 0} \frac{(-4)^n}{(2n)!} B_{2n} E_{2n} x^{2n-1} (4^n - 1) \right) \left(\sum_{m\geq 0} \frac{(-1)^m}{(2m)!} T_{2m} x^{2m} \right) = \sum_{j\geq 0} \frac{(-1)^{j+1}}{(2j+1)!} T_{2j+2} x^{2j+1}. \tag{188}$$

Comparing the coefficients of x^{2n+1} results in the equation

$$\begin{aligned} \frac{(-1)^{n+1}}{(2n+1)!} T_{2n+2} =& \frac{(-4)}{(2)!} B_2 E_2 (4-1) \frac{(-1)^n}{(2n)!} T_{2n} + \frac{(-4)^2}{(4)!} B_4 E_4 (4^2 - 1) \frac{(-1)^{n-1}}{(2n-2)!} T_{2n-2} + \cdots \\ &+ \frac{(-4)^n}{(2n)!} B_{2n} E_{2n} (4^n - 1) \frac{(-1)}{(2)!} T_2 + \frac{(-4)^{n+1}}{(2n+2)!} B_{2n+2} E_{2n+2} (4^{n+1} - 1) T_0. \end{aligned} \tag{189}$$

Solving for T_{2n+2}, one gets

$$\begin{aligned} T_{2n+2} =& \frac{4(4-1)}{2!} (2n+1) B_2 E_2 T_{2n} + \frac{4^2(4^2-1)}{4!} (2n+1)(2n)(2n-1) B_4 E_4 T_{2n-2} + \cdots \\ &+ \frac{4^n(4^n-1)}{2!} (2n+1) B_{2n} E_{2n} T_2 + \frac{4^{n+1}(4^{n+1}-1)}{2n+2} B_{2n+2} E_{2n+2} T_0. \end{aligned} \tag{190}$$

Multiplying everything by 1 in the form of $(2n+2)/(2n+2)$, we at last get the recurrence relation:

$$T_{2n+2} = \frac{1}{2n+2} \sum_{k=0}^{n} \binom{2n+2}{2k} 4^{n-k+1} (4^{n-k+1} - 1) B_{2(n-k+1)} E_{2(n-k+1)} T_{2k}. \tag{191}$$

For $n = 0, 1, 2, 3$ we have, in terms of T_0:

$$T_2 = (1/2)(1)4(3) B_2 E_2 T_0 = T_0 E_2 \tag{192}$$

$$T_4 = [4(15)B_4E_4T_0 + 6(3)B_2E_2T_2] = T_0[3E_2^2 - 2E_4] \tag{193}$$

$$T_6 = 16E_6T_0 - 20E_4T_2 + 5E_2T_4 = T_0[15E_2^3 - 30E_2E_4 + 16E_6] \tag{194}$$

$$T_8 = T_0[105E_2^4 - 420E_2^2E_4 + 448E_2E_6 + 140E_4^2 - 272E_8] \tag{195}$$

Similar to the S_{2n+1}, by definition:

$$24q\frac{dT_{2m}}{dq} = 24q \cdot 2\sum_{n=-\infty}^{\infty}(-1)^n(6n+1)^{2m}\frac{(6n+1)^2}{24}q^{\frac{(6n+1)^2}{24}-1} = T_{2m+2}. \tag{196}$$

Then, one has

$$24q\frac{d}{dq}(T_0[15E_2^3 - 30E_2E_4 + 16E_6]) = T_0[105E_2^4 - 420E_2^2E_4 + 448E_2E_6 + 140E_4^2 - 272E_8] \tag{197}$$

which expands to:

$$\begin{aligned} &15E_2^4 - 30E_2^2E_4 + 16E_2E_6 + 1080E_2^2q\frac{dE_2}{dq} - 720E_2q\frac{dE_4}{dq} - 720E_4q\frac{dE_2}{dq} + 384q\frac{dE_6}{dq} \\ &\quad = 105E_2^4 - 420E_2^2E_4 + 448E_2E_6 + 140E_4^2 - 272E_8. \end{aligned} \tag{198}$$

Using the previous results for E_2 and E_4, one obtains after some simplification:

$$24q\frac{dE_6}{dq} + 17E_8 - 12E_2E_6 - 5E_4^2 = 0. \tag{199}$$

With this and the similar result from the S_{2n+1} recurrence, the E_8 term can be removed from the system, resulting in:

$$24q\frac{dE_6}{dq} + (17/27)\left(20E_2E_6 + 7E_4^2 - 40q\frac{dE_6}{dq}\right) - 12E_2E_6 - 5E_4^2 = 0 \tag{200}$$

$$-(32/27)q\frac{dE_6}{dq} + (16/27)E_2E_6 - (16/27)E_4^2 = 0 \tag{201}$$

$$q\frac{dE_6}{dq} = \frac{E_2E_6 - E_4^2}{2} \tag{202}$$

which proves the last equation and therefore the theorem. $\square$

One thing to note is that we could also remove the E_4^2 term, and solving for the derivative then shows in fact $E_4^2 = E_8$.

Corollary 3.6:

$$1728\eta^{24}(\tau) = E_4^3 - E_6^2. \tag{203}$$

Proof: By Theorem 3.5,

$$q\frac{d}{dq}(E_4^3 - E_6^2) = 3E_4^2 \cdot \frac{E_2E_4 - E_6}{3} - 2E_6 \cdot \frac{E_2E_6 - E_4^2}{2} = E_2(E_4^3 - E_6^2). \tag{204}$$

This implies

$$\frac{d}{dq}\log(E_4^3 - E_6^2) = E_2/q = q^{-1} - 24\sum_{n\geq 1}\frac{nq^{n-1}}{1-q^n}. \tag{205}$$

Integrating both sides with respect to q,

$$\log(E_4^3 - E_6^2) = \log q + 24\sum_{n\geq 1}\log(1-q^n) + \log C \tag{206}$$

for some constant C, which implies

$$E_4^3 - E_6^2 = C\eta^{24}(\tau). \tag{207}$$

Using the geometric series for $q/(1-q)$ and comparing the coefficients of q shows:

$$[3(240) - 2(-504)]q = 1728q = Cq. \ \square \tag{208}$$

The Taylor series expansions of $n^{2k-1}q^n/(1-q^n)$ actually lead to another, equivalent definition of the normalized Eisenstein series. For observe that

$1/(1-q^n) = 1 + q^n + q^{2n} + \cdots$ and so

$$\sum_{n\geq 1} \frac{n^{2k-1}q^n}{1-q^n} = \sum_{n\geq 1} \left(n^{2k-1}q^n + n^{2k-1}q^{2n} + n^{2k-1}q^{3n} + \cdots\right). \tag{209}$$

Combining powers of q leads to:

$$\sum_{n\geq 1} \frac{n^{2k-1}q^n}{1-q^n} = q + (1+2^{2k-1})q^2 + (1+3^{2k-1})q^3 + (1+2^{2k-1}+4^{2k-1})q^4 + \cdots \tag{210}$$

which, upon recalling the divisor sum function $\sigma_k(n)$ from elementary number theory, shows the Eisenstein series can be defined as:

$$E_{2k} = 1 - \frac{4k}{B_{2k}} \sum_{n\geq 1} \sigma_{2k-1}(n)q^n. \tag{211}$$

4 Modular Forms

Let $\mathbf{H} = \{z \in \mathbb{C} : \Im(z) > 0\}$ denote the upper half-plane of $\mathbb{C}$ and let $\mathbf{\Gamma}$ denote $\mathrm{PSL}_2(\mathbb{Z}) = \mathrm{SL}_2(\mathbb{Z})/\{I, -I\}$; call it the ***modular group***. Consider the group action $\mathbf{\Gamma} \times \mathbf{H} \longrightarrow \mathbf{H}$ defined by

$$gz = \begin{pmatrix} a & b \\ c & d \end{pmatrix} z = \frac{az+b}{cz+d}. \tag{212}$$

To begin we show this is in fact a group action. First,

$$Iz = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z = \frac{1z+0}{0z+1} = z. \tag{213}$$

Next let $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $h = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$. Then,

$$\begin{aligned}(gh)z &= \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix} z = \frac{(ae+bg)z+af+bh}{(ce+dg)z+cf+dh} = \frac{a(ez+f)+b(gz+h)}{c(ez+f)+d(gz+h)} \\ &= \frac{a(\frac{ez+f}{gz+h})+b}{c(\frac{ez+f}{gz+h})+d} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \frac{ez+f}{gz+h} = g(hz).\end{aligned} \tag{214}$$

Finally, let $z = x+iy$. We have

$$\begin{aligned}\Im(gz) &= \Im\Big(\frac{az+b}{cz+d}\Big) = \Im\left(\frac{(az+b)\overline{(cz+d)}}{(cz+d)\overline{(cz+d)}}\right) = \Im\Big(\frac{(ax+b+iay)(cx+d-icy)}{(cz+d)\overline{(cz+d)}}\Big) \\ &= \frac{\Im(ac(x^2+y^2)+x(ad+bc)+bd+iy(ad-bc))}{|cz+d|^2} = \frac{y(ad-bc)}{|cz+d|^2} = \frac{y}{|cz+d|^2} > 0.\end{aligned} \tag{215}$$

Definition 4.1 (Modular Function): *A modular function $f : \mathbf{H} \longrightarrow \mathbb{C}$ is one meromorphic in $\mathbf{H}$ where for any $g \in \mathbf{\Gamma}, f(gz) = f(z)$.*

More generally, we define a modular form for $\mathbf{\Gamma}$ of weight k to be a function $f : \mathbf{H} \longrightarrow \mathbb{C}$ holomorphic in $\mathbf{H}$ with f bounded as $\Im(z) \to \infty$, where for any $g \in \mathbf{\Gamma}$, $f(gz) = (cz+d)^k f(z)$. In this respect a modular function can be viewed as a modular form of weight 0 with the holomorphic restriction slightly relaxed.

The first example of a modular form is $\eta(\tau)$, and is especially important as many other modular forms can be defined in terms of it.

Theorem 4.2: *$\eta(\tau)$ is a modular form of weight 1/2.*

Proof: Recall a function is holomorphic if in its domain it can be written as a convergent power series. But this is obvious for η by the pentagonal number theorem so holomorphicity is not a concern. As well, η is bounded as $\Im(\tau) \to \infty$ due to $q = e^{2\pi i\tau}$ being in the unit circle.

In order to show η has weight $\frac{1}{2}$, we will prove a useful lemma and then show that η^{24} is a modular form of weight 12. By the definition, taking the 24th root of everything then results in $k = 1/2$.

Lemma 4.3: *The condition on a modular form of weight k that $f(gz) = (cz + d)^k f(z)$ is equivalent to the following:*

$$f(z+1) = f(z) \tag{216}$$

$$f\Big(-\frac{1}{z}\Big) = z^k f(z). \tag{217}$$

Proof: Let

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \; T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{218}$$

Both are in $\mathrm{PSL}_2(\mathbb{Z})$, and we have

$$Tz = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} z = z + 1 \tag{219}$$

$$Sz = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} z = -1/z \tag{220}$$

Now if S, T generate $\mathbf{\Gamma}$, then any $g \in \mathbf{\Gamma}$ can be represented by a finite product of S's and T's. Thus the condition that $f(gz) = (cz + d)^k f(z)$ would be the same as

requiring

$$f(Tz) = f(z+1) = (0z+1)^k f(z) = f(z) \tag{221}$$

$$f(Sz) = f(-1/z) = (1z+0)^k f(z) = z^k f(z). \tag{222}$$

From algebra[2] it is known the group presentation of $\mathbf{\Gamma} = \mathrm{PSL}_2(\mathbb{Z})$ is $\langle a, b : a^2 = b^3 = e \rangle$. Then it is enough to show that S and ST satisfy a, b. The relation can be shown by doing the matrix multiplication, and recalling that in $\mathbf{\Gamma}, -A = A$:

$$\begin{aligned}
S^2 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = I \\
(ST)^3 &= \left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \right)^3 = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix}^3 \\
&= \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} \\
&= \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} = I.
\end{aligned} \tag{223}$$

With the lemma established, Consider now η^{24}. This is simply

$$\eta^{24}(\tau) = q \prod_{n \geq 1} (1 - q^n)^{24}. \tag{224}$$

Notice that

$$e^{2\pi i(\tau+1)} = e^{2\pi i \tau} e^{2\pi i} = e^{2\pi i \tau} \tag{225}$$

and it is clear now that $\eta^{24}(\tau+1) = \eta^{24}(\tau)$. For $\eta^{24}(-1/\tau)$, we show now that $\eta(-1/\tau) = \sqrt{-i\tau}\,\eta(\tau)$. Taking that to the 24$^{\text{th}}$ power then yields

$\eta^{24}(-1/\tau) = \tau^{12}\eta(\tau)$. First recall the definition given of E_{2k} in terms of the divisor sum function σ_k. Note $|q| < 1$ so $E_2(\tau)$ and thus $G_2(\tau)$ is analytic in $\mathbf{H}$, and also by the discussion above $G_2(\tau+1) = G_2(\tau)$. By use of the product rule, the logarithmic derivative of $\eta(\tau)$ is

$$\begin{aligned}\frac{\eta'(\tau)}{\eta(\tau)} &= \frac{\left(q^{1/24}(1-q)(1-q^2)(1-q^3)\cdots\right)'}{q^{1/24}(1-q)(1-q^2)(1-q^3)\cdots} = \frac{2\pi i}{24} - \frac{2\pi i q}{1-q} - \frac{4\pi i q^2}{1-q^2} - \frac{6\pi i q^3}{1-q^3} - \cdots \\ &= \frac{2\pi i}{24}\left(1 - 24\sum_{n\geq 1}\frac{nq^n}{1-q^n}\right) = \frac{\pi i}{12}E_2(\tau) = \frac{i}{4\pi}G_2(\tau).\end{aligned} \tag{226}$$

From this it is clear that

$$\frac{\left(\sqrt{-i\tau}\eta(\tau)\right)'}{\sqrt{-i\tau}\eta(\tau)} = \frac{1}{2\tau} + \frac{\eta'(\tau)}{\eta(\tau)} = \frac{1}{2\tau} + \frac{i}{4\pi}G_2(\tau). \tag{227}$$

Now by the chain rule, the logarithmic derivative of $\eta(-1/\tau)$ is

$$\frac{\eta'(-1/\tau)}{\eta(-1/\tau)} = \frac{1}{\tau^2}\frac{\eta(-1/\tau)'}{\eta(-1/\tau)}. \tag{228}$$

If $\eta(-1/\tau)$ and $\sqrt{-i\tau}\eta(\tau)$ have the same logarithmic derivative, then they must differ only by some constant factor. In fact taking $\tau = i$ we see that constant factor is 1, by

$$\sqrt{-i(i)}e^{2n\pi i(i)} = e^{-2n\pi} = e^{-2n\pi i/i}. \tag{229}$$

So all that remains to show that η is a modular form of weight $\frac{1}{2}$ is to prove:

$$\frac{1}{\tau^2}\frac{\eta(-1/\tau)'}{\eta(-1/\tau)} = \frac{1}{2\tau} + \frac{i}{4\pi}G_2(\tau) \tag{230}$$

or equivalently that

Lemma 4.4:

$$G_2(-1/\tau) = \tau^2 G_2(\tau) - 2\pi i\tau. \tag{231}$$

Proof: There is another way to define the Eisenstein series G_{2n} than given previously. However for $n = 1$ this definition is only conditionally convergent, so care must be taken when re-indexing terms. Let

$$G_{2n}(\tau) = \sum_{(c,d)\neq(0,0)} (c\tau + d)^{-2n} \tag{232}$$

for integers c, d. The equality of this definition with the one given before will be shown in the next section. Assuming this for now, and observing that the series is not defined when $c = d = 0$, for $n = 1$ we have[16]:

$$\begin{aligned} G_2\Big(-\frac{1}{\tau}\Big) &= \sum_{c=-\infty}^{\infty}\sum_{d=-\infty}^{\infty}\Big(-\frac{c}{\tau}+d\Big)^{-2} = \sum_{c=-\infty}^{\infty}\sum_{d=-\infty}^{\infty}\frac{1}{c^2/\tau^2 - 2cd/\tau + d^2} \\ &= \sum_{c=-\infty}^{\infty}\sum_{d=-\infty}^{\infty}\frac{\tau^2}{c^2 - 2cd\tau + d^2\tau^2} = \tau^2\sum_{c=-\infty}^{\infty}\sum_{d=-\infty}^{\infty}(d\tau - c)^{-2}. \end{aligned} \tag{233}$$

Replacing d with $-d$,

$$G_2(-1/\tau) = \tau^2 \sum_{c=-\infty}^{\infty}\sum_{d=-\infty}^{\infty}(d\tau + c)^{-2} \tag{234}$$

and re-indexing by switching c and d,

$$G_2\Big(-\frac{1}{\tau}\Big) = \tau^2 \sum_{d=-\infty}^{\infty}\sum_{c=-\infty}^{\infty}(c\tau + d)^{-2}. \tag{235}$$

Denote that last double sum by $G_2'(\tau)$, so $G_2\Big(-\frac{1}{\tau}\Big) = \tau^2 G_2'(\tau)$. Note that if $G_2'(\tau) = G_2(\tau) - \frac{2\pi i}{\tau}$, then

$$G_2\Big(-\frac{1}{\tau}\Big) = \tau^2 G_2'(\tau) = \tau^2 G_2 - 2\pi i\tau \tag{236}$$

as desired. To this end, define[16]

$$\begin{aligned} H(\tau) &= \sum_{c=-\infty}^{\infty} \sum_{d=-\infty}^{\infty} \frac{1}{(c\tau+d)(c\tau+d-1)} \\ H'(\tau) &= \sum_{d=-\infty}^{\infty} \sum_{c=-\infty}^{\infty} \frac{1}{(c\tau+d)(c\tau+d-1)} \end{aligned} \tag{237}$$

Note that in $H(\tau)$ if $c = 0$, d cannot be 0 or 1, and vice versa in $H'(\tau)$. Now,

$$\begin{aligned} H(\tau) - G_2(\tau) &= \sum_{c=-\infty}^{\infty} \sum_{d=-\infty}^{\infty} \left(\frac{1}{(c\tau+d)(c\tau+d-1)} - \frac{1}{(c\tau+d)^2} \right) - 1 \\ &= \sum_{c=-\infty}^{\infty} \sum_{d=-\infty}^{\infty} \left(\frac{(c\tau+d)^2 - (c\tau+d)(c\tau+d-1)}{(c\tau+d)^3(c\tau+d-1)} \right) - 1 \\ &= \sum_{c=-\infty}^{\infty} \sum_{d=-\infty}^{\infty} \left(\frac{(c\tau+d) - (c\tau+d-1)}{(c\tau+d)^2(c\tau+d-1)} \right) - 1 \\ &= \sum_{c=-\infty}^{\infty} \sum_{d=-\infty}^{\infty} \left(\frac{1}{(c\tau+d)^2(c\tau+d-1)} \right) - 1 \end{aligned} \tag{238}$$

where the double sum has $d \neq 0, 1$ when $c = 0$. $H(\tau) - G_2(\tau)$ converges unconditionally, so re-indexing c and d will not change the sum and we have

$$H(\tau) - G_2(\tau) = H'(\tau) - G_2'(\tau), \tag{239}$$

or

$$G_2(\tau) - G_2'(\tau) = H(\tau) - H'(\tau). \tag{240}$$

It then remains to show that $H(\tau) - H'(\tau) = \frac{2\pi i}{\tau}$. To start with,

$$\begin{aligned}
H(\tau) &= \sum_{c=-\infty}^{\infty}\sum_{d=-\infty}^{\infty} \frac{1}{(c\tau+d)(c\tau+d-1)} = \sum_{c=-\infty}^{\infty}\sum_{d=-\infty}^{\infty}\left(\frac{1}{c\tau+d-1} - \frac{1}{c\tau+d}\right) \\
&= \sum_{d=-\infty}^{\infty}\left(\frac{1}{d-1} - \frac{1}{d}\right) = \lim_{N\to\infty}\sum_{d=-N+1}^{-1}\left(\frac{1}{d-1} - \frac{1}{d}\right) + \lim_{N\to\infty}\sum_{d=2}^{N}\left(\frac{1}{d-1} - \frac{1}{d}\right) \\
&= \lim_{N\to\infty}\left(\frac{1}{-N} - \frac{1}{-N+1}\right) + \left(\frac{1}{-N+1} - \frac{1}{-N+2}\right) + \cdots + \left(\frac{1}{-2} - \frac{1}{-1}\right) \\
&\qquad + \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots + \left(\frac{1}{N-1} - \frac{1}{N}\right) \\
&= \lim_{N\to\infty}\left(\frac{1}{-N} - \frac{1}{-1}\right) + \left(\frac{1}{1} - \frac{1}{N}\right) = 1 + 1 = 2
\end{aligned} \tag{241}$$

where all the other terms cancel out by the 'telescoping series' trick. $H'(\tau)$ is slightly more complicated, but using the same technique:

$$\begin{aligned}
H'(\tau) &= \sum_{d=-\infty}^{\infty}\sum_{c=-\infty}^{\infty}\left(\frac{1}{c\tau+d-1} - \frac{1}{c\tau+d}\right) = \lim_{N\to\infty}\sum_{d=-N+1}^{N}\sum_{c=-\infty}^{\infty}\left(\frac{1}{c\tau+d-1} - \frac{1}{c\tau+d}\right) \\
&= \lim_{N\to\infty}\Bigg[\sum_{d=-N+1}^{-1}\sum_{c=-\infty}^{\infty}\left(\frac{1}{c\tau+d-1} - \frac{1}{c\tau+d}\right) + \sum_{d=2}^{N}\sum_{c=-\infty}^{\infty}\left(\frac{1}{c\tau+d-1} - \frac{1}{c\tau+d}\right) \\
&\qquad + \sum_{c=-\infty}^{\infty}\left(\frac{1}{c\tau-1} - \frac{1}{c\tau}\right) + \sum_{c=-\infty}^{\infty}\left(\frac{1}{c\tau} - \frac{1}{c\tau+1}\right)\Bigg] \\
&= \lim_{N\to\infty}\Bigg[\sum_{c=-\infty}^{\infty}\left(\frac{1}{c\tau-N} - \frac{1}{c\tau-N+1}\right) + \sum_{c=-\infty}^{\infty}\left(\frac{1}{c\tau-N+1} - \frac{1}{c\tau-N+2}\right) + \cdots \\
&\qquad + \sum_{c=-\infty}^{\infty}\left(\frac{1}{c\tau-2} - \frac{1}{c\tau-1}\right) + \sum_{c=-\infty}^{\infty}\left(\frac{1}{c\tau+1} - \frac{1}{c\tau+2}\right) + \\
&\qquad + \cdots + \sum_{c=-\infty}^{\infty}\left(\frac{1}{c\tau+N-1} - \frac{1}{c\tau+N}\right) + \sum_{c=-\infty}^{\infty}\left(\frac{1}{c\tau-1} - \frac{1}{c\tau+1}\right)\Bigg] \\
&= \lim_{N\to\infty}\Bigg[\sum_{c=-\infty}^{\infty}\left(\frac{1}{c\tau-N} - \frac{1}{c\tau-1}\right) + \sum_{c=-\infty}^{\infty}\left(\frac{1}{c\tau+1} - \frac{1}{c\tau+N}\right)
\end{aligned} \tag{242}$$

$$
\begin{aligned}
&\qquad + \sum_{c=-\infty}^{\infty}\Big(\frac{1}{c\tau-1}-\frac{1}{c\tau+1}\Big)\Bigg]\\
&= \lim_{N\to\infty}\Bigg[\sum_{c=-\infty}^{\infty}\Big(\frac{1}{c\tau-N}-\frac{1}{c\tau+N}\Big)\Bigg]+\Big(-\frac{1}{-1}\Big)+\Big(\frac{1}{1}\Big)\\
&= \lim_{N\to\infty}\Bigg[\sum_{c=-\infty}^{\infty}\Big(\frac{1}{c\tau-N}-\frac{1}{c\tau+N}\Big)\Bigg]+2
\end{aligned}
$$

where if one was keeping track of where the series were and weren't defined, this last sum takes values $c \neq 0$. Now, it was proved in the previous section that the partial fraction expansion of $\pi\cot(\pi x)$ is

$$
\pi\cot(\pi x) = \frac{1}{x} + 2x\sum_{n\geq 1}\frac{1}{x^2-n^2}. \tag{243}
$$

With that in mind[16],

$$
\begin{aligned}
&\lim_{N\to\infty}\sum_{c=-\infty}^{\infty}\Big(\frac{1}{c\tau-N}-\frac{1}{c\tau+N}\Big) = \frac{2}{\tau}\lim_{N\to\infty}\sum_{c\geq 1}\Big(\frac{1}{c-N/\tau}-\frac{1}{c+N/\tau}\Big)\\
&= \frac{2}{\tau}\lim_{N\to\infty}\sum_{c\geq 1}\frac{2N/\tau}{c^2-N^2/\tau^2} = \frac{2}{\tau}\lim_{N\to\infty}\Bigg[\pi\cot\Big(-\pi\frac{N}{\tau}\Big)+\frac{\tau}{N}\Bigg]\\
&= \frac{2\pi}{\tau}\lim_{N\to\infty}\cot\Big(-\pi\frac{N}{\tau}\Big)+\lim_{N\to\infty}\frac{\tau}{N} = \frac{2\pi}{\tau}\lim_{N\to\infty} i\frac{e^{-i\pi N/\tau}+e^{i\pi N/\tau}}{e^{-i\pi N/\tau}-e^{i\pi N/\tau}}\\
&= \frac{2\pi i}{\tau}\lim_{N\to\infty}\frac{e^{-2\pi i N/\tau}+1}{e^{-2\pi i N/\tau}-1} = -\frac{2\pi i}{\tau}.
\end{aligned} \tag{244}
$$

Hence $H(\tau)-H'(\tau) = 2-(2-\frac{2\pi i}{\tau}) = \frac{2\pi i}{\tau}$, and so the lemma and consequently the claim that $\eta(\tau)$ is a modular form of weight $\frac{1}{2}$ is proved. $\square$

Since $G_2(-1/\tau) \neq \tau^2 G_2(\tau)$, G_2 is not a modular form of weight 2. For $k \geq 2$ however, to show G_{2k} is a modular form of weight $2k$ it suffices to show

$$
G_{2k}(-1/\tau) = \tau^{2k}G_{2k}(\tau). \tag{245}
$$

But this is obvious by the same argument that showed $G_2(-1/\tau) = \tau^2 G_2'(\tau)$, and noting that because for $k \geq 2$, the double sum G_{2k} is absolutely convergent, so $G_{2k}(\tau) = G_{2k}'(\tau)$. Therefore G_{2k} is a modular form of weight $2k$ for $k > 1$. Note this also means E_{2k} is a modular form in the same vein.

Although we will not discuss this further, one of the first modular functions studied intensely was *elliptic modular function*, or the ***j-invariant***, which is defined as follows:

$$j(\tau) = \frac{E_4^3(\tau)}{\eta^{24}(\tau)} = 1728\frac{E_4^3}{E_4^3 - E_6^2}. \tag{246}$$

Being a quotient of two holomorphic functions, j is meromorphic in $\mathbf{H}$. As E_4^3 and η^{24} are both modular forms of weight 12, j is of weight 0, and j is a well-defined modular function. When expanded out in powers of q, it has the form[2]:

$$j(\tau) = q^{-1} + 744 + 196884q + \cdots \tag{247}$$

This function is related to invariants of classes of elliptic curves and also to the Weierstrass elliptic function, which we introduce shortly.

From the equation $\eta(-1/\tau) = \sqrt{-i\tau}\eta(\tau)$, some transformation formulas for certain eta quotients can be proven. One example is the following[12]:

Proposition 4.5: *Writing $\theta_3(q)$ in terms of τ,*

$$\theta_3(-1/\tau) = \sqrt{-i\tau}\theta_3(\tau). \tag{248}$$

Proof: By Proposition 1.12, observe that

$$\theta_3(\tau) = \phi(q^{1/2}) = \frac{\eta^5(\tau)}{\eta^2(2\tau)\eta^2(\tau/2)} \tag{249}$$

Then, by the formula proven in Lemma 4.4:

$$\theta_3(-1/\tau) = \frac{\eta^5(-1/\tau)}{\eta^2(-2/\tau)\eta^2(-1/(2\tau))} = \frac{\sqrt{-i\tau}^5}{\sqrt{-i\tau}^4}\frac{\eta^5(\tau)}{\eta^2(2\tau)\eta^2(\tau/2)} = \sqrt{-i\tau}\theta_3(\tau). \square \quad (250)$$

5 Elliptic Functions

Definition 5.1 (Elliptic Function): *For two complex numbers ω_1, ω_2 with $\Im(\omega_1/\omega_2) > 0$, a function f with $f(z) = f(z+\omega_1) = f(z+\omega_2)$ is called a **doubly-periodic function**. An **elliptic function** is a meromorphic doubly-periodic function.*

If every period of f is of the form $x\omega_1 + y\omega_2$ for x, y integers, then the set

$$P(\omega_1, \omega_2) = \{x\omega_1 + y\omega_2 \mid 0 \leq x, y < 1\} \quad (251)$$

is called a fundamental parallelogram of f.

Theorem 5.2: *If an elliptic function f has no poles in some $P(\omega_1, \omega_2)$ of f, then f is a constant.*

Proof: Being an elliptic function f is continuous, and by the assumption of having no poles in $P(\omega_1, \omega_2)$, f is bounded on the closure of the parallelogram. By the periodicity of f its values everywhere are determined by its values in the fundamental parallelogram and hence f is entire. But By Liouville's Theorem[19] a bounded entire function is constant. $\square$

We now prove some transformation formulas for $\theta_1(q, x)$ that will be useful later. Similar results for the other theta functions can be proven in the same manner and

will be stated as needed.

Proposition 5.3:

$$\theta_1(q, x+\pi) = -\theta_1(q, x). \tag{252}$$

Proof: By definition,

$$\theta_1(q, x+\pi) = -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+1/2)^2} e^{(2n+1)i(x+\pi)} = -\theta_1(q, x). \ \Box \tag{253}$$

Proposition 5.4:

$$\theta_1(q, x+\pi\tau) = -q^{-1} e^{-2ix} \theta_1(q, x). \tag{254}$$

Proof:

$$\begin{aligned} \theta_1(q, x+\pi\tau) &= -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+1/2)^2} e^{(2n+1)ix} e^{(2n+1)i\pi\tau} \\ &= -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+1/2)^2} e^{(2n+1)ix} q^{2n+1} \\ &= q^{-1} e^{-2ix} (-i) \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+3/2)^2} e^{(2n+3)ix} \end{aligned} \tag{255}$$

Setting $m = n+1$ and simplifying finishes the proof. $\Box$

Proposition 5.5:

$$\theta_1(q, x+\pi/2) = \theta_2(q, x). \tag{256}$$

Proof:

$$\theta_1(q, x + \pi/2) = -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+1/2)^2} e^{(2n+1)ix} e^{ni\pi} e^{i\pi/2} = -i(i) \sum_{n=-\infty}^{\infty} (-1)^{2n} q^{(n+1/2)^2}$$
$$= \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2} e^{(2n+1)ix} = \theta_2(q, x). \ \square \tag{257}$$

Proposition 5.6

$$\theta_1(q, x + \pi\tau/2) = iq^{-1/4} e^{-ix} \theta_4(q, x). \tag{258}$$

Proof:

$$\begin{aligned} \theta_1(q, x + \pi\tau/2) &= -i \sum_{n=-\infty}^{\infty} (-1)^n q^{(n+1/2)^2} e^{(2n+1)ix} e^{ni\pi\tau} e^{i\pi\tau/2} \\ &= -i \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2+2n+3/4} e^{2nix} e^{ix} \\ &= iq^{-1/4} e^{-ix} \sum_{n=-\infty}^{\infty} (-1)^{n+1} q^{(n+1)^2} e^{2(n+1)ix} \end{aligned} \tag{259}$$

Re-index $n + 1$ by n and the result follows. $\square$

Theorem 5.7: *Let $C = P(\pi, \pi\tau)$ denote the fundamental parallelogram with vertices t, $t + \pi$, $t + \pi\tau$, and $t + \pi + \pi\tau$ where the $\theta_i(q, x)$ have no zeroes in ∂C. Then there is exactly one zero in C for each of $\theta_i(q, x)$.*

Proof[12]: By the product form of $\theta_1(q, x)$ it is clear a zero occurs when $x = 0$. For a parallelogram with dimensions described in C with 0 in the interior, by the properties of the sine function that is the only zero in C. By Propositions 5.3 and 5.5 this implies the only fundamental zero of $\theta_2(q, x)$ is $\pi/2$. Similarly Propositions 5.4 and 5.6 show the only zero of $\theta_4(q, x)$ to be $\pi\tau/2$. Finally, it can be shown in the

same way as Proposition 5.5 that

$$\theta_4(q, x + \pi/2) = \theta_3(q, x) \tag{260}$$

and thus the only zero in C for $\theta_3(q, x)$ is $(\pi\tau + \pi)/2$. $\square$

Definition 5.8 (Weierstrass $\wp$-Function): *Let $\omega_1, \omega_2 \in \mathbb{C}$ with $\omega_1/\omega_2 = ci \neq 0$ and let $[\omega_1, \omega_2]$ denote the lattice generated by ω_1, ω_2 Also let $L = [\omega_1, \omega_2]\backslash\{0\}$. Define the Weierstrass $\wp$-Function as:*

$$\wp(z, \omega_1, \omega_2) = \frac{1}{z^2} + \sum_{u \in L} \left(\frac{1}{(z-u)^2} - \frac{1}{u^2} \right). \tag{261}$$

For short we write $\wp(z)$ when the ω_i are clear. By its definition as a series, $\wp(z)$ is meromorphic, having double poles at points in L. It can be shown that the $\wp$-function converges absolutely, and with that property that $\wp(z)$ is an even function. For with the capability to rearrange terms in the lattice sum,

$$\begin{aligned} \wp(-z) &= 1/(-z)^2 + \sum_{u \in L} (1/(-z-u)^2 - 1/u^2) = 1/z^2 + \sum_{u \in L} (1/(z+u)^2 - 1/u^2) \\ &= 1/z^2 + \sum_{u \in L} (1/(z-u)^2 - 1/u^2) = \wp(z). \end{aligned} \tag{262}$$

We now explain our current interest in $\wp(z)$:

Theorem 5.9: *$\wp(z, \omega_1, \omega_2)$ is an elliptic function with periods ω_1 and ω_2.*

Proof: We show $\wp(z + \omega_1) = \wp(z)$. The process can be repeated to show $\wp(z + \omega_2) = \wp(z)$. As before $L = [\omega_1, \omega_2]\backslash\{0\}$. Additionally set

$L_\omega = [\omega_1, \omega_2] \backslash \{0, \omega_1\}$. Then,

$$\begin{aligned}
\wp(z+\omega_1) &= 1/(z+\omega_1)^2 + \sum_{u\in L}(1/(z+\omega_1-u)^2 - 1/u^2) \\
&= 1/(z+\omega_1)^2 + \sum_{u\in L_\omega}(1/(z+\omega_1-u)^2 - 1/u^2) + 1/z^2 - 1/\omega_1^2 \\
&= 1/(z+\omega_1)^2 + \sum_{u\in L_\omega}(1/(z+\omega_1-u)^2 - 1/(u-\omega_1)^2) \\
&\quad - \sum_{u\in L_\omega}(1/u^2 - 1/(u-\omega_1)^2) + 1/z^2 - 1/\omega_1^2.
\end{aligned} \tag{263}$$

But now observe that

$$\begin{aligned}
\sum_{u\in L_\omega}(1/u^2 - 1/(u-\omega_1)^2) &= \sum_{u'=\omega_1-u\in L_\omega}(1/(\omega_1-u')^2 - 1/(-u')^2) \\
&= -\sum_{u'\in L_\omega}(1/(u')^2 - 1/((u')-\omega_1)^2)
\end{aligned} \tag{264}$$

so the whole sum is zero and

$$\begin{aligned}
\wp(z+\omega_1) &= 1/(z+\omega_1)^2 + \sum_{u\in L_\omega}(1/(z+\omega_1-u)^2 - 1/(u-\omega_1)^2) + 1/z^2 - 1/\omega_1^2 \\
&= 1/z^2 + \sum_{u'=u-\omega_1\in L_\omega}(1/(z-u')^2 - 1/(u')^2) + 1/(z+\omega_1)^2 - 1/\omega_1^2 \\
&= 1/z^2 + \sum_{u'\in L}(1/(z-u')^2 - 1/(u')^2) = \wp(z). \ \square
\end{aligned} \tag{265}$$

Definition 5.10: *Define the Eisenstein series G_{2n} with respect to a lattice $[\omega_1, \omega_2]$ as:*

$$G_{2n}([\omega_1, \omega_2]) = \sum_{u\in L} u^{-2n}. \tag{266}$$

When the lattice is clear we omit it and just write G_{2n}.

Theorem 5.11:

$$\wp'(z)^2 = 4\wp^3(z) - 60G_4\wp(z) - 140G_6. \tag{267}$$

Proof: Recall the geometric series $(1-x)^{-1} = \sum_{n\geq 0} x^n$. Differentiate both sides to obtain $(1-x)^{-2} = 1 + \sum_{n\geq 1}(n+1)x^n$. Now let $x = z/u$ where $|z| < |u|$. Then upon dividing both sides of the equation by $1/u^2$ we get

$$1/(z-u)^2 - 1/u^2 = \sum_{n\geq 1}(n+1)z^n/u^{n+2}. \tag{268}$$

Summing over all $u \in L$ gives:

$$\wp(z) = 1/z^2 + \sum_{n\geq 1}(n+1)G_{n+2}z^n \tag{269}$$

But since $\wp(z)$ is an even function,

$$\wp(z) = 1/z^2 + \sum_{n\geq 1}(2n+1)G_{2n+2}z^{2n}. \tag{270}$$

Differentiating term by term results in:

$$\wp'(z) = -2/z^3 + \sum_{n\geq 1} 2n(2n+1)G_{2n+2}z^{2n-1}. \tag{271}$$

With these expressions we can compute the first few terms of $\wp'(z)^2$, $4\wp^3(z)$, and $60G_4\wp(z)$[19]:

$$\wp'(z)^2 = 4/z^6 - 24G_4/z^2 - 80G_6 - \cdots \tag{272}$$

$$4\wp^3(z) = 4/z^6 + 36G_4/z^2 + 60G_6 + \cdots \tag{273}$$

$$60G_4\wp(z) = 60G_4/z^2 + 180G_4z^2 + 300G_6z^4 + \cdots \tag{274}$$

From this it is clear that

$$\wp'(z)^2 - 4\wp^3(z) + 60G_4\wp(z) = -140G_6 + \cdots \tag{275}$$

but the equation on the right is an elliptic function with no poles so by Theorem 5.2 is the constant $-140G_6$ and the result follows. □

Definition 5.12: *Define a new function $J(q,x)$ by*[12]

$$J(q,x) = (\theta_1'(q,x)/\theta_1(q,x))'. \tag{276}$$

Recall from Proposition 3.4 the expression found for $\theta_1'(q,x)/\theta_1(q,x)$. Differentiating term by term with respect to x gives

$$J(q,x) = \sum_{n\geq 0}(2n-1)\frac{(-4)^n}{(2n)!}B_{2n}E_{2n}x^{2n-2}. \tag{277}$$

Then clearly $J(q,x)$ is an even function with a double pole at $x=0$.

Theorem 5.13: *$J(q,x)$ is an elliptic function with periods $\pi, \pi\tau$.*

Proof: Differentiating Propositions 5.3 and 5.4 result in:

$$\theta_1'(q,x+\pi) = -\theta_1'(q,x) \tag{278}$$

$$\theta_1'(q,x+\pi\tau) = q^{-1}e^{-2ix}(2i\theta_1(q,x) - \theta_1'(q,x)) \tag{279}$$

The first expression implies

$$\theta_1'(q,x+\pi)/\theta_1(q,x+\pi) = \theta_1'(q,x)/\theta_1(q,x) \tag{280}$$

and taking the derivative of both sides shows $J(q, x+\pi) = J(q, x)$. Similarly, the second equation implies:

$$\begin{aligned} \theta_1'(q, x+\pi\tau)/\theta_1(q, x+\pi\tau) &= q^{-1}e^{-2ix}(2i\theta_1(q,x) - \theta_1'(q,x))/(-q^{-1}e^{-2ix}\theta_1(q,x)) \\ &= \theta_1'(q,x)/\theta_1(q,x) - 2i \end{aligned} \tag{281}$$

and likewise differentiating both sides with respect to x leads to
$J(q, x+\pi\tau) = J(q, x)$. $\square$

With these results we see that $J(q, x)$ is an even elliptic function that has a double pole at $x = 0$ in the fundamental parallelogram $P(\pi, \pi\tau)$. But $\wp(z, \pi\tau, \pi)$ is also an even elliptic function with a double pole at $z = 0$ and the same periods. So then $J(q, x) - C\wp(z)$ for some C is an entire elliptic function and so a constant by Theorem 5.2. So we have that

$$J(q, x) = C\wp(z) + d \tag{282}$$

Using the expansions derived above for $J(q, x)$ and $\wp(x)$ and comparing powers of x, the coefficients are clear and we have:

Theorem 5.14

$$J(q, x) = -\wp(x) - E_2/3 \tag{283}$$

where E_2 is as given in Section 3.

The above result also enables a proof of the assumption earlier that the definitions of G_{2n} given in Definition 3.2 and Lemma 4.4 are equivalent. Let G^*_{2n} denote the G_{2n} given in Definition 5.10 to distinguish from the G_{2n} in Definition 3.2.

Comparing the x^{2n} terms in $J(q,x)$ and $\wp(x)$ gives:

$$-G^*_{2n+2} = \frac{(-4)^{n+1}}{(2n+2)!} B_{2n+2} E_{2n+2}, \tag{284}$$

or

$$G^*_{2n} = (-1)^{n-1} \frac{(-4)^n}{(2n)!} B_{2n} E_{2n}. \tag{285}$$

Recalling the relation between G_{2n} and E_{2n} and that between $\zeta(2n)$ and B_{2n}, we get

$$G^*_{2n} = (-1)^{n-1} \frac{(-4)^n}{(2n)!} B_{2n} (-1)^{n+1} \frac{(2n)!}{B_{2n}(2\pi)^{2n}} G_{2n} \tag{286}$$

which upon simplifying is:

$$G^*_{2n} = G_{2n}/\pi^{2n}. \tag{287}$$

Now observe that since G^*_{2n} as given has periods $\pi\tau, \pi$, it can be written as:

$$G^*_{2n} = \sum_{(c,d)\neq(0,0)} (c\pi\tau + d\pi)^{-2n} \tag{288}$$

where as before $(c,d) \in \mathbb{Z}^2$. Pulling out the π^{-2n} from the sum and cancelling completes the desired equivalence.

Definition 5.15: *Define a new function $B(q,x)$ as[12]:*

$$B(q,x) = (\theta_4(q,x)/\theta_1(q,x))^2. \tag{289}$$

Similar to Proposition 5.3, it can be shown that $\theta_4(q, x+\pi) = \theta_4(q,x)$. By this and Proposition 5.3 then it is clear $B(q, x+\pi) = B(q,x)$. In a similar vein it can be shown like Proposition 5.4 that $\theta_4(q, x+\pi\tau) = -q^{-1}e^{-2ix}\theta_4(q,x)$. Then $B(q, x+\pi\tau) = B(q,x)$ and so $B(q,x)$ is an elliptic function with periods $\pi, \pi\tau$. By how everything is squared $B(q,x)$ is also an even function, and because from

Theorem 5.7 we know $\theta_1(q,x)$ has a zero at $x=0$ in $P(\pi\tau,\pi)$, it follows that $B(q,x)$ has a double pole at $x=0$.

Since $\theta_4(q,x)/\theta_1(q,x)$ has a pole of order 1 at $x=0$, suppose its expansion is of the form

$$\theta_4(q,x)/\theta_1(q,x) = c_{-1}/x + c_0 + \cdots \tag{290}$$

Multiplying both sides by x and taking the limit as $x \to 0$,

$$c_{-1} = \lim_{x\to 0} x(\theta_4(q,x)/\theta_1(q,x)) = \theta_4(q) \lim_{x\to 0} x/\theta_1(q,x) = \theta_4(q)/\theta_1'(q). \tag{291}$$

By Proposition 2.5 this simplifies to $c_{-1} = 1/(\theta_2(q)\theta_3(q))$. Now for notation let $\zeta = \theta_3^2(q)$. Then in the expansion of $\theta_4(q,x/\zeta)/\theta_1(q,x/\zeta)$ we find the $1/x$ coefficient to be:

$$c_{-1}\zeta = \theta_3(q)/\theta_2(q). \tag{292}$$

From this then the function

$$(\theta_2(q)\theta_4(q,x/\zeta))/(\theta_3(q)\theta_1(q,x/\zeta)) \tag{293}$$

has $c_{-1} = 1$.

Definition 5.16: *Define a new function $S(q,x)$ in the following way[12]:*

$$S(q,x) = \sqrt{\lambda(q)}B(q,x/\zeta). \tag{294}$$

By the discussion above, $S(q,x)$ is an even function with a double pole at $x=0$, and this $1/x^2$ term has coefficient 1. Also, in exactly the same manner that $B(q,x)$ was shown to be a elliptic function, $S(q,x)$ is an elliptic function with periods $\pi\zeta, \pi\zeta\tau$.

Just like $J(q,x)$, the properties of $S(q,x)$ indicate that it relates to $\wp(x, \pi\tau\zeta, \pi\zeta)$ as

$$S(q,x) = \wp(x, \pi\tau\zeta, \pi\zeta) + C. \tag{295}$$

Plugging in $x = \pi\tau\zeta/2$ reveals the constant C, for see that:

$$S(q, \pi\tau\zeta/2) = \sqrt{\lambda(q)}(\theta_4^2(q, \pi\tau/2))/(\theta_1^2(q, \pi\tau/2)). \tag{296}$$

It can be proven in the same way as Proposition 5.6 that
$\theta_4(q, x + \pi\tau/2) = iq^{-1/4}e^{-ix}\theta_1(q,x)$. But $\theta_1(q,0) = 0$ and so $S(q, \pi\tau\zeta/2) = 0$. From this we conclude that, with $\wp(x) = \wp(x, \pi\tau\zeta, \pi\zeta)$,

$$S(q,x) = \wp(x) - \wp(\pi\tau\zeta/2). \tag{297}$$

Now from Definition 5.8 and the above discussion involving G_{2n}, it is clear that

$$\wp(x, \pi\tau\zeta, \pi\zeta) = \zeta^{-2}\wp(x/\zeta, \pi\tau, \pi) \tag{298}$$

and so with $\wp(x) = \wp(x, \pi\tau, \pi)$:

$$\zeta^2 S(q,x) = \wp(x/\zeta) - \wp(\pi\tau/2). \tag{299}$$

Theorem 5.17:

$$S'(q,x)^2 = 4S(q,x)(S(q,x) - 1)(S(q,x) - \lambda(q)). \tag{300}$$

Proof: When τ is fixed write $S(x)$ for $S(q,x)$. Since $S(x)$ is an even elliptic function, differentiating term by term implies $S'(x)$ is an odd elliptic function. Now

for either period ω of $S'(x)$,

$$S'(\omega/2) = S'(\omega/2 - \omega) = S'(-\omega/2) = -S(\omega/2) \tag{301}$$

and so $S'(\omega/2) = 0$. This implies the zeros in $P(\pi\zeta\tau, \pi\zeta)$ are $\pi\zeta/2$, $\pi\zeta\tau/2$, and $\pi(\tau+1)\zeta/2$. By the fundamental theorem of algebra then, we have, in terms of $S(x)$:

$$S'(x)^2 = C(S(x) - S(\pi\zeta/2))(S(x) - S(\pi\zeta\tau/2))(S(x) - S(\pi(\tau+1)\zeta/2)). \tag{302}$$

Now for some lemmas to simplify the above equation.

Lemma 5.18:

$$S(q, \pi\zeta/2) = 1. \tag{303}$$

Proof:

$$S(q, \pi\zeta/2) = \sqrt{\lambda(q)}\theta_4^2(q, \pi/2)/\theta_1^2(q, \pi/2) = \sqrt{\lambda(q)}\theta_3^2(q)/\theta_2^2(q) = 1. \ \square \tag{304}$$

Lemma 5.19:

$$S(q, \pi(\tau+1)\zeta/2) = \lambda(q). \tag{305}$$

Proof:

$$\begin{aligned} S(q, \pi(\tau+1)\zeta/2) &= \sqrt{\lambda(q)}\theta_4^2(q, \pi/2 + \pi\tau/2)/\theta_1^2(q, \pi/2 + \pi\tau/2) \\ &= \sqrt{\lambda(q)}\theta_1(q, \pi/2)/\theta_4(q, \pi/2) = \sqrt{\lambda(q)}\theta_2^2(q)/\theta_3^2(q). \ \square \end{aligned} \tag{306}$$

Putting it all together, we have

$$S'(x)^2 = CS(x)(S(x) - 1)(S(x) - \lambda(q)). \tag{307}$$

To obtain the constant C, observe that since $S'(x) = \wp'(x)$ and so $S'(x)^2 = \wp'(x)^2$, from Theorem 5.11 it is clear that the x^{-6} term of $\wp'(x)^2$ has coefficient 4 and thus by equating powers of x so must the x^{-6} term of $S'(x)^2$ have coefficient 4. $\square$

Theorem 5.17 can be used to give an expansion of $S(x)$. From now on when the value of q is clear we denote $\lambda = \lambda(q)$. Now differentiating both sides of Theorem 5.17 leads to

$$2S'(x)S''(x) = 4S'(x)(S(x)-1)(S(x)-\lambda)+4S(x)S'(x)(S(x)-\lambda)+4S(x)(S(x)-1)S'(x) \tag{308}$$

or upon simplifying:

$$S''(x) = 2(S(x)-1)(S(x)-\lambda) + 2S(x)(S(x)-\lambda) + 2S(x)(S(x)-1). \tag{309}$$

Assuming $S(x) = 1/x^2 + c_0 + c_2x^2 + c_4x^4 + \cdots$, differentiating term by term twice gives

$$S''(x) = 6/x^4 + 2c_2 + 12c_4x^2 + \cdots \tag{310}$$

Plugging in these expansions to the expression for $S''(x)$ reveals:

$$\begin{aligned} &6/x^4 + 2c_2 + 12c_4x^2 + \cdots \\ &\quad = 2(1/x^2 + c_0 - 1 + c_2x^2 + c_4x^4 + \cdots)(1/x^2 + c_0 - \lambda + c_2x^2 + c_4x^4 + \cdots) \\ &\quad + 2(1/x^2 + c_0 + c_2x^2 + c_4x^4 + \cdots)(1/x^2 + c_0 - \lambda + c_2x^2 + c_4x^4 + \cdots) \\ &\quad + 2(1/x^2 + c_0 + c_2x^2 + c_4x^4 + \cdots)(1/x^2 + c_0 - 1 + c_2x^2 + c_4x^4 + \cdots) \end{aligned} \tag{311}$$

Comparing powers of x can lead to the solution of the c_i. For instance, comparing the coefficients of x^{-2} on both sides of the equation one gets $0 = 12c_0 - 4(\lambda + 1)$, which upon simplifying leads to $c_0 = (\lambda + 1)/3$. Similarly, comparing the constant

terms shows

$$2c_2 = 12c_2 + 6c_0^2 - 4c_0(\lambda + 1) + 2\lambda \tag{312}$$

which implies $c_2 = (\lambda^2 - \lambda + 1)/15$. One is unable to find c_4 in the same manner, however by using Theorem 5.17 the same method can be applied. The expansion becomes:

$$\begin{aligned}(-2/x^3 + 2c_2x + 4c_4x^3 + \cdots)^2 &= 4(1/x^2 + c_0 + c_2x^2 + c_4x^4 + \cdots) \\ &\times (1/x^2 + c_0 - 1 + c_2x^2 + c_4x^4 + \cdots)(1/x^2 + c_0 - \lambda + c_2x^2 + c_4x^4 + \cdots)\end{aligned} \tag{313}$$

Equating the constant terms on both sides gives

$$-16c_4 = 4(3c_4 + 6c_0c_2 - (2c_2 + c_0^2)(\lambda + 1) + c_0^3 + c_0\lambda) \tag{314}$$

and solving for c_4 results in $c_4 = (\lambda + 1)(\lambda - 2)(2\lambda - 1)/189$.

From the relation between $S(q, x)$ and $\wp(x, \pi\tau, \pi)$ derived earlier, the expansions for both functions and thus their coefficients of powers of x can be equated. For example, comparing the x^2 terms of $\zeta^2 S(q, x) = \wp(x/\zeta) - \wp(\pi\tau/2)$[12]:

$$\zeta^2(\lambda^2 - \lambda + 1)/15 = 3G_4/\zeta^2 = -2E_4B_4/\zeta^2 = E_4/(15\zeta^2). \tag{315}$$

This implies E_4 can be written in terms of theta functions $\theta_i(q)$ as $E_4 = \zeta^4(\lambda^2 - \lambda + 1)$. Likewise for the x^4 terms we have:

$$\zeta^2(\lambda + 1)(\lambda - 2)(2\lambda - 1)/189 = 5G_6/\zeta^4 = 4E_6B_6/(9\zeta^4) = 10E_6/(945\zeta^4) \tag{316}$$

and solving for E_6 gives $E_6 = \zeta^6(\lambda+1)(\lambda-1/2)(\lambda-2)$.

Lemmas 5.18 and 5.19 in conjunction with the relation between $S(x)$ and $\wp(x)$ lead to interesting additive results for $\wp(x)$. With $\wp(x)$ using the lattice $[\pi\tau, \pi]$, from Lemma 5.18 we see that $\zeta^2 = \wp(\pi/2) - \wp(\pi\tau/2)$. Similarly, by subtracting Lemma 5.19 from Lemma 5.18 and the result obtained for $S(\pi\tau\zeta/2)$,

$$\wp(\pi/2) - \wp(\pi(\tau+1)/2) = \zeta^2(1-\lambda) \tag{317}$$

$$\wp(\pi\tau/2) - \wp(\pi(\tau+1)/2) = -\zeta^2\lambda \tag{318}$$

In the same way that $\wp(x, \pi\tau\zeta, \pi\zeta) = \wp(x/\zeta, \pi\tau, \pi)/\zeta^2$, one can pull out the τ and obtain

$$\wp(x, \pi, \pi\tau) = \wp(x/\tau, \pi/\tau, \pi)/\tau^2. \tag{319}$$

Also, since the summation in $\wp(x)$ occurs over the whole lattice it is true that $[\omega, \omega\tau] = [-\omega, \omega\tau]$. From these two observations and that $\wp(x)$ is an even function it follows that:

$$\wp(\pi\tau/2, \pi\tau, \pi) = \wp(\pi/2, \pi, -\pi/\tau)/\tau^2 \tag{320}$$

$$\wp(\pi/2, \pi\tau, \pi) = \wp(-\pi/(2\tau), \pi, -\pi/\tau)/\tau^2 \tag{321}$$

From this, observe that as ζ depends on τ we can write $\zeta = \zeta(\tau)$. Then see:

$$\begin{aligned} \zeta^2(\tau) &= \wp(\pi/2, \pi\tau, \pi) - \wp(\pi\tau/2, \pi\tau, \pi) \\ &= (1/\tau^2)(\wp(\pi(-1/\tau)/2, \pi, \pi(-1/\tau)) - \wp(\pi/2, \pi, \pi(-1/\tau))) \\ &= (-1/\tau^2)\zeta^2(-1/\tau). \end{aligned} \tag{322}$$

Likewise,

$$\begin{aligned}(\zeta^2(1-\lambda))(\tau) &= \wp(\pi/2, \pi\tau, \pi) - \wp(\pi(1+\tau)/2, \pi\tau, \pi) \\ &= \wp(-\pi/(2\tau), \pi, -\pi/\tau)/\tau^2 - \wp((\pi/2)(-1-1/\tau)+\pi, \pi, -\pi/\tau)/\tau^2 \\ &= (1/\tau^2)(-\zeta^2\lambda)(-1/\tau).\end{aligned} \tag{323}$$

By the definition of ζ it is clear that $\zeta^2(\tau) = \theta_3(q)^4 = \theta_3(\tau)^4$. This also implies $\zeta^2(1-\lambda)(\tau) = \theta_4^4(\tau)$ and $\zeta^2\lambda(\tau) = \theta_2^4(\tau)$. Then the above results imply:

Proposition 5.20:

$$\theta_3^4(-1/\tau) = -\tau^2\theta_3^4(\tau) \tag{324}$$

$$\theta_2^4(-1/\tau) = -\tau^2\theta_4^4(\tau) \tag{325}$$

Here the θ_i are given in terms of τ instead of q to make the connection more obvious.

Corollary 5.21:

$$\lambda(e^{-i\pi/\tau}) = 1 - \lambda(e^{i\pi\tau}). \tag{326}$$

Proof: By Proposition 5.20 and Theorem 2.10,

$$\lambda(e^{-i\pi/\tau}) = \theta_2^4(-1/\tau)/\theta_3^4(-1/\tau) = \theta_4^4(\tau)/\theta_3^4(\tau) = 1 - \theta_2^4(\tau)/\theta_3^4(\tau) = 1 - \lambda(e^{i\pi\tau}). \ \square \tag{327}$$

6 Hypergeometric Series

Recall the derivations of E_4 and E_6 in terms of λ, ζ. There is a slight issue in that E_{2n} is given in terms of $q = q^* = e^{2\pi i\tau}$ while ζ and λ are given in terms of

$q = q^\dagger = e^{\pi i \tau}$. With this in mind, let $q = q^\dagger$ and observe that:

$$(q^2)\frac{dE_4(q^2)}{d(q^2)} = q\frac{dE_4}{2dq} = 2\zeta^3(\lambda^2 - \lambda + 1)q\frac{d\zeta}{dq} + (\zeta^4/2)(2\lambda - 1)q\frac{d\lambda}{dq} \tag{328}$$

$$q\frac{dE_6}{2dq} = 3\zeta^5(\lambda + 1)(\lambda - 1/2)(\lambda - 2)q\frac{d\zeta}{dq} + (3/2)\zeta^6(\lambda^2 - \lambda - 1/2)q\frac{d\lambda}{dq} \tag{329}$$

Theorem 6.1:

$$q\frac{d\lambda}{dq} = \zeta^2\lambda(1 - \lambda). \tag{330}$$

Proof: By Theorem 3.5 and the derivations of E_4 and E_6 in the previous section,

$$\begin{aligned} 3E_6 q\frac{dE_4}{2dq} - 2E_4 q\frac{dE_6}{2dq} &= E_4^3 - E_6^2 = \zeta^{12}(\lambda^2 - \lambda + 1)^3 - \zeta^{12}(\lambda + 1)^2(\lambda - 1/2)^2(\lambda - 2)^2 \\ &= (27/4)\zeta^{12}\lambda^2(\lambda - 1)^2. \end{aligned} \tag{331}$$

But by the derivatives given above now,

$$\begin{aligned} 3E_6 q\frac{dE_4}{2dq} - 2E_4 q\frac{dE_6}{2dq} &= 3\zeta^{10}(\lambda + 1)(\lambda - 1/2)^2(\lambda - 2)q\frac{d\lambda}{dq} \\ &\quad - 3\zeta^{10}(\lambda^2 - \lambda + 1)(\lambda^2 - \lambda - 1/2)q\frac{d\lambda}{dq} \\ &= (27/4)\zeta^{10}\lambda(1 - \lambda)q\frac{d\lambda}{dq}. \end{aligned} \tag{332}$$

Equating the two expressions and solving for $q\frac{d\lambda}{dq}$ completes the proof. $\square$

Combining Theorems 6.1 and 3.5 lead to a differential equation relating ζ and λ that will be useful later.

Theorem 6.2:

$$\lambda(1 - \lambda)\frac{d^2\zeta}{d\lambda^2} + (1 - 2\lambda)\frac{d\zeta}{d\lambda} - \zeta/4 = 0. \tag{333}$$

Proof: Writing Theorem 6.1 as

$$(q^2)\frac{d\lambda}{d(q^2)} = \zeta^2\lambda(1-\lambda)/2, \tag{334}$$

we see from Theorem 3.5 that:

$$(1/(q^2))\frac{d(q^2)}{d\lambda}(q^2)\frac{dE_4}{d(q^2)} = \frac{2}{\zeta^2\lambda(1-\lambda)}\frac{E_2E_4 - E_6}{3}, \tag{335}$$

or equivalently[12]

$$(\zeta^2\lambda(1-\lambda)/2)\frac{dE_4}{d\lambda} = (E_2E_4 - E_6)/3. \tag{336}$$

By the definition of E_4, it is clear that

$$\frac{dE_4}{d\lambda} = 4\zeta^3(\lambda^2 - \lambda + 1)\frac{d\zeta}{d\lambda} + \zeta^4(2\lambda - 1). \tag{337}$$

Plugging that into the above equation and solving for E_2 yields

$$E_2 = 6\zeta\lambda(1-\lambda)\frac{d\zeta}{d\lambda} + \zeta^2(1-2\lambda) \tag{338}$$

and differentiating term by term:

$$\frac{dE_2}{d\lambda} = 6\left(\frac{d\zeta}{d\lambda}\right)^2\lambda(1-\lambda) + 6\zeta(1-\lambda)\frac{d\zeta}{d\lambda} - 6\zeta\lambda\frac{d\zeta}{d\lambda} + 6\zeta\lambda(1-\lambda)\frac{d^2\zeta}{d\lambda^2} + 2\zeta\frac{d\zeta}{d\lambda}(1-2\lambda) - 2\zeta^2. \tag{339}$$

Then the expression

$$(\zeta^2\lambda(1-\lambda)/2)\frac{dE_4}{d\lambda} = (E_2^2 - E_4)/12 \tag{340}$$

can be written in terms of ζ, λ. First off,

$$\frac{\zeta^2\lambda(1-\lambda)}{2}\left[6\left(\frac{d\zeta}{d\lambda}\right)^2\lambda(1-\lambda)+6\zeta(1-\lambda)\frac{d\zeta}{d\lambda}-6\zeta\lambda\frac{d\zeta}{d\lambda}+6\zeta\lambda(1-\lambda)\frac{d^2\zeta}{d\lambda^2}\right.$$
$$\left.+2\zeta\frac{d\zeta}{d\lambda}(1-2\lambda)-2\zeta^2\right] \tag{341}$$
$$=\frac{\left(6\zeta\lambda(1-\lambda)\frac{d\zeta}{d\lambda}+\zeta^2(1-2\lambda)\right)^2-\zeta^4(\lambda^2-\lambda+1)}{12}$$

After factoring everything out and combining like terms, we get:

$$3\zeta^3\lambda(1-\lambda)(1-2\lambda)\frac{d\zeta}{d\lambda}+3\zeta^3\lambda^2(1-\lambda)^2\frac{d^2\zeta}{d\lambda^2}=\frac{3\zeta^4\lambda(1-\lambda)}{4}. \tag{342}$$

Dividing both sides by $3\zeta^3\lambda(1-\lambda)$ obtains the result. $\square$

If we define an operator $D_\lambda = \lambda\frac{d}{d\lambda}$, then by the product rule

$$D_\lambda^2\zeta = D_\lambda\zeta + \lambda^2\frac{d^2\zeta}{d\lambda}. \tag{343}$$

Then Theorem 6.2 can be written as:

$$(1-\lambda)(D_\lambda^2\zeta - D_\lambda\zeta)+(1-2\lambda)D_\lambda\zeta-\lambda\zeta/4=0. \tag{344}$$

Distributing out and solving for $D_\lambda^2\zeta$ gives:

$$D_\lambda^2\zeta=\lambda(D_\lambda+1/2)^2\zeta. \tag{345}$$

From this equation, a useful connection between theta functions and the hypergeometric series ${}_2F_1(a,b;c;z)$ defined in Definition 1.2 can be established.

Theorem 6.3 (Jacobi Inversion Theorem):

$$\zeta = {}_2F_1(1/2, 1/2; 1; \lambda). \tag{346}$$

Proof: Dividing Theorem 2.10 through by ζ^2 suggests that ζ can be written as a polynomial in λ. Assuming this, Theorem 6.2 becomes

$$D_\lambda^2\left(\sum_{n\geq 0} a_n\lambda^n\right) = \lambda(D_\lambda + 1/2)^2\left(\sum_{n\geq 0} a_n\lambda^n\right) \tag{347}$$

and comparing powers of λ shows:

$$n^2 a_n = (n - 1/2)^2 a_{n-1}. \tag{348}$$

But after some consideration, it becomes clear that this implies $\zeta = a_0 \cdot {}_2F_1(1/2, 1/2; 1; \lambda)$. Comparing the expansion of ζ as given in Proposition 2.6 with the expansion of ${}_2F_1(1/2, 1/2; 1; \lambda)$ shows $a_0 = 1$. $\square$

To make things less cumbersome, some notation is useful. Let ${}_2F_1(1/2, 1/2; 1; \lambda)$ be denoted as simply $F(\lambda)$. Also, by $F'(x)$ is meant $\frac{dF(x)}{dx}$. Now we prove an interesting identity between λ and $F(\lambda) = \zeta$.

Proposition 6.4: *Let $\lambda = \lambda(q)$ and $\lambda^* = \lambda(q^2)$. Then[11],*

$$2\lambda(1-\lambda)F'(\lambda)(1+\sqrt{1-\lambda}) - \lambda\sqrt{1-\lambda}F(\lambda) = 2\lambda^*(1-\lambda^*)F'(\lambda^*)(1+\sqrt{1-\lambda})^2. \tag{349}$$

Proof: Note that $\zeta^2 = \theta_3^4(q)$ and $\zeta^2\lambda = \theta_2^4(q)$. Then by Theorem 2.10,

$$\theta_4^4(q) = \zeta^2(1-\lambda). \tag{350}$$

By Theorem 6.3, Proposition 2.8 can be written:

$$2F(\lambda^*) = F(\lambda)(1+\sqrt{1-\lambda}). \tag{351}$$

Taking the derivative of both sides respect to λ results in

$$2F'(\lambda^*)\frac{d\lambda^*}{d\lambda} = F'(\lambda)(1+\sqrt{1-\lambda}) - F(\lambda)/(2\sqrt{1-\lambda}). \tag{352}$$

Now Theorem 6.1 implies

$$q\frac{d\lambda^*}{dq} = 2F^2(\lambda^*)\lambda^*(1-\lambda^*) \tag{353}$$

and dividing this by Theorem 6.1 gives:

$$\frac{d\lambda^*}{d\lambda} = 2\frac{F^2(\lambda^*)\lambda^*(1-\lambda^*)}{F^2(\lambda)\lambda(1-\lambda)}. \tag{354}$$

Plugging this in above, we see:

$$\begin{aligned}
2F'(\lambda^*)\left(2\frac{F^2(\lambda^*)\lambda^*(1-\lambda^*)(1+\sqrt{1-\lambda})^2}{4F^2(\lambda^*)\lambda(1-\lambda)}\right) &= F'(\lambda)(1+\sqrt{1-\lambda}) - F(\lambda)/(2\sqrt{1-\lambda}) \\
2F'(\lambda^*)\lambda^*(1-\lambda^*)(1+\sqrt{1-\lambda})^2 &= 2F'(\lambda)(1+\sqrt{1-\lambda})\lambda(1-\lambda) - F(\lambda)\lambda(1-\lambda)/\sqrt{1-\lambda} \\
2\lambda^*(1-\lambda^*)F'(\lambda^*)(1+\sqrt{1-\lambda})^2 &= 2\lambda(1-\lambda)F'(\lambda)(1+\sqrt{1-\lambda}) - \lambda\sqrt{1-\lambda}F(\lambda). \ \square
\end{aligned} \tag{355}$$

We are now able to begin to relate theta and hypergeometric functions to π. Letting $\tau = ir$ for some positive real number r, $e^{i\pi\tau}$ becomes $e^{-\pi r}$. Note that $\tau \in \mathbf{H}$ so this is still < 1, and calling it q leaves things convergent thus previous results hold. Denote $\lambda(e^{-\pi r})$ as λ_r for short.

Theorem 6.5: *For any $r \in \mathbb{R}^+$,*

$$1/\pi = \lambda_r(1-\lambda_r)(F'(1-\lambda_r)F(\lambda_r) + F'(\lambda_r)F(1-\lambda_r)). \tag{356}$$

Proof: From Corollary 5.21 and Theorem 6.3, it is clear that $\theta_3^2(e^{-\pi/r}) = F(1-\lambda_r)$. By Proposition 4.5, it is also true that $\theta_3^2(e^{-1/(ir)}) = rF(\lambda_r)$. It follows then that $r = F(1-\lambda_r)/F(\lambda_r)$. Differentiating both sides with respect to λ_r gives:

$$\frac{dr}{d\lambda_r} = -\frac{F'(1-\lambda_r)}{F(\lambda_r)} - \frac{F(1-\lambda_r)F'(\lambda_r)}{F^2(\lambda_r)}. \tag{357}$$

With $q = e^{-\pi r}$, note $\frac{dr}{d\lambda_r} = \frac{dr}{dq}\frac{dq}{d\lambda_r}$. But $\frac{dr}{dq} = \left(\frac{dq}{dr}\right)^{-1} = -1/(q\pi)$, and thus:

$$\frac{1}{q\pi}\frac{dq}{d\lambda_r} = \frac{F'(1-\lambda_r)}{F(\lambda_r)} + \frac{F(1-\lambda_r)F'(\lambda_r)}{F^2(\lambda_r)}. \tag{358}$$

Multiplying both sides by $q\frac{d\lambda_r}{dq}$, which by Theorem 6.1 is $F^2(\lambda_r)\lambda_r(1-\lambda_r)$, gives the result after simplifying[12]:

$$\begin{aligned} 1/\pi &= \left(\frac{F'(1-\lambda_r)}{F(\lambda_r)} + \frac{F(1-\lambda_r)F'(\lambda_r)}{F^2(\lambda_r)}\right)\left(F^2(\lambda_r)\lambda_r(1-\lambda_r)\right) \\ &= \lambda_r(1-\lambda_r)(F'(1-\lambda_r)F(\lambda_r) + F(1-\lambda_r)F'(\lambda_r)). \ \square \end{aligned} \tag{359}$$

7 The AGM and π

For $a, b \in \mathbb{R}^+$, recall the arithmetic mean $(a+b)/2$ and the geometric mean $\sqrt{ab}$. Define two sequences $\{a_n\}$ and $\{b_n\}$ by:

$$a_{n+1} = (a_n + b_n)/2 \tag{360}$$

$$b_{n+1} = \sqrt{a_n b_n} \tag{361}$$

where $n \geq 0$.

Proposition 7.1: *For $n > 0$,*

$$b_n \leq b_{n+1} \leq a_{n+1} \leq a_n. \tag{362}$$

Proof: $b_{n+1} \leq a_{n+1}$ is simply the arithmetic-geometric inequality. Now even though it may not be true that $b_0 \leq a_0$, by that same inequality it is true $b_1 \leq a_1$. With this in mind, for $n \geq 1$:

$$a_{n+1} = (a_n + b_n)/2 \leq (a_n + a_n)/2 = a_n \tag{363}$$

$$b_{n+1} = \sqrt{a_n b_n} \geq \sqrt{b_n b_n} = b_n \tag{364}$$

and the result follows. $\square$.

Proposition 7.1 implies $\{a_n\}$ is a monotone decreasing sequence bounded below by b_1 and that $\{b_n\}$ is a monotone increasing sequence bounded above by a_1. By the monotone convergence theorem then both sequences have limits, say $\lim_{n\to\infty}\{a_n\} = A$ and $\lim_{n\to\infty}\{b_n\} = B$. But by taking the limit for either of the two sequences defined above, it is clear that $A = B$.

Definition 7.2 (The AGM): *The arithmetic-geometric mean, or AGM, is the common limit of $\{a_n\}$ and $\{b_n\}$ described above. For $a_0 = a$ and $b_0 = b$, the AGM is denoted $M(a, b)$.*

Proposition 7.3: *For $r \in \mathbb{R}^+$,*

$$M(ra, rb) = rM(a, b). \tag{365}$$

Proof: Let $\{a_n\}$ ($\{b_n\}$) denote the arithmetic (geometric) sequence with $a_0 = ra$ and $b_0 = rb$. In addition, let $\{a'_n\}$ ($\{b'_n\}$) denote the arithmetic (geometric) sequence with $a_0 = a$ and $b_0 = b$. Observe that $a_1 = (ra + rb)/2 = r(a + b)/2 = ra'_1$ and $b_1 = \sqrt{rarb} = r\sqrt{ab} = rb'_1$. By induction this holds for a_n and so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} ra'_n = r \lim_{n\to\infty} a'_n = rM(a, b). \square \tag{366}$$

Proposition 7.4:

$$M(a, b) = ((a + b)/2)M(1, (2\sqrt{ab})/(a + b)). \tag{367}$$

Proof: Note that $\{a_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=1}^{\infty}$ converge to the same limit, and similarly for $\{b_n\}$. Then by this and Proposition 7.3,

$$M(a, b) = M((a + b)/2, \sqrt{ab}) = ((a + b)/2)M(1, (2\sqrt{ab})/(a + b)). \square \tag{368}$$

As an interesting example of two sequences satisfying the AGM requirements, let $r \in \mathbb{R}^+$ and let $q = e^{-\pi r}$. Set

$$a_n = \theta_3^2(q^{2^n}) \tag{369}$$

$$b_n = \theta_4^2(q^{2^n}) \tag{370}$$

By Proposition 2.6 we have, for $n \geq 1$, $a_{n+1} = (a_n + b_n)/2$ and by Theorem 2.4 we have $b_{n+1} = \sqrt{a_n b_n}$. Thus $\{a_n\}$ and $\{b_n\}$ satisfy the AGM and converge to a common limit. Additionally, $M(a_0, b_0) = M(a_1, b_1) = M(a_n, b_n)$. These sequences lead to another proof of a transformation formula for $F(\lambda)$.

Theorem 7.5: *With $\lambda^* = \lambda(q^2)$ and $q = e^{-\pi r}$,*

$$F(\lambda) = 2F(\lambda^*)/(1 + \sqrt{1-\lambda}). \tag{371}$$

Proof: Since $M(a_0, b_0) = M(a_n, b_n)$, Proposition 7.3 implies

$$\theta_3^2(q)M(1, \theta_4^2(q)/\theta_3^2(q)) = \theta_3^2(q^{2n})M(1, \theta_4^2(q^{2n})/\theta_3^2(q^{2n})). \tag{372}$$

Since $\lim_{n\to\infty} \theta_3^2(q^{2n}) = 1$ by q tending to 0, we have

$$\theta_3^2(q) = 1/M(1, \theta_4^2(q)/\theta_3^2(q)). \tag{373}$$

In the proof of Theorem 2.12 it was observed that $\theta_4^2(q)/\theta_3^2(q) = \sqrt{1-\lambda}$. Then it follows that:

$$\begin{aligned} F(\lambda) &= \frac{1}{M(1, \sqrt{1-\lambda})} = \frac{1}{M((1+\sqrt{1-\lambda})/2, \sqrt[4]{1-\lambda})} \\ &= \frac{2}{1+\sqrt{1-\lambda}} \frac{1}{M(1, 2\sqrt[4]{1-\lambda}/(1+\sqrt{1-\lambda}))} = \frac{2}{1+\sqrt{1-\lambda}} F(x) \end{aligned} \tag{374}$$

where $\sqrt{1-x} = 2\sqrt[4]{1-\lambda}/(1+\sqrt{1-\lambda})$. But this implies
$1 - x = 4\sqrt{1-\lambda}/(1+\sqrt{1-\lambda})^2$, or upon solving for x[12],

$$x = \frac{(1+\sqrt{1-\lambda})^2 - 4\sqrt{1-\lambda}}{(1+\sqrt{1-\lambda})^2} = \frac{1 + (1-\lambda) - 2\sqrt{1-\lambda}}{(1+\sqrt{1-\lambda})^2} = \left(\frac{1-\sqrt{1-\lambda}}{1+\sqrt{1-\lambda}}\right)^2 = \lambda(q^2) \tag{375}$$

by Theorem 2.12. □

With the relation between theta functions and the AGM established, an algorithm for computing π can be proven. This algorithm is attributed to many mathematicians, notably Gauss, Legendre, Brent and Salamin.

Theorem 7.6 (Brent-Salamin Algorithm): *Let $a_0 = 1, b_0 = 1/\sqrt{2}$. Let $\{a_n\}$ be the arithmetic mean sequence and $\{b_n\}$ the geometric mean sequence. Then, the sequence*

$$\pi_n = \frac{2a_n^2}{1 - \sum_{k=0}^{n} 2^k(a_k^2 - b_k^2)} \tag{376}$$

converges to π.

Proof: An outline of this can be found in [12]. From Corollary 5.21 with $\tau = i$, $\lambda(e^{-\pi}) = 1 - \lambda(e^{-\pi})$ and so $\lambda_1 = 1/2$. By Theorem 6.5, this implies

$$1/\pi = (1/2)(1-1/2)(F'(1-1/2)F(1/2) + F'(1/2)F(1-1/2)), \tag{377}$$

or

$$2/\pi = F'(1/2)F(1/2). \tag{378}$$

Let $G(\lambda_r) = 2\lambda_r(1-\lambda_r)F'(\lambda_r) + (1-\lambda_r)F(\lambda_r)$. Subtracting both sides by $(1-\lambda_r)F(\lambda_r)$, Proposition 6.4 can be written as:

$$\begin{aligned}
&\Big(G(\lambda_r) - (1-\lambda_r)F(\lambda_r)\Big) - \frac{\lambda_r\sqrt{1-\lambda_r}}{1+\sqrt{1-\lambda_r}}F(\lambda_r) = \Big(G(\lambda_r^*) - (1-\lambda_r^*)F(\lambda_r^*)\Big)(1+\sqrt{1-\lambda_r})\\
&G(\lambda_r) - F(\lambda_r)(1+\sqrt{1-\lambda_r} - \lambda_r) = (1+\sqrt{1-\lambda_r})G(\lambda_r^*) - 2F(\lambda_r)\sqrt{1-\lambda_r}\\
&G(\lambda_r) = (1+\sqrt{1-\lambda_r})G(\lambda_r^*) + F(\lambda)\left(\frac{1+\sqrt{1-\lambda_r} - \lambda_r - 2\sqrt{1-\lambda_r}(1+\sqrt{1-\lambda_r})}{1+\sqrt{1-\lambda_r}}\right)\\
&G(\lambda_r) = (1+\sqrt{1-\lambda_r})G(\lambda_r^*) - \sqrt{1-\lambda_r}F(\lambda_r)
\end{aligned} \tag{379}$$

where $\lambda_r^* = \lambda(e^{-2\pi r})$.

Now, let $A_n = \theta_3^2(q^{2^n})$ and $B_n = \theta_4^2(q^{2^n})$ as before. Then by Theorem 6.3, $A_n = F(\lambda_{2^n})$. In addition, by the proof of Proposition 6.4 we know that

$B_n/A_n = \sqrt{1-\lambda_{2^n}}$. Putting this into the rewrite of Proposition 6.4 above,

$$G(\lambda_{2^n}) = (1 + B_n/A_n)G(\lambda_{2^{n+1}}) - B_nF(\lambda_{2^n})/A_n = (2A_{n+1}/A_n)G(\lambda_{2^{n+1}}) - B_n, \quad (380)$$

or equivalently:

$$A_nB_n = 2A_{n+1}G(\lambda_{2^{n+1}}) - A_nG(\lambda_{2^n}). \quad (381)$$

Now, making use of the identity

$$ab = 2((a + b)/2)^2 - a^2 + (a^2 - b^2)/2, \quad (382)$$

The above equation becomes:

$$2A_{n+1}^2 - A_n^2 + (A_n^2 - B_n^2)/2 = 2A_{n+1}G(\lambda_{2^{n+1}}) - A_nG(\lambda_{2^n}) \quad (383)$$

which after multiplying both sides by 2^n simplifies to

$$2^{n+1}(A_{n+1}G(\lambda_{2^{n+1}}) - A_{n+1}^2) - 2^n(A_nG(\lambda_{2^n}) - A_n^2) = 2^{n-1}(A_n^2 - B_n^2). \quad (384)$$

Summing both sides from 0 to N causes most of the terms on the left to telescope and cancel out, leading to:

$$2^{N+1}(A_{N+1}G(\lambda_{2^{N+1}}) - A_{N+1}^2) - (A_0G(\lambda_1) - A_0^2) = \sum_{n=0}^{N} 2^{n-1}(A_n^2 - B_n^2). \quad (385)$$

By the product form of λ, it is clear that $\lambda_{2^n} \to 0$ as $n \to \infty$. Then by the definition of $G(\lambda_r)$, $G(\lambda_{2^n}) \to F(0) = 1$ as $n \to \infty$. Additionally, by the equation given for θ_3^2 in Proposition 2.6, $A_n \to 1$ as $n \to \infty$. Also observe that 2^n grows more slowly than e^{-2^n} falls to 0, so the limit is dominated by the behavior of A_n and $G(\lambda_{2^n})$. With

this in mind, passing to limit reveals:

$$-A_0 G(\lambda_1) + A_0^2 = \sum_{n\geq 0} 2^{n-1}(A_n^2 - B_n^2) \tag{386}$$

or equivalently

$$A_0 G(\lambda_1) = A_0^2\left(1 - \sum_{n\geq 0} 2^{n-1}((A_n/A_0)^2 - (B_n/A_0)^2)\right). \tag{387}$$

Let $a_n = A_n/A_0$ and $b_n = B_n/A_0$. Then $a_0 = 1$ and $b_0 = \sqrt{1-\lambda_1} = \sqrt{1-1/2} = 1/\sqrt{2}$. Since $A_0 = \theta_3^2(e^{-\pi}) = F(\lambda(e^{-\pi})) = F(1/2)$ and $G(\lambda_1) = G(1/2) = F'(1/2)/2 + F(1/2)/2$, the equation can be written as:

$$F(1/2)(F'(1/2)/2+F(1/2)/2) = 1/\pi + F^2(1/2)/2 = F(1/2)^2\left(1 - \sum_{n\geq 0} 2^{n-1}(a_n^2 - b_n^2)\right). \tag{388}$$

Subtracting both sides by $F^2(1/2)/2$ and multiplying everything by 2 yields

$$2/\pi = F^2(1/2)\left(1 - \sum_{n\geq 0} 2^n(a_n^2 - b_n^2)\right). \tag{389}$$

Since $A_n = F(\lambda_{2^n})$, it follows that $a_n = F(\lambda_{2^n})/F(1/2)$. This implies that as $n \to \infty$, $a_n \to 1/F(1/2)$. From this the theorem follows, as

$$1/\pi = \lim_{N\to\infty}(1/(2a_N^2))\left(1 - \sum_{n=0}^{N} 2^n(a_n^2 - b_n^2)\right). \;\square \tag{390}$$

It is possible to perform two iterations of the above algorithm at once, creating a pi algorithm that converges twice as fast. The following iteration leads to another such algorithm.

Theorem 7.7: *Let $a_0 = 1, b_0 = 1/\sqrt[4]{2}$. Let $\{a_n\}$ be the arithmetic mean sequence and let*

$$b_{n+1} = \sqrt[4]{\frac{a_n^3 b_n + a_n b_n^3}{2}}. \tag{391}$$

for all $n \geq 0$. Then, the sequence[11]

$$\pi_n = \frac{4a_n^4}{2 - \sum_{k=0}^{n} 4^k (a_k^2 - b_k^2)(b_k^2 + 3a_k^2)} \tag{392}$$

converges to π.

The $\{b_n\}$ sequence is obtained by two iterations of the AGM, which has quadratic convergence. This is further detailed in[6] and implies Theorem 7.7 converges to π quartically.

As the proof of the Brent-Salamin algorithm made use of the relations between $\lambda(q)$ and $\lambda(q^2)$, before proving the theorem we will express $\lambda(q^4)$ in terms of $\lambda(q)$ and derive similar relations. By Theorem 2.12,

$$\lambda(q^4) = \left(\frac{1 - \sqrt{1 - \lambda(q^2)}}{1 + \sqrt{1 - \lambda(q^2)}}\right)^2 = \left(\frac{1 - \sqrt{1 - \left(\frac{1-\sqrt{1-\lambda(q)}}{1+\sqrt{1-\lambda(q)}}\right)^2}}{1 + \sqrt{1 - \left(\frac{1-\sqrt{1-\lambda(q)}}{1+\sqrt{1-\lambda(q)}}\right)^2}}\right)^2 = \left(\frac{1 - \sqrt{\frac{4\sqrt{1-\lambda(q)}}{\left(1+\sqrt{1-\lambda(q)}\right)^2}}}{1 + \sqrt{\frac{4\sqrt{1-\lambda(q)}}{\left(1+\sqrt{1-\lambda(q)}\right)^2}}}\right)^2$$
$$= \left(\frac{1 - \frac{2\sqrt[4]{1-\lambda(q)}}{1+\sqrt{1-\lambda(q)}}}{1 + \frac{2\sqrt[4]{1-\lambda(q)}}{1+\sqrt{1-\lambda(q)}}}\right)^2 = \left(\frac{\frac{\left(1-\sqrt[4]{1-\lambda(q)}\right)^2}{1+\sqrt{1-\lambda(q)}}}{\frac{\left(1+\sqrt[4]{1-\lambda(q)}\right)^2}{1+\sqrt{1-\lambda(q)}}}\right)^2 = \left(\frac{1 - \sqrt[4]{1 - \lambda(q)}}{1 + \sqrt[4]{1 - \lambda(q)}}\right)^4. \tag{393}$$

Now let λ denote $\lambda(q)$ and λ^* denote $\lambda(q^2)$ as before, and $\lambda^\dagger$ denote $\lambda(q^4)$. Recalling (eq. 351), we have:

$$(1+\sqrt{1-\lambda^*})F(\lambda^*) = 2F(\lambda^\dagger) \tag{394}$$

$$(1+\sqrt{1-\lambda^*})F(\lambda)(1+\sqrt{1-\lambda}) = 4F(\lambda^\dagger) \tag{395}$$

$$(1+\sqrt[4]{1-\lambda})^2 F(\lambda) = 4F(\lambda^\dagger) \tag{396}$$

where the last equality follows from (eq. 393). As in Proposition 6.4, we can take the derivative of both sides with respect to λ. This results in

$$F'(\lambda)(1+\sqrt[4]{1-\lambda})^2 - F(\lambda)\frac{(1+\sqrt[4]{1-\lambda})}{2\sqrt[4]{1-\lambda}^3} = 4F'(\lambda^\dagger)\frac{d\lambda^\dagger}{d\lambda}. \tag{397}$$

Proposition 7.8:

$$\frac{d\lambda(q^n)}{d\lambda(q)} = n \cdot \frac{F^2(\lambda(q^n))\lambda(q^n)(1-\lambda(q^n))}{F^2(\lambda(q))\lambda(q)(1-\lambda(q))} \tag{398}$$

Proof: By Theorems 6.1 and 6.3, $q\frac{d\lambda}{dq} = F^2(\lambda)\lambda(1-\lambda)$. Now,

$$\frac{d\lambda(q^n)}{d\lambda(q)} = q\frac{d\lambda(q^n)}{dq} \cdot q^{-1}\frac{dq}{d\lambda(q)} = \frac{d\lambda(q^n)}{dq} \cdot \frac{dq}{d\lambda(q)} \tag{399}$$

Additionally,

$$(q^n)\frac{d\lambda(q^n)}{d(q^n)} = q^n\frac{d\lambda(q^n)}{nq^{n-1}dq} = q\frac{d\lambda(q^n)}{ndq} \tag{400}$$

implies

$$q\frac{d\lambda(q^n)}{dq} = n \cdot F^2(\lambda(q^n))\lambda(q^n)(1-\lambda(q^n)). \tag{401}$$

Dividing these results completes the proof. □

With this Proposition established, (eq. 397) becomes

$$F'(\lambda)(1+\sqrt[4]{1-\lambda})^2 - F(\lambda)\frac{(1+\sqrt[4]{1-\lambda})}{2\sqrt[4]{1-\lambda}^3} = 4F'(\lambda^\dagger)\cdot 4\frac{F^2(\lambda^\dagger)\lambda^\dagger(1-\lambda^\dagger)}{F^2(\lambda)\lambda(1-\lambda)}. \tag{402}$$

But by (eq. 396) $F^2(\lambda^\dagger) = F^2(\lambda)(1+\sqrt[4]{1-\lambda})^4/16$ and so:

$$F'(\lambda)(1+\sqrt[4]{1-\lambda})^2 - F(\lambda)\frac{(1+\sqrt[4]{1-\lambda})}{2\sqrt[4]{1-\lambda}^3} = F'(\lambda^\dagger)\frac{\lambda^\dagger(1-\lambda^\dagger)}{\lambda(1-\lambda)}(1+\sqrt[4]{1-\lambda})^4. \tag{403}$$

This equation is a direct analogue of Proposition 6.4. Using $G(\lambda_r) = 2\lambda_r(1-\lambda_r)F'(\lambda_r) + (1-\lambda_r)F(\lambda_r)$ as in Theorem 7.6, the above analogue can be written:

$$F'(\lambda)(1+\sqrt[4]{1-\lambda})^2 - F(\lambda)\frac{(1+\sqrt[4]{1-\lambda})}{2\sqrt[4]{1-\lambda}^3} = F'(\lambda^\dagger)\frac{\lambda^\dagger(1-\lambda^\dagger)}{\lambda(1-\lambda)}(1+\sqrt[4]{1-\lambda})^4 \tag{404}$$

$$G(\lambda) - F(\lambda)(1-\lambda) - F(\lambda)\frac{\lambda(1-\lambda)}{\sqrt[4]{1-\lambda}^3(1+\sqrt[4]{1-\lambda})} = \Big[G(\lambda^\dagger) - F(\lambda^\dagger)(1-\lambda^\dagger)\Big](1+\sqrt[4]{1-\lambda})^2 \tag{405}$$

$$\begin{aligned} G(\lambda) - F(\lambda)(1-\lambda) - F(\lambda)\frac{\lambda\sqrt[4]{1-\lambda}}{(1+\sqrt[4]{1-\lambda})} + F(\lambda)\frac{(1+\sqrt[4]{1-\lambda})^4 - (1-\sqrt[4]{1-\lambda})^4}{4} \\ = G(\lambda^\dagger)(1+\sqrt[4]{1-\lambda})^2 \end{aligned} \tag{406}$$

$$\begin{aligned} G(\lambda) - F(\lambda)(1-\lambda) - F(\lambda)\frac{\lambda\sqrt[4]{1-\lambda}}{(1+\sqrt[4]{1-\lambda})} + F(\lambda)\cdot 2\sqrt[4]{1-\lambda}(1+\sqrt{1-\lambda}) \\ = G(\lambda^\dagger)(1+\sqrt[4]{1-\lambda})^2 \end{aligned} \tag{407}$$

$$G(\lambda) + F(\lambda)\sqrt[4]{1-\lambda}\Big[2(1+\sqrt{1-\lambda}) - \sqrt[4]{1-\lambda}^3 - \frac{\lambda}{(1+\sqrt[4]{1-\lambda})}\Big] = G(\lambda^\dagger)(1+\sqrt[4]{1-\lambda})^2 \tag{408}$$

$$G(\lambda) + F(\lambda)\sqrt[4]{1-\lambda}\Big[\frac{(1+\sqrt[4]{1-\lambda})^2 + \sqrt{1-\lambda} + \sqrt[4]{1-\lambda}^3}{(1+\sqrt[4]{1-\lambda})}\Big] = G(\lambda^\dagger)(1+\sqrt[4]{1-\lambda})^2 \tag{409}$$

$$G(\lambda) + F(\lambda)\sqrt[4]{1-\lambda}(1+\sqrt[4]{1-\lambda}+\sqrt{1-\lambda}) = G(\lambda^\dagger)(1+\sqrt[4]{1-\lambda})^2 \tag{410}$$

With this result we are able to prove a second π-algorithm.

Proof of Theorem 7.7: In a similar manner to the proof of Theorem 7.6, let $A_n = \theta_3(e^{-4^n\pi})$, $B_n = \theta_4(e^{-4^n\pi})$ and $\lambda_r = \lambda(e^{-\pi r})$. Then

$$\lambda_{4^n} = 1 - \left(\frac{B_n}{A_n}\right)^4 \tag{411}$$

and so $B_n/A_n = \sqrt[4]{1-\lambda_{4^n}}$. Plugging this in to (eq. 410),

$$G(\lambda_{4^n}) + F(\lambda_{4^n})\frac{B_n}{A_n}(1 + B_n/A_n + B_n^2/A_n^2) = G(\lambda_{4^{n+1}})(1 + B_n/A_n)^2. \tag{412}$$

Multiplying both sides by $A_n^3/A_n = A_n^2$ and noticing that by definition $F(\lambda_{4^n}) = A_n^2$, this simplifies to:

$$4G(\lambda_{4^{n+1}})A_{n+1}^2 = G(\lambda_{4^n})A_n^2 + A_nB_n(A_n^2 + A_nB_n + B_n^2). \tag{413}$$

Now, by expanding and combining terms we see that

$$\begin{aligned} A_nB_n(A_n^2 + A_nB_n + B_n^2) &= (A_n + B_n)^4/4 - A_n^4/4 - An^2B_n^2/2 - B_n^4/4 \\ &= 4A_{n+1}^4 - A_n^4 + \frac{1}{4}(A_n^2 - B_n^2)(B_n^2 + 2A_n^2) \end{aligned} \tag{414}$$

and so (eq. 413) is equivalent to

$$4G(\lambda_{4^{n+1}})A_{n+1}^2 = G(\lambda_{4^n})A_n^2 + 4A_{n+1}^4 - A_n^4 + \frac{1}{4}(A_n^2 - B_n^2)(B_n^2 + 3A_n^2) \tag{415}$$

$$4(G(\lambda_{4^{n+1}})A_{n+1}^2 - A_{n+1}^4) - (G(\lambda_{4^n})A_n^2 - A_n^4) = \frac{1}{4}(A_n^2 - B_n^2)(B_n^2 + 3A_n^2) \tag{416}$$

Multiply all terms by 4^n,

$$4^{n+1}(G(\lambda_{4^{n+1}})A_{n+1}^2 - A_{n+1}^4) - 4^n(G(\lambda_{4^n})A_n^2 - A_n^4) = 4^{n-1}(A_n^2 - B_n^2)(B_n^2 + 3A_n^2), \tag{417}$$

and then sum both sides from 0 to N as was done in Theorem 7.6:

$$4^{N+1}(G(\lambda_{4^{N+1}})A_{N+1}^2 - A_{N+1}^4) - (G(\lambda_1)A_0^2 - A_0^4) = \sum_{n=0}^{N} 4^{n-1}(A_n^2 - B_n^2)(B_n^2 + 3A_n^2) \quad (418)$$

Taking $\lim_{N\to\infty}$ and recalling the values obtained in the proof of Theorem 7.6,

$$-G(\lambda_1)A_0^2 + A_0^4 = \sum_{n=0}^{\infty} 4^{n-1}(A_n^2 - B_n^2)(B_n^2 + 3A_n^2), \quad (419)$$

or:

$$G(\lambda_1)A_0^2 = A_0^4 - \sum_{n=0}^{\infty} 4^{n-1}(A_n^2 - B_n^2)(B_n^2 + 3A_n^2) \quad (420)$$

$$G(\lambda_1)A_0^2 - A_0^4/2 = A_0^4\left(\frac{1}{2} - \sum_{n=0}^{\infty} 4^{n-1}(a_n^2 - b_n^2)(b_n^2 + 3a_n^2)\right) \quad (421)$$

where $a_n = A_n/A_0$ and $b_n = B_n/B_0$. Note $a_0 = 1$ and $b_0 = \sqrt[4]{1-\lambda_1} = 1/\sqrt[4]{2}$. Continuing on by multiplying everything by 4,

$$4G(\lambda_1)A_0^2 - 2A_0^4 = A_0^4\left(2 - \sum_{n=0}^{\infty} 4^{n}(a_n^2 - b_n^2)(b_n^2 + 3a_n^2)\right) \quad (422)$$

Using the values for $G(\lambda_1)$ and A_0^2 obtained earlier, this simplifies to[11]

$$\frac{4}{\pi} = A_0^4\left(2 - \sum_{n=0}^{\infty} 4^{n}(a_n^2 - b_n^2)(b_n^2 + 3a_n^2)\right) \quad (423)$$

$$\frac{1}{\pi} = \lim_{N\to\infty} \frac{1}{4a_N^4}\left(2 - \sum_{n=0}^{N} 4^{n}(a_n^2 - b_n^2)(b_n^2 + 3a_n^2)\right) \quad (424)$$

which upon truncating is the reciprocal of the desired result. $\Box$

8 Concluding Remarks

We conclude with a couple of comments on a few directions the theories developed here have gone recently. In the introduction it was remarked that most calculations for π involve the use of a Ramanujan-esque infinite series. This is not due to such series converging faster than the iterative techniques developed here, but rather because iterative algorithms need to store the previous information and this leads to inordinate computer memory usage. The most impressive of such series discovered so far is due to the Chudnovskys[14],

$$\frac{1}{\pi} = 12 \sum_{n \geq 0} \frac{(-1)^n (6n)!(545140134n + 13591409)}{(3n)!(n!)^3 (640320)^{3n+3/2}} \tag{425}$$

and yields 14 more digits correct per term added. This formula has been the basis for all record calculations of π since 2010. In 2022 an algorithm for this formula was used to calculate π to 100 trillion digits, the current record. On the theoretical side, recently q-analogues of such series to π have been discovered[13]; the derivation of such formulae use transformation and inversion identities belonging firmly in the realm of basic hypergeometric series.

The Borweins developed analogues of Ramanujan's theta functions $\phi(q), \psi(q)$ and used them along with elliptic integral theory to create several iterative algorithms for π of a different nature than Theorems 7.6 and 7.7. Their algorithms have various convergence rates that range from quadratic to nonic (9th order) convergence. A discussion of these π algorithms and their bit complexity can be found in [4],[6].

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