2023

The Lie algebra sl2(C) and Krawtchouk polynomials

NKosi Alexander
n01067718@unf.edu

The Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and Krawtchouk polynomials

by

NKosi Alexander-Williams

A thesis submitted to the Department of Mathematics and Statistics in partial fulfillment of the requirements for the degree of

Master of Science in Mathematics

UNIVERSITY OF NORTH FLORIDA

COLLEGE OF ARTS AND SCIENCES

April, 2023

Acknowledgements

I would like to start by thanking my mother and sister for being with me every step of the way in preparing for this thesis. Every single day, they would find ways to boost my confidence levels and help me get the job done. No matter how stressful the thesis process became, they were right there by my side, helping me develop the work-life balance required to succeed. I am truly thankful to have such love. Of course, I would like to thank UNF's Mathematics and Statistics department. While numerous professors from the department helped me reach my academic goals, I would particularly like to thank Dr. Jae-Ho Lee, Dr. Ognjen Milatovic, Dr. Mei-Qin Zhan, and Dr. Yisu Jia. I thank Dr. Lee for being in my corner ever since I took him for Abstract Algebra I during my senior year of undergrad. I appreciate how he took me on as one of his thesis students, giving us the opportunity to produce this awesome paper. Dr. Ognjen Milatovic has also greatly contributed to my academic success. I had the pleasure of being his student during my first semester at the university. From that semester onwards, he has given me plenty of advice regarding my career, and was generous enough to be part of my thesis committee. Milatovic thoroughly read my paper, giving key pointers that contributed to our prodigious outcome. I would also like to thank Dr. Mei-Qin Zhan. Zhan was the professor that started my entire college journey after my high school graduation. Since my very first college class at the university, he has provided me with great emotional and professional support. I undoubtedly asked him to be part of my thesis committee, and he immediately agreed. Akin to Milatovic, Zhan read through the paper and ensured that all material presented was clear and smooth; this helped produce the great outcome. In addition, I want to thank Dr. Yisu Jia. Whenever she saw me stressing out, she told me how I would always conquer, giving me confidence to keep going. Not only did she help me throughout the thesis process, but my undergraduate and graduate journeys, as well. Furthermore, I would like to thank UNF's Department of Teaching, Learning and Curriculum. In particular, I thank Dr. Lunetta Williams. Multiple times each

week, Williams would check-in with me to see how I was doing emotionally. She would always provide encouragement, and told me how I would complete the thesis when I did not think that I could. A superhero in the world of academia, she has no problem proving it.

Contents

Abstract

The Lie algebra $L = \mathfrak{sl}_2(\mathbb{C})$ consists of the 2×2 complex matrices that have trace zero, together with the Lie bracket $[y, z] = yz - zy$. In this thesis we study a relationship between L and Krawtchouk polynomials. We consider a type of element in L said to be normalized semisimple. Let a, a^* be normalized semisimple elements that generate L. We show that a, a^* satisfy a pair of relations, called the Askey-Wilson relations. For a positive integer N, we consider an $(N+1)$-dimensional irreducible L-module V consisting of the homogeneous polynomials in two variables that have total degree N. We define a certain nondegenerate symmetric bilinear form $\langle\ ,\ \rangle$ on V. We display two bases for V, denoted $\{v_i\}_{i=1}^N$ and $\{v_i^*\}_{i=1}^N$, each basis diagonalizes a and a^*, respectively. We show that each of these bases is orthogonal with respect to $\langle\ ,\ \rangle$ and also show that $\langle v_i, v_j^* \rangle = K_i(j; p, N)$, $i, j = 0, 1, 2, \ldots, N$, where $K_i(j; p, N)$ is the ith Krawtchouk polynomial with parameters N and p. Using these results we find some well-known facts about Krawtchouk polynomials including the three-term recurrence, the orthogonality, the difference equation, and the generating function.

1. Introduction

In this thesis, we explore the connection between the special linear Lie algebra $\mathfrak{sl}_2$ and Krawtchouk polynomials. Throughout our paper, we use $\mathbb{C}$ to refer to the complex field. We begin by recalling the definition of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and the Krawtchouk polynomial.

The *Lie algebra* $\mathfrak{sl}_2(\mathbb{C})$ is a three-dimensional $\mathbb{C}$-vector space of complex 2×2 matrices with a zero trace, given by

$$\mathfrak{sl}_2(\mathbb{C}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a, b, c, d \in \mathbb{C} \text{ and } a + d = 0 \right\}$$

and with the binary operation $[X, Y] = XY - YX$.

Let x be an indeterminate and $\mathbb{C}[x]$ be the $\mathbb{C}$-algebra that contains all the polynomials in x with coefficients in $\mathbb{C}$. Fix a nonnegative integer $N \geq 0$ and let $0 \neq p \in \mathbb{C}$. For $i = 0, 1, 2, \ldots, N$ define a polynomial K_i in $\mathbb{C}[x]$ by

$$K_i = K_i(x) = K_i(x; p, N) = \sum_{n=0}^{\infty} \frac{(-i)_n(-x)_n}{(-N)_n} \cdot \frac{p^{-n}}{n!}, \tag{1}$$

where $(\alpha)_n$ is the shifted factorial

$$(\alpha)_n = \alpha \cdot (\alpha + 1) \cdot (\alpha + 2) \ldots (\alpha + n - 1), \text{ where } n = 0, 1, 2, \ldots.$$

We interpret $(\alpha)_0 = 1$. These polynomials are well-defined. Observe that $(-i)_n$ vanishes for $n > i$, so the n-summand in (1) is zero for $n > i$. Observe also that $(-N)_n$ is nonzero for $n = 0, 1, 2, \ldots, i$, so the n-summand in (1) has nonzero denominator for $n = 0, 1, 2, \ldots, n$. We note that the polynomial K_i has degree i and the coefficient of x^i is $\frac{1}{(-N)_i p^i}$. We call K_i the ith *Krawtchouk polynomial* with parameters N and p. From construction, we find

$$K_i(j) = K_j(i) \qquad \text{for } i, j = 0, 1, 2, \ldots, N. \tag{2}$$

This is a special case of an example known as Askey-Wilson duality [**?**].

The relationship between Krawtchouk polynomials and $\mathfrak{sl}_2$ has a long-standing history of being studied. It was first discovered by Willard Miller Jr. in 1969 when he noticed that the difference equations for Krawtchouk polynomials come from the irreducible representations of $\mathfrak{sl}_2$. In 1982, Tom H. Koornwinder discovered that the matrix elements of a finite-dimensional irreducible representation of the group $SU(2)$ can be expressed in terms of Krawtchouk polynomials. Since the irreducible representations of $SU(2)$ and $\mathfrak{sl}_2$ are essentially the same, Koornwinder's discovery established a connection between Krawtchouk polynomials and $\mathfrak{sl}_2$. Several articles, such as those by Feinsilver ([4], [5], and [3]), subsequently provided a connection between Krawtchouk polynomials and $\mathfrak{sl}_2$, with the pair

$$S = \begin{bmatrix} 0 & 1 & & & & \\ N & 0 & 2 & & & \\ & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \\ & & & 2 & 0 & N \\ & & & & 1 & 0 \end{bmatrix}, \qquad D = \begin{bmatrix} N & & & & & \\ & N-2 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & 2-N & \\ & & & & & -N \end{bmatrix}$$

acting as a bridge between the two. From one perspective, the matrix S (resp. D) represents the action of $e+f$ (resp. h) on the irreducible $\mathfrak{sl}_2$-module with dimension $N+1$, where e, f, and h denote a standard Chevalley basis for $\mathfrak{sl}_2$. Alternatively, James Joseph Sylvester found that the matrix S has distinct and non-repeating eigenvalues $\{N-2i\}_{i=0}^{N}$. This was later recalled by Richard Askey in the early 2000s, who noted that because S has such eigenvalues, there exists an invertible matrix P such that $PSP^{-1} = D$. In the 1940s, Mark Kac discovered that after suitable normalization, the entries of P are given by:

$$Pij = \binom{N}{i} K_i\left(j; \tfrac{1}{2}, N\right), \qquad \text{for} \quad i, j = 0, 1, \ldots, N,$$

where $K_i(j; 1/2, N)$ is the Krawtchouk polynomial of degree i. Interestingly, this discovery was initially made in probability theory and later found applications in

combinatorics in the context of the Hamming association scheme. In 2012, Nomura and Terwilliger described a connection between $\mathfrak{sl}_2$ and the Krawtchouk polynomial based on the theory of Leonard pairs [2]. This thesis is based on the paper [2].

We summarize the results of this thesis as follows. We discuss a particular element in $\mathfrak{sl}_2(\mathbb{C})$ that is known as a normalized semisimple element. The thesis focuses on a pair of normalized semisimple elements a and a^* that generate $\mathfrak{sl}_2(\mathbb{C})$. We show that a and a^* satisfy a pair of relations, which are called the Askey-Wilson relations:

$$[a,[a,a^*]] = 4(2p-1)a + 4a^*,$$

$$[a^*,[a^*,a]] = 4(2p-1)a^* + 4a.$$

We note that the scalar p depends on the $\mathfrak{sl}_2(\mathbb{C})$ Killing form applied to a and a^*. Moreover, we prove that a and a^* generate $\mathfrak{sl}_2(\mathbb{C})$. We also show that $\mathfrak{sl}_2(\mathbb{C})$ induces an antiautomorphism $\dagger$ that fixes a and a^*.

Next, we consider an $(N+1)$-dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$-module V consisting of homogeneous polynomials in two variables of total degree N. We define a nondegenerate symmetric bilinear form $\langle\ ,\ \rangle$ on V such that $\langle \varphi.u, v\rangle = \langle u, \varphi^\dagger.v\rangle$ for all $\varphi \in \mathfrak{sl}_2(\mathbb{C})$ and $u, v \in V$. We present two bases for V, denoted $\{v_i\}_{i=1}^N$ and $\{v_i^*\}_{i=1}^N$, each of which diagonalizes a and a^*, respectively. We also show that each basis is orthogonal with respect to $\langle\ ,\ \rangle$. Furthermore, we demonstrate that

$$\langle v_i, v_j^*\rangle = K_i(j;p,N) \qquad\qquad i,j = 0,1,2,\dots,N,$$

where $K_i(j;p,N)$ is the ith Krawtchouk polynomial with parameters N and p. Using these results, we can recover well-known properties of Krawtchouk polynomials, including the three-term recurrence, orthogonality, the difference equation, and the generating function. We often display this information through matrices.

This thesis is organized as follows. In Section 2, we provide some background information about the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ and its representation. In Section 3, we discuss normalized semisimple elements of $\mathfrak{sl}_2(\mathbb{C})$ and show how these elements can

be used to obtain a new basis for $\mathfrak{sl}_2(\mathbb{C})$. In Section 4, we present three different bases for $\mathfrak{sl}_2(\mathbb{C})$ and discuss the transition matrices between two of the three bases. In Section 5, we introduce an antiautomorphism of $\mathfrak{sl}_2(\mathbb{C})$ and discuss its properties. In Section 6, we construct an irreducible $\mathfrak{sl}_2(\mathbb{C})$-module V consisting of homogeneous polynomials in two variables of total degree N. We discuss the actions of the standard basis for $\mathfrak{sl}_2(\mathbb{C})$ on V. In Section 7, we demonstrate the action of the dual standard basis for $\mathfrak{sl}_2(\mathbb{C})$ on the module V. We define a nondegenerate symmetric bilinear form $\langle\ ,\ \rangle$ on V and construct two bases $\{y^{N-i}z^i\}_{i=0}^N$ and $\{y^{*N-i}z^{*i}\}_{i=0}^N$ on V. We show that each of these bases is orthogonal with respect to $\langle\ ,\ \rangle$. In Section 8, we determine the inner products between the bases $\{y^{N-i}z^i\}_{i=0}^N$ and $\{y^{*N-i}z^{*i}\}_{i=0}^N$ and show how the Krawtchouk polynomials act on V. Finally, in Section 9, we conclude by recapturing well-known properties of the Krawtchouk polynomials using the results in Section 8.

2. The Lie Algebra $\mathfrak{sl}_2(\mathbb{C})$

In this section, we recall some background about Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. We begin with the definition of Lie algebra.

Definition 2.1. A *Lie algebra* $\mathfrak{g}$ is a $\mathbb{C}$-vector space, together with a bilinear map, called the Lie bracket

$$\mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}, \qquad (x, y) \longmapsto [x, y],$$

satisfying the following properties:

(i) $[x, x] = 0$ for all x in $\mathfrak{g}$,

(ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

The condition (ii) is known as the *Jacobi identity*. Since the Lie bracket $[-,-]$ is bilinear, by the condition (i) we have

$$[x, y] = -[y, x] \quad \text{for all } x, y \in L.$$

We define the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$.

Definition 2.2. The *Lie algebra* $\mathfrak{sl}_2(\mathbb{C})$ is the $\mathbb{C}$-vector space consisting of 2×2 complex matrices with trace zero, where the Lie bracket is defined by

$$[x, y] = xy - yx.$$

Throughout this thesis, we abbreviate $L = \mathfrak{sl}_2(\mathbb{C})$. We note that the Lie algebra L is three dimensional and has a basis $\{e, h, f\}$, where

$$e : \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad h : \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad f : \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \tag{3}$$

Therefore, for any element $y \in L$, we write

$$y = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} = \beta e + \alpha h + \gamma f. \tag{4}$$

We call the basis $\{e, h, f\}$ the *standard basis* for L.

Lemma 2.3. *We have*

$$[h, e] = 2e, \qquad [h, f] = -2f, \qquad [e, f] = h. \tag{5}$$

Proof. Using (3) we have

$$[h,e] = he - eh$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 0 \end{bmatrix} = 2e,$$
$$[h,f] = hf - fh$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & 0 \end{bmatrix} = -2f,$$
$$[e,f] = ef - fe$$
$$= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = h.$$

□

For $y \in L$, define the $\mathbb{C}$-linear map

$$ad_y : L \longrightarrow L, \qquad ad_y(z) = [y,z]. \tag{6}$$

We call ad_y the *adjoint* map with respect to y. For example, for the basis element $h \in L$ we have

$$ad_h(e) = [h,e] = 2e,$$
$$ad_h(h) = [h,h] = 0$$
$$ad_h(f) = [h,f] = -2f.$$

For $y \in L$, let $[ad_y]$ denote the matrix representing ad_y with respect to the standard basis $\{e, h, f\}$.

Lemma 2.4. *For each $y \in \{e, h, f\}$, the matrix representing ad_y with respect to $\{e, h, f\}$ is given as follows.*

$$[ad_e] = \begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad [ad_h] = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \qquad [ad_f] = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}.$$

Proof. Use (5). $\square$

We now recall the *K*illing form. The Killing form is a bilinear form

$$(\ ,\ ) : L \times L \longrightarrow \mathbb{C}, \qquad (x, y) = tr([ad_x][ad_y]),$$

where tr means trace. We compute several values of the Killing form $(\ ,\ )$ on the elements (3). We have

$$\begin{aligned}(e, f) = tr([ad_e][ad_f]) &= tr\left(\begin{bmatrix} 0 & -2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 2 & 0 \end{bmatrix}\right) \\ &= tr\begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 4.\end{aligned}$$

In a similar way, we have

$$(e, h) = tr([ad_e][ad_h]) = 0,$$

$$(f, h) = tr([ad_f][ad_h]) = 0,$$

$$(h, h) = tr([ad_h][ad_h]) = 8.$$

Computing all possible Killing form combinations, we get

(,)	e	h	f
e	0	0	4
h	0	8	0
f	4	0	0

. (7)

For computational convenience in the thesis, we define a bilinear form by normalizing the Killing form:

$$\langle\ ,\ \rangle : L \times L \longrightarrow \mathbb{C}, \qquad \langle y, z\rangle = \frac{1}{8}(y, z). \tag{8}$$

We abbreviate $\|y\|^2 = \langle y, y\rangle$ for $y \in L$. With respect to the bilinear form $\langle\ ,\ \rangle$ the table (7) is given as

$\langle\ ,\ \rangle$	e	h	f
e	0	0	1/2
h	0	1	0
f	1/2	0	0

. (9)

Lemma 2.5. *Let $y \in L$ be as in* (4)*. Then we have*

$$\|y\|^2 = \alpha^2 + \beta\gamma = -\det(y). \tag{10}$$

Proof. We note that $y = \beta e + \alpha h + \gamma f$. Using (9) we have

$$\|y\|^2 = \langle \beta e + \alpha h + \gamma f, \beta e + \alpha h + \gamma f\rangle = \alpha^2 + \beta\gamma.$$

Since $\det(y) = -\alpha^2 - \beta\gamma$, the result follows. □

Corollary 2.6. *Let λ, μ denote the eigenvalue of an element y in L. Then $\lambda+\mu = 0$ and $\lambda\mu = -\|y\|^2$.*

Proof. By Linear algebra, we know that $\lambda + \mu = -tr(y)$ and $\lambda\mu = \det(y)$. Since $y \in L$ and by Lemma 2.5, the results follow. □

We recall the notion of an automorphism. By an *a*utomorphism of L we mean a $\mathbb{C}$-vector space isomorphism $\sigma : L \longrightarrow L$ such that

$$[y, z]^\sigma = [y^\sigma, z^\sigma]$$

for $y, z \in L$.

Lemma 2.7. *Let σ denote an automorphism of L. Then for $y \in L$*

$$ad_{y^\sigma} = \sigma ad_y \sigma^{-1}. \tag{11}$$

Proof. Let $z \in L$. Then $ad_{y^\sigma}(z) = [y^\sigma, z]$. Furthermore,

$$\sigma ad_y \sigma^{-1}(z) = \sigma ad_y(z^{\sigma^{-1}}) = [y, z^{\sigma^{-1}}]^\sigma = [y^\sigma, z^{\sigma^{-1}\sigma}] = [y^\sigma, z].$$

We have proven the lemma. □

By Lemma 2.7, we find

$$\begin{aligned}
\langle y^\sigma, z^\sigma\rangle &= (1/8)tr\left([ad_y^\sigma][ad_z^\sigma]\right)\\
&= (1/8)tr\left([\sigma ad_y \sigma^{-1}][\sigma ad_z \sigma^{-1}]\right)\\
&= (1/8)tr\left([\sigma][ad_y][\sigma^{-1}][\sigma][ad_z][\sigma^{-1}]\right)\\
&= (1/8)tr\left([\sigma][ad_y][ad_z][\sigma^{-1}]\right)\\
&= (1/8)tr\left([ad_y][ad_z][\sigma^{-1}][\sigma]\right)\\
&= (1/8)tr\left([ad_y][ad_z]\right)\\
&= \langle y, z\rangle.
\end{aligned}$$

In particular, we have $\langle y, y\rangle = \langle y^\sigma, y^\sigma\rangle$ which implies that $\|y\|^2 = \|y^\sigma\|^2$.

We finish this section with the following lemma, which is the well-known result.

Lemma 2.8. [1, Section 2.3] *The following hold.*

(i) *Let M denote an invertible matrix in $Mat_2(\mathbb{C})$. Then the map $L \longrightarrow L$, $y \longmapsto MyM^{-1}$ is an automorphism of L.*

(ii) *Let σ denote an automorphism of L. Then there exists an invertible $M \in Mat_2(\mathbb{C})$ such that $y^\sigma = MyM^{-1}$ for $y \in L$.*

3. Normalized Semisimple Elements in $\mathfrak{sl}_2(\mathbb{C})$

Recall the Lie algebra $L = \mathfrak{sl}_2(\mathbb{C})$. In this section, we discuss normalized semisimple elements of L and show that these elements yield a new basis for L. We begin by recalling a few definitions concerning linear algebra.

Definition 3.1. Let V denote a finite-dimensional vector space over $\mathbb{C}$. Let A denote a linear transformation from $V \longrightarrow V$. We say that A is *diagonalizable* whenever V has a basis consisting of eigenvectors of A.

Definition 3.2. Let $\{u_i\}_{i=1}^n$ denote a basis for V. For $B \in Mat_n(\mathbb{C})$, we say B *represents A with respect to $\{u_i\}_{i=1}^n$* whenever $Au_j = \sum_{i=1}^n B_{ij}u_i$ for $1 \leq j \leq n$.

Lemma 3.3. *Recall the standard basis $\{e, h, f\}$ for L. Let $y \in L$ be as in (4). Then the matrix representing ad_y with respect to $\{e, h, f\}$ is*

$$\begin{bmatrix} 2\alpha & -2\beta & 0 \\ -\gamma & 0 & \beta \\ 0 & 2\gamma & -2\alpha \end{bmatrix}. \tag{12}$$

Proof. Write $y = \beta e + \alpha h + \gamma f$. First, we find the first column of the matrix (12). Using the definition of ad_y, we have

$$\begin{aligned} ad_y(e) &= [y, e] \\ &= [\beta e + \alpha h + \gamma f, e] \\ &= \beta[e, e] + \alpha[h, e] + \gamma[f, e] \\ &= 0 + \alpha \cdot 2e + \gamma \cdot (-h) && \text{(By (5))} \\ &= (2\alpha) \cdot e + (-\gamma) \cdot h + 0 \cdot f. \end{aligned}$$

Similarly, we find the second and third columns of (12). The result follows. $\square$

Let A be a linear transformation form V to V. By an eigenvalue of A, we mean an eigenvalue of the matrix representation of A.

Corollary 3.4. *Let $y \in L$ and let λ and $-\lambda$ be eigenvalues of y. Then, the eigenvalues of the linear transformation $ad_y : L \longrightarrow L$ are 2λ, 0, and -2λ.*

Proof. By Lemma 2.5 and Corollary 2.6, we have $\lambda^2 = \alpha^2 + \beta\gamma$. Let $\phi(x)$ denote the characteristic polynomial of the matrix in (12). Then one routinely finds

$$\phi(x) = -x(x^2 - 4(\alpha^2 + \beta\gamma)).$$

Substituting $\alpha^2 + \beta\gamma$ for λ^2 and simplifying the result, we get

$$\phi(x) = -x(x - 2\lambda)(x + 2\lambda).$$

The result follows. $\square$

Lemma 3.5. *Let $0 \neq y \in L$ be as in (4). Let λ, $-\lambda$ denote the eigenvalues of y.*

(i) *If $\lambda = 0$, then $y^2 = 0$, $\|y\|^2 = 0$, and* $\det(y) = 0$.

(ii) *If $\lambda \neq 0$, then y is diagonalizable, $\|y\|^2 \neq 0$, and* $\det(y) \neq 0$.

Proof. (i): If $\lambda = 0$, then $y^2 = \begin{bmatrix} \alpha^2 + \beta\gamma & 0 \\ 0 & \alpha^2 + \beta\gamma \end{bmatrix}$ and by Corollary 2.6 we have $y^2 = 0$. Moreover, by Lemma 2.5, the remaining results follow.

(ii): If $\lambda \neq 0$, then y has two distinct eigenvalues λ, $-\lambda$. Thus, y is diagonalizable on L. By Corollary 2.6, the remaining results follow. $\square$

Let $y \in L$. Then we say y is *semisimple* if the $\mathbb{C}$-linear map $ad_y : L \longrightarrow L$ is diagonalizable. We note that $y \in L$ is semisimple if and only if y^σ is semisimple, where σ is an automorphism of L. We give a characterization of a semisimple element in L.

Lemma 3.6. [1, Section 4.2] *For $y \in L$, the following are equivalent:*

(i) *y is semisimple.*

(ii) *y is diagonalizable.*

Lemma 3.7. *For $y \in L$, the following are equivalent:*

(i) $||y||^2 = 1$.

(ii) $det(y) = -1$.

(iii) *y is diagonalizable with eigenvalues 1 and -1.*

(iv) *There exists an automorphism: $L \longrightarrow L$ that sends $y \longmapsto h$.*

Proof. (i)$\Longrightarrow$(ii): By Lemma 2.5.

(ii)$\Longrightarrow$(iii): By Lemma 3.5.

(iii)$\Longrightarrow$(iv): By linear algebra.

(iv)$\Longrightarrow$(i): Let σ denote an automorphism of L that sends $y \longmapsto h$. Then $\|y\|^2 = \|y^\sigma\|^2 = \|h\|^2 = 1$. $\square$

Definition 3.8. Let $y \in L$ be semisimple. We say that y is *normalized* if $\|y\|^2 = 1$.

Definition 3.9. Let a and a^* be normalized semisimple elements in L. Define $p \in \mathbb{C}$ such that $\langle a, a^* \rangle = 1 - 2p$. We call p the *corresponding parameter* of a and a^*.

Example 3.10. Let

$$a = \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix}, \qquad a^* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \tag{13}$$

where we assume that $\alpha^2 + \beta\gamma = 1$. By Lemma 3.7, we observe that each of a and a^* is a normalized semisimple element of L. We also observe that

$$\begin{aligned} \langle a, a^* \rangle &= \langle \beta e + \alpha h + \gamma f, h \rangle \\ &= \beta \langle e, h \rangle + \alpha \langle h, h \rangle + \gamma \langle f, h \rangle \\ &= \alpha. \end{aligned}$$

Set $p = (1 - \alpha)/2$. Then the corresponding parameter for the pair a, a^* is p. We note that

$$\alpha = 1 - 2p, \qquad \beta\gamma = 4p(1 - p). \tag{14}$$

Example 3.11. Let

$$a = \begin{bmatrix} 1-2p & 2(1-p) \\ 2p & 2p-1 \end{bmatrix}, \qquad a^* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

where $p \in \mathbb{C}$. Then each of a, a^* is a normalized semisimple element of L, with p being the corresponding parameter. We note that this is a special case of Example 3.10 with $\beta = 2(1-p)$ and $\gamma = 2p$.

Lemma 3.12. *Let a and a^* denote a pair of normalized semisimple elements of L, and let p denote the corresponding parameter. Then the values of $\langle\ ,\ \rangle$ on the elements of a, a^*, and $[a, a^*]$ result in the following table*

$\langle\ ,\ \rangle$	a	a^*	$[a, a^*]$
a	1	$1-2p$	0
a^*	$1-2p$	1	0
$[a, a^*]$	0	0	$-16p(1-p)$

. (15)

Proof. By Lemma 3.7(iv) we may assume that a, a^* are the matrices as in Example 3.10. Using the matrices along with (9), we compute $\langle a, a^* \rangle$:

$$\begin{aligned} \langle a, a^* \rangle &= \langle \beta e + \alpha h + \gamma f, h \rangle \\ &= \beta \langle e, h \rangle + \alpha \langle h, h \rangle + \gamma \langle f, h \rangle \\ &= \beta \cdot 0 + \alpha \cdot 1 + \gamma \cdot 0 \\ &= \alpha = 1 - 2p. \end{aligned}$$

Similarly we have

$$\langle a, a \rangle = \langle \beta e + \alpha h + \gamma f, \beta e + \alpha h + \gamma f \rangle = \alpha^2 + \beta\gamma = 1,$$

$$\langle [a, a^*], a \rangle = \langle -2\beta e + 2\gamma f, \beta e + \alpha h + \gamma f \rangle = \gamma\beta - \gamma\beta = 0.$$

The remaining entries are computed using a similar method. $\square$

Lemma 3.13. *Let M denote the matrix with respect to $\langle\ ,\ \rangle$ in* (15)*. Then*

$$\det(M) = -64p^2(1-p)^2.$$

Proof. Routine. □

Lemma 3.14. *Let a and a^* be normalized semisimple elements in L and p be the corresponding parameter for a and a^*. Then the following are equivalent.*

(i) $p \neq 0$ *and* $p \neq 1$.

(ii) $a, a^*, [a, a^*]$ *is a basis for* L.

(iii) a *and* a^* *generate* L.

Proof. We recall the matrix M from Lemma 3.13.

(i) $\Longrightarrow$ (ii): Since $p \neq 0$ and $p \neq 1$, by Lemma 3.13 the matrix M is invertible, resulting in the columns being linearly independent. Thus, a, a^*, and $[a, a^*]$ are linearly independent. Since $\dim(L) = 3$, the result (ii) follows.

(ii) $\Longrightarrow$ (i): Since a, a^*, and $[a, a^*]$ is a basis for L, the matrix M is nonsingular. Therefore, by Lemma 3.13 $\det(M) = -64p^2(1-p)^2 \neq 0$. Hence, $p \neq 0$ and $p \neq 1$.

(ii) $\Longrightarrow$ (iii): Trivial.

(iii) $\Longrightarrow$ (ii): Suppose a and a^* generate L. Consider the element $[a, a^*] \in L$. We claim that $[a, a^*] \notin span\{a, a^*\}$. Without loss of generality, we may assume that a, a^* are from Example 3.10. We claim that $[a, a^*] \notin span\{a, a^*\}$. If $[a, a^*] \in span\{a, a^*\}$, then we can write $[a, a^*] = k_1 a + k_2 a^*$ for some $k_1, k_2 \in \mathbb{C}$. Observe that

$$[a, a^*] = -2\beta e + 2\gamma f = \begin{bmatrix} 0 & -2\beta \\ 2\gamma & 0 \end{bmatrix},$$

$$k_1 a + k_2 a^* = k_1 \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} + k_2 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} k_1\alpha + k_2 & k_1\beta \\ k_1\gamma & -k_1\alpha - k_2 \end{bmatrix}.$$

So, the equation $[a, a^*] = k_1 a + k_2 a^*$ implies that $k_1\alpha + k_2 = 0$ and $\beta = \gamma = 0$. By (14), we have $a = \pm a^*$. This contradicts our assumption. Thus we have shown our

claim. Next, we find that a and a^* are linearly independent. If this was not the case, then $[a, a^*] = 0$, a contradiction. By these comments, it follows that a, a^*, and $[a, a^*]$ are linearly independent. □

Lemma 3.15. *Let a, a^* and b, b^* be pairs of normalized semi-simple elements of L. Assume each of the pairs generate L. Then, the following are equivalent.*

(i) $\langle a, a^* \rangle = \langle b, b^* \rangle$

(ii) *There exists an automorphism of L that maps $a \longmapsto b$ and $a^* \longmapsto b^*$.*

Proof. (ii) $\longrightarrow$ (i): Let σ denote an automorphism of L that sends $a \longmapsto b$ and $a^* \longmapsto b^*$. Then we have $\langle b, b^* \rangle = \langle a^\sigma, a^{*\sigma} \rangle = \langle a, a^* \rangle$.

(i) $\longrightarrow$ (ii): Since $\langle a, a^* \rangle = \langle b, b^* \rangle$, the pairs a, a^* and b, b^* have the same corresponding parameter, say p. Since a and a^* generate L, it follows that $p \neq 0$ and $p \neq 1$ by Lemma 3.7. We claim that there exists an automorphism of L that maps

$$a \longmapsto \begin{bmatrix} 1-2p & 2(1-p) \\ 2p & 2p-1 \end{bmatrix} \qquad a^* \longmapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Since a, a^* are normalized semisimple, by Lemma 3.7(iv) we may assume that a, a^* are as in (13). Since $p \neq 0$ and $p \neq 1$, we have $\beta\gamma = 4p(1-p) \neq 0$. Thus we have $\gamma \neq 0$. Define the matrix

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 2p\gamma^{-1} \end{bmatrix}.$$

Obviously, Q is invertible. Define the map $f : L \longmapsto L$ by $f(y) = QyQ^{-1}$. Observe that $f([x, y]) = [f(x), f(y)]$ for $x, y \in L$ and f is bijective. Thus, f is an

automorphism of L. We have

$$f(a) = QaQ^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2p\gamma^{-1} \end{bmatrix} \begin{bmatrix} \alpha & \beta \\ \gamma & -\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{\gamma}{2p} \end{bmatrix}$$
$$= \begin{bmatrix} \alpha & \beta \\ 2p & -2p\gamma^{-1}\alpha \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{\gamma}{2p} \end{bmatrix}$$
$$= \begin{bmatrix} 1-2p & 2(1-p) \\ 2p & 2p-1 \end{bmatrix}.$$

Moreover, we have

$$f(a^*) = Qa^*Q^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 2p\gamma^{-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \frac{\gamma}{2p} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The result follows. □

Lemma 3.16. *Let a and a^* be a pair of normalized semisimple elements of L and let p denote the corresponding parameter. Then,*

(i) $[a, [a, a^*]] = 4(2p-1)a + 4a^*$.

(ii) $[a^*, [a^*, a]] = 4(2p-1)a^* + 4a$.

Proof. (i): By Lemma 3.15 we assume that a, a^* is the pair of matrices from Example 3.11. Since $a = (1-2p)h + 2(1-p)e + 2pf$ and $a^* = h$, we have

$$[a, a^*] = -4(1-p)e + 4pf.$$

So,

$$[a, [a, a^*]] = [(1-2p)h + 2(1-p)e + 2pf, -4(1-p)e + 4pf]$$
$$= 4(2p-1)\Big(2(1-p)e + 2pf\Big) + 16p(1-p)h$$
$$= 4(2p-1)\Big((1-2p)h + 2(1-p)e + 2pf\Big) + 4h$$
$$= 4(2p-1)a + 4a^*.$$

(ii): Similar to (i). □

Lemma 3.17. *Fix $p \in \mathbb{C}$ such that $p \neq 0$ and $p \neq 1$. Let $\mathcal{L}$ denote the Lie algebra defined by generators u and v and relations*

(i) $[u, [u, v]] = 4(2p - 1)u + 4v$,

(ii) $[v, [v, u]] = 4(2p - 1)v + 4u$.

Then $\mathcal{L}$ is isomorphic to L. Additionally, u and v are normalized semisimple elements, with p being the corresponding parameter.

Proof. Let a and a^* be elements in Example 3.11. Note that a and a^* is a pair of normalized semisimple elements in L, with p being the corresponding parameter. We construct a Lie algebra homomorphism $\phi : \mathcal{L} \longrightarrow L$ that sends $u \longrightarrow a$ and $v \longrightarrow a^*$. Observe that a and a^* satisfy relations (i) and (ii) in Lemma 3.16. Compare these relations with the relations (i), (ii) in the present lemma. It follows that the Lie algebra homomorphism ϕ exists. As a and a^* generate L, ϕ is surjective. As a, a^*, and $[a, a^*]$ is a basis for L, $\dim(\mathcal{L}) \geq 3$. Moreover, by given relations (i) and (ii) in the present lemma, every element of $\mathcal{L}$ is expressed as a linear combination of u, v, and $[u, v]$. Therefore, $\dim(\mathcal{L}) \leq 3$. By these comments, we have $\dim(\mathcal{L}) = 3$. Consequently, ϕ is bijective, implying that ϕ is an isomorphism of Lie algebras. □

We finish this section with some comments. Let a and a^* denote normalized semisimple elements that generate L, with p denoting the corresponding parameter. Note that $p \neq 0$ and $p \neq 1$. By Lemma 3.15, there exists an automorphism of L sending a and a^* to the pair of matrices from Example 3.11. We can assume that a and a^* is the pair of matrices from Example 3.11 where $p \neq 0$ and $p \neq 1$, without loss of generality. So,

$$a = 2(1 - p)e + (1 - 2p)h + 2pf, \quad a^* = h. \tag{16}$$

We see that,

$$[a, a^*] = -4(1 - p)e + 4pf \tag{17}$$

By Lemma 3.14, we have a basis $a, a^*, [a, a^*]$ for L. Note that there exists an automorphism of L that sends $a \longmapsto a^*$ and $a^* \longmapsto a$. This is because by Lemma 3.17, the relations (i), (ii) are invariant under the mapping that sends $u \longmapsto v$ and $v \longmapsto u$. This automorphism is unique since a, a^* generates L. We denote this automorphism by $*$. Note that $(y^*)^* = y$ for all $y \in L$. We will use this automorphism in the next section.

4. Matrices

In this section, we display three bases for L and discuss the transition matrices between two bases among the three. We first define the matrices U and W in $Mat_2(\mathbb{C})$ by

$$U = \begin{bmatrix} 1 & 1 \\ 1 & 1-p^{-1} \end{bmatrix} \qquad \text{and} \qquad W = \begin{bmatrix} 1-p & 0 \\ 0 & p \end{bmatrix}. \tag{18}$$

Define $R = WU$. Observe that

$$R = \begin{bmatrix} 1-p & 0 \\ 0 & p \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1-p^{-1} \end{bmatrix} = \begin{bmatrix} 1-p & 1-p \\ p & p-1 \end{bmatrix}. \tag{19}$$

So, we have

$$R^2 = \begin{bmatrix} 1-p & 1-p \\ p & p-1 \end{bmatrix} \begin{bmatrix} 1-p & 1-p \\ p & p-1 \end{bmatrix} = \begin{bmatrix} 1-p & 0 \\ 0 & 1-p \end{bmatrix} = (1-p)I. \tag{20}$$

Thus. R is invertible and its inverse matrix is

$$R^{-1} = \frac{1}{(1-p)(p-1) - p(1-p)} \begin{bmatrix} p-1 & p-1 \\ -p & 1-p \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ \frac{p}{1-p} & -1 \end{bmatrix}.$$

Lemma 4.1. *Recall the automorphism $*$ of L from the below line* (17). *For $y \in L$, $y^* = RyR^{-1}$.*

Proof. Let a, a^* be the normalized semisimple elements in L, with the corresponding parameter p, with $p \neq 0$ and $p \neq 1$. We may assume that a, a^* are matrices

from Example 3.11, respectively. Since a and a^* generate L, it suffices to check $a^* = RaR^{-1}$ and $a = Ra^*R^{-1}$. We first verify the equation $a^* = RaR^{-1}$. Using (19), (20) we have

$$RaR^{-1} = \begin{bmatrix} 1-p & 1-p \\ p & p-1 \end{bmatrix} \begin{bmatrix} 1-2p & 2(1-p) \\ 2p & 2p-1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ \frac{p}{1-p} & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} = a^*.$$

Similarly, we routinely verify that $a = Ra^*R^{-1}$. The result follows. □

Recall the standard basis e, h, f for L. Applying the automorphism $*$ to this basis, we get a basis e^*, h^*, f^*, called the *dual standard basis* for L.

Lemma 4.2. *Recall the dual basis e^*, h^*, f^* for L. Then we have*

$$e^* = (p-1)e + ph + \frac{p^2}{1-p}f,$$

$$h^* = 2(1-p)e + (1-2p)h + 2pf,$$

$$f^* = (1-p)e + (1-p)h + (p-1)f.$$

Note that $h^* = a$, where a is the matrix in Example 3.11.

Proof. For each $y \in \{e, h, f\}$, evaluate $y^* = RyR^{-1}$ using (3), (19), and (20). □

We recall the definition of a transition matrix.

Definition 4.3. Let V be a finite dimensional vector space over $\mathbb{C}$. Let $\{u_i\}_{i=1}^n$ and $\{v_i\}_{i=1}^n$ be bases for V. By the *transition matrix* from $\{u_i\}_{i=1}^n$ to $\{v_i\}_{i=1}^n$, we mean the matrix T such that

$$v_j = \sum_{i=1}^{n} T_{ij} u_i,$$

for $1 \leq j \leq n$.

We review some properties about the transition matrices. Let V be a finite dimensional $\mathbb{C}$-vector space. Let $\{u_i\}_{i=1}^n$ and $\{v_i\}_{i=1}^n$ be bases for V. Let T be the transition matrix from $\{u_i\}_{i=1}^n$ to $\{v_i\}_{i=1}^n$. Then the inverse of T exists and it

turns out that T^{-1} is the transition matrix from $\{v_i\}_{i=1}^n$ to $\{u_i\}_{i=1}^n$. Suppose that $\{w_i\}_{i=1}^n$ is a basis for V. Let S be the transition matrix from $\{v_i\}_{i=1}^n$ to $\{w_i\}_{i=1}^n$. Then the transition matrix from from $\{u_i\}_{i=1}^n$ to $\{w_i\}_{i=1}^n$ is TS. Let $A : V \longrightarrow V$ be a $\mathbb{C}$-linear transformation and B (resp. C) denote the matrix representing A with respect to $\{u_i\}_{i=1}^n$ (resp. $\{v_i\}_{i=1}^n$). Then $BT = TC$ so it follows that $C = T^{-1}BT$.

We recall three basis for L:

$$\{e, h, f\}, \qquad \{a, a^*, [a, a^*]\}, \qquad \{e^*, h^*, f^*\}. \tag{21}$$

The following lemmas 4.4–4.6 display the transition matrices between two bases taken from (21).

Lemma 4.4. *The transition matrix from* $\{e, h, f\}$ *to* $\{a, a^*, [a, a^*]\}$ *is*

$$\begin{bmatrix} 2(1-p) & 0 & 4(p-1) \\ 1-2p & 1 & 0 \\ 2p & 0 & 4p \end{bmatrix} \tag{22}$$

The transition matrix from $\{a, a^*, [a, a^*]\}$ *to* $\{e, h, f\}$ *is*

$$\begin{bmatrix} \frac{1}{4(1-p)} & 0 & \frac{1}{4p} \\ \frac{2p-1}{4(1-p)} & 1 & \frac{2p-1}{4p} \\ \frac{1}{8(p-1)} & 0 & \frac{1}{8p} \end{bmatrix} \tag{23}$$

Proof. Recall from (16), (17) that

$$a = 2(1-p)e + (1-2p)h + 2pf, \qquad a^* = h,$$

$$[a, a^*] = 4(p-1)e + 4pf.$$

From these equations, we obtain the matrix (22). To get the matrix (23), find the inverse matrix of (22). The result follows. $\square$

Lemma 4.5. *The transition matrix from* $\{e^*, h^*, f^*\}$ *to* $\{a, a^*, [a, a^*]\}$ *is*

$$\begin{bmatrix} 0 & 2(1-p) & 4(p-1) \\ 1 & 1-2p & 0 \\ 0 & 2p & -4p \end{bmatrix} \tag{24}$$

The transition matrix from $\{a, a^*, [a, a^*]\}$ *to* $\{e^*, h^*, f^*\}$ *is*

$$\begin{bmatrix} \frac{2p-1}{4(1-p)} & 1 & \frac{2p-1}{4p} \\ \frac{1}{4(1-p)} & 0 & \frac{1}{4p} \\ \frac{1}{8(1-p)} & 0 & -\frac{1}{8p} \end{bmatrix} \tag{25}$$

Proof. We apply the map * to each of a, a^*, and $[a, a^*]$ in (16), (17) to get

$$a^* = 2(1-p)e^* + (1-2p)h^* + 2pf^*, \qquad (a^*)^* = a = h*,$$

$$[a, a^*]^* = [a^*, a] = -[a, a^*] = 4(p-1)e^* + 4pf^*.$$

From these equations, we obtain the matrix (24). To get the matrix (25), find the inverse matrix of (24). The result follows. □

Lemma 4.6. *The transition matrix from* $\{e, h, f\}$ *to* $\{e^*, h^*, f^*\}$ *is*

$$\begin{bmatrix} p-1 & 2(1-p) & 1-p \\ p & 1-2p & 1-p \\ \frac{p^2}{1-p} & 2p & p-1 \end{bmatrix}. \tag{26}$$

The transition matrix from $\{e^*, h^*, f^*\}$ *to* $\{e, h, f\}$ *is*

$$\begin{bmatrix} p-1 & 2(1-p) & 1-p \\ p & 1-2p & 1-p \\ \frac{p^2}{1-p} & 2p & p-1 \end{bmatrix}. \tag{27}$$

Proof. By Lemma 4.2 we obtain the matrix (26). To get the matrix (27), find the inverse matrix of (26). The result follows. □

Recall the bilinear form (8) on L. For given two bases for L, say $\{u_i\}_{i=1}^3$ and $\{v_i\}_{i=1}^3$, the matrix representing $\langle\ ,\ \rangle$ is defined by the 3×3 matrix M such that $M_{ij} = \langle u_i, v_j\rangle$. In the following lemma, we display the matrices representing $\langle\ ,\ \rangle$ with respect to given two bases for L.

Lemma 4.7. *For a given pair of bases within* (21)*, the matrix representing* $\langle\ ,\ \rangle$ *is described as follows:*

$\langle\ ,\ \rangle$	e	h	f
e	0	0	$1/2$
h	0	1	0
f	$1/2$	0	0

$\langle\ ,\ \rangle$	e^*	h^*	f^*
e^*	0	0	$1/2$
h^*	0	1	0
f^*	$1/2$	0	0

$\langle\ ,\ \rangle$	e^*	h^*	f^*
e	$\frac{p^2}{2(1-p)}$	p	$\frac{p-1}{2}$
h	p	$1-2p$	$1-p$
f	$\frac{p-1}{2}$	$1-p$	$\frac{1-p}{2}$

$\langle\ ,\ \rangle$	a	a^*	$[a,a^*]$
a	1	$1-2p$	0
a^*	$1-2p$	1	0
$[a,a^*]$	0	0	$-16p(1-p)$

$\langle\ ,\ \rangle$	a	a^*	$[a,a^*]$
e	p	0	$2p$
h	$1-2p$	1	0
f	$1-p$	0	$2(p-1)$

$\langle\ ,\ \rangle$	a	a^*	$[a,a^*]$
e^*	0	p	$-2p$
h^*	1	$1-2p$	0
f^*	0	$1-p$	$2(1-p)$

.

Proof. We get the first matrix using (7) and the fourth table using Lemma (3.12). We obtain the rest of the tables using (16), (17), and Lemma 4.2. □

We recall the adjoint map (6).

Lemma 4.8. *The matrices representing* ad_a *and* ad_{a^*} *with respect to* $\{a, a^*, [a,a^*]\}$ *are given as*

$$ad_a : \begin{bmatrix} 0 & 0 & 4(2p-1) \\ 0 & 0 & 4 \\ 0 & 1 & 0 \end{bmatrix}, \qquad ad_{a^*} : \begin{bmatrix} 0 & 0 & -4 \\ 0 & 0 & 4(1-2p) \\ -1 & 0 & 0 \end{bmatrix}$$

Proof. We first find the matrix representing ad_a with respect to $\{a, a^*, [a, a^*]\}$. Evaluating $ad_a(a)$, $ad_a(a^*)$, $ad_a([a, a^*])$ we get

$$ad_a(a) = [a, a] = 0 = 0a + 0a^* + 0[a, a^*],$$

$$ad_a(a^*) = [a, a^*] = 0a + 0a^* + 1[a, a^*],$$

$$ad_a([a, a^*]) = [a, [a, a^*]] = 4(2p - 1)a + 4a^*.$$

Thus, we obtain the matrix representation of ad_a. The matrix representing ad_{a^*} is similarly obtained. □

Lemma 4.9. *The matrices representing ad_a and ad_{a^*} with respect to e, h, f are given as*

$$ad_a : \begin{bmatrix} 2(1-2p) & 4(p-1) & 0 \\ -2p & 0 & 2(1-p) \\ 0 & 4p & 2(2p-1) \end{bmatrix}, \qquad ad_{a^*} : \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}.$$

Proof. Use (16) and (5). □

Lemma 4.10. *The matrices representing ad_a and ad_{a^*} with respect to e^*, h^*, f^* are given as*

$$ad_a : \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \qquad ad_{a^*} : \begin{bmatrix} 2(1-2p) & 4(p-1) & 0 \\ -2p & 0 & 2(1-p) \\ 0 & 4p & 2(2p-1) \end{bmatrix}.$$

Proof. Use Lemma 4.2 and (5). □

5. Antiautomorphisms

In this section, we discuss an antiautomorphism of L, which plays a role in our paper.

Definition 5.1. By an *antiautomorphism* of L we mean an isomorphism of $\mathbb{C}$-vector spaces $\sigma : L \longrightarrow L$ such that

$$[y,z]^\sigma = [z^\sigma, y^\sigma], \qquad \text{for every } y \text{ and } z \text{ in } L.$$

For example, the map $L \longrightarrow L$ that sends $y \longmapsto -y$ is an antiautomorphism since

$$\sigma([y,z]) = -[y,z] = [z,y] = [-z,-y] = [\sigma(z), \sigma(y)],$$

for $y, z \in L$. Additionally, the map $L \longrightarrow L$ that sends $y \longmapsto y^t$ is an antiautomorphism since

$$\sigma([y,z]) = [y,z]^t = [z^t, y^t] = [\sigma(z), \sigma(y)].$$

Consider two maps $\sigma : L \longrightarrow L$ and $\tau : L \longrightarrow L$, where we assume that each of which is an automorphism or an antiautomorphism). If exactly one is an antiautomorphism, then $\sigma\tau$ is an antiautomorphism. Otherwise, $\sigma\tau$ is an automorphism. For example, if σ is an antiautomorphism and τ is an automorphism, we have,

$$\sigma\tau([y,z]) = \sigma([\tau(y), \tau(z)]) = [\sigma\tau(z), \sigma\tau(y)] \qquad (y, z \in L).$$

Lemma 5.2. *The following hold:*

(i) *Let M be an invertible matrix in $Mat_2(\mathbb{C})$. Then the map $\sigma : L \longrightarrow L$ defined by $\sigma(y) = My^tM^{-1}$ is an antiautomorphism of L.*

(ii) *Let σ be an antiautomorphism of L. Then, there exists an invertible $M \in Mat_2(\mathbb{C})$ such that $\sigma(y) = My^tM^{-1}$ for $y \in L$.*

Proof. (i): Recall the map $L \longrightarrow L$, $y \longmapsto MyM^{-1}$ is an automorphism and the map $L \longrightarrow L$ and $y \longmapsto y^t$ is an antiautomorphism. Therefore, by the comments above Lemma 5.2, their composition σ is an antiautomorphism.
(ii): Since σ is an automorphism and the map $L \longrightarrow L$, $y \longmapsto y^t$ is an antiautomorphism, their composition map $L \longrightarrow L$, $y \longmapsto (y^t)^\sigma$ is an automorphism of L. By Lemma 2.8, there exists an invertible matrix $M \in Mat_2(\mathbb{C})$ such that $(y^t)^\sigma = MyM^{-1}$ for $y \in L$. The result follows. $\square$

Lemma 5.3. *Let σ be an antiautomorphism of L. Then,*

$$\langle y, z\rangle = \langle y^\sigma, z^\sigma\rangle \quad \textit{for } y, z \in L.$$

Proof. Define $\tau : L \longrightarrow L$ such that $u^\tau = -u^\sigma$ for $u \in L$. Then τ is an automorphism since

$$\tau([u, v]) = -[u, v]^\sigma = -[v^\sigma, u^\sigma] = [u^\sigma, v^\sigma] = [-u^\sigma, -v^\sigma] = [\tau(u), \tau(v)].$$

Thus, for $y, z \in L$ we have

$$\langle y, z\rangle = \langle y^\tau, z^\tau\rangle = \langle -y^\sigma, -z^\sigma\rangle = \langle y^\sigma, z^\sigma\rangle.$$

The result follows. □

Lemma 5.4. *There exists a unique automorphism $\dagger$ of L such that $a^\dagger = a$ and $a^{*\dagger} = a^*$. Moreover, we have $(y^\dagger)^\dagger = y$ for $y \in L$.*

Proof. We first show the existence of an automorphism. Define the map $\dagger : L \longrightarrow L$ by

$$y \longmapsto W y^t W^{-1}, \quad \text{where } W = \begin{bmatrix} 1-p & 0 \\ 0 & p \end{bmatrix}.$$

Then, $\dagger$ is an antiautomorphism by Lemma 5.2(i). Recall that a and a^* are normalized semisimple elements that generates L, with the corresponding parameter $p \neq 1$ and $p \neq 0$. View a, a^* as

$$a = \begin{bmatrix} 1-2p & 2(1-p) \\ 2p & 2p-1 \end{bmatrix}, \qquad a^* = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Then,

$$a^\dagger = Wa^tW^{-1} = \begin{bmatrix} 1-p & 0 \\ 0 & p \end{bmatrix} \begin{bmatrix} 1-2p & 2p \\ 2(1-p) & 2p-1 \end{bmatrix} \begin{bmatrix} \frac{1}{1-p} & 0 \\ 0 & \frac{1}{p} \end{bmatrix}$$
$$= \begin{bmatrix} 1-2p & 2(1-p) \\ 2p & 2p-1 \end{bmatrix}$$
$$= a.$$

Also,

$$a^{*\dagger} = Wa^{*t}W^{-1} = \begin{bmatrix} 1-p & 0 \\ 0 & p \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \frac{1}{1-p} & 0 \\ 0 & \frac{1}{p} \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$
$$= a^*.$$

This implies that $\dagger$ fixes a and a^*.

We now show the uniqueness. Let $\dagger'$ denote an antiautomorphism of L that fixes a and a^*. Then, its inverse ${\dagger'}^{-1}$ is also an antiautomorphism of L that fixes a and a^*. This implies that the composition $\dagger{\dagger'}^{-1}$ is an automorphism of L that fixes each of a and a^*. Since L is generated by a and a^*, $\dagger{\dagger'}^{-1}$ is the identity map on L, we have $\dagger = \dagger^{-1}$.

Finally, since $(a^\dagger)^\dagger = a^\dagger = a$ and $(a^{*\dagger})^\dagger = (a^*)^\dagger = a^*$, it follows that $(y^\dagger)^\dagger = y$ for all $y \in L$. $\square$

We give a comment on Lemma 5.4. Let $\dagger$ be an antiautomorphism as in Lemma 5.4. Then for $y \in L$, we have $y^\dagger = Wy^tW^{-1}$, where W is from (18). This is because that the map $L \longrightarrow L$, $y \longmapsto Wy^tW^{-1}$ is an antiautomorphism that fixes each of a, a^*, and by Lemma 5.4 such an antiautomorphism is unique.

Lemma 5.5. *The maps $*$ and $\dagger$ commute.*

Proof. One finds that $(a^*)^\dagger = (a^\dagger)^*$ and $(a^*)^{*\dagger} = (a^*)^{\dagger *}$. Since a, a^* generate L, the result follows. $\square$

Lemma 5.6. *The antiautomorphism $\dagger$ acts on e, h, f and e^*, h^*, f^* in the following way:*

$$e^\dagger = \frac{p}{1-p} f, \qquad e^{*\dagger} = \frac{p}{1-p} f^*, \tag{28}$$

$$h^\dagger = h, \qquad h^{*\dagger} = h^*, \tag{29}$$

$$f^\dagger = \frac{1-p}{p} e, \qquad f^{*\dagger} = \frac{1-p}{p} e^*. \tag{30}$$

Proof. We describe the action of $\dagger$ on e. By the comment above Lemma 5.6 we have

$$e^\dagger = We^tW^{-1} = WfW^{-1} = \frac{p}{1-p} f.$$

By similar argument, the other cases follow. $\square$

6. The algebra $\mathcal{A}$

Recall the Lie algebra $L = \mathfrak{sl}_2(\mathbb{C})$ and recall Krawtchouk polynomials. Our main goal is to study how L and Krawtchouk polynomials are related. To this end, in this section, we construct a certain L-module. Let $\mathbb{C}[y, z]$ be the $\mathbb{C}$-algebra consisting of the polynomials in y, z, where y and z are commuting indeterminates. For notational convenience, we denote by $\mathcal{A} = \mathbb{C}[y, z]$. View $\mathcal{A}$ as a vector space. Then $\mathcal{A}$ has a basis

$$\{y^r z^s \mid r, s = 0, 1, 2, \ldots\}.$$

For an integer $n \geq 0$ we define the subspace H_n of $\mathcal{A}$ by

$$H_n = Span\{y^{n-i} z^i\}_{i=0}^n.$$

We give some comments on H_n. We observe that $\dim(H_n) = n+1$. We also observe that $H_0 = \mathbb{C}1$ and $H_1 = \mathbb{C}y + \mathbb{C}z$. By construction, we have

$$\mathcal{A} = \sum_{n=0}^{\infty} H_n.$$

Consider the subspace $H_nH_m = Span\{fg \mid f \in H_n, g \in H_m\}$ for $n, m \geq 0$. Since every element of H_nH_m is the $n+m$-th homogeneous component of $\mathcal{A}$, it follows $H_nH_m = H_{n+m}$. where $m, n \geq 0$.

Let V be a nonzero vector space over $\mathbb{C}$. Let $\mathcal{A}$ denote the $\mathbb{C}$-algebra consisting of all $\mathbb{C}$-linear transformation from V to V. Then $\mathcal{A}$ becomes the Lie algebra together with Lie bracket $[\varphi, \phi] = \varphi\phi - \phi\varphi$.

Definition 6.1. A *derivation* of $\mathcal{A}$ is an element $\partial \in \mathcal{A}$ such that

$$\partial(bc) = \partial(b)c + b\partial(c) \qquad \text{for} \quad b, c \in \mathcal{A}.$$

We denote $Der(\mathcal{A})$ by the set of all derivations of $\mathcal{A}$.

Lemma 6.2. *$Der(\mathcal{A})$ is the Lie subalgebra of $\mathcal{A}$.*

Proof. Clearly $Der(\mathcal{A})$ is a subspace of $\mathcal{A}$. We check to see if $Der(\mathcal{A})$ is closed under the Lie bracket, that is, $[\partial_1, \partial_2] \in Der(\mathcal{A})$ for $\partial_1, \partial_2 \in Der(\mathcal{A})$. For $b, c \in \mathcal{A}$ we show that

$$[\partial_1, \partial_2](bc) = [\partial_1, \partial_2](b)c + b[\partial_1, \partial_2](c). \tag{31}$$

Computing the left-hand side of (31), we have

$$\begin{aligned}
[\partial_1, \partial_2](bc) &= \partial_1\partial_2(bc) - \partial_2\partial_1(bc) \\
&= \partial_1\Big(\partial_2(b)\cdot c + b\cdot\partial_2(c)\Big) - \partial_2\Big(\partial_1(b)\cdot c + b\cdot\partial_1(c)\Big) \\
&= \partial_1\Big(\partial_2(b)\cdot c\Big) + \partial_1\Big(b\cdot\partial_2(c)\Big) - \partial_2\Big(\partial_1(b)\cdot c\Big) - \partial_2\Big(b\cdot\partial_1(c)\Big) \\
&= \partial_1\partial_2(b)\cdot c + b\cdot\partial_1\partial_2(c) - \partial_2\partial_1(b)\cdot c - b\cdot\partial_2\partial_1(c).
\end{aligned}$$

Computing the right-hand side of (31), we have

$$\begin{aligned}[\partial_1, \partial_2](b)c + b[\partial_1, \partial_2](c) &= (\partial_1\partial_2 - \partial_2\partial_1)(b) \cdot c + b \cdot (\partial_1\partial_2 - \partial_2\partial_1)(c) \\ &= \partial_1\partial_2(b) \cdot c + b \cdot \partial_1\partial_2(c) - \partial_2\partial_1(b) \cdot c - b \cdot \partial_2\partial_1(c).\end{aligned}$$

By these comments, the result follows. □

Lemma 6.3. *The following (i) – (iii) hold.*

(i) $\partial(1) = 0$.

(ii) $\partial(b^n) = nb^{n-1}\partial(b)$ *for* $b \in \mathcal{A}$, $n = 1, 2, \ldots$.

(iii) $\partial(y^r z^s) = ry^{r-1}z^s\partial(y) + sy^r z^{s-1}\partial(z)$.

Proof. (i): Trivial.

(ii): Use induction on n. Then

$$\begin{aligned}\partial(b^n) &= \partial(b^{n-1}) \cdot b + b^{n-1} \cdot \partial(b) \\ &= (n-1)b^{n-1}\partial(b) + b^{n-1} \cdot \partial(b) \qquad \text{(by Induction hypothesis)} \\ &= nb^{n-1}\partial(b).\end{aligned}$$

(iii): Use part (ii). □

Corollary 6.4. *Let* $\partial \in Der(\mathcal{A})$. *Then* $\partial = 0$ *if and only if* ∂ *vanishes on* H_1.

Proof. By Lemma 6.3(iii), a derivation $\partial \in Der(\mathcal{A})$ is determined by $\partial(y)$ and $\partial(z)$. This implies that ∂ is determined by its action on H_1. Therefore, the result follows. □

Lemma 6.5. *Let* φ *be a* $\mathbb{C}$*-linear transformation from* $H_1 \longrightarrow \mathcal{A}$. *Then there exists a unique derivation* $\partial = \partial_\varphi \in Der(\mathcal{A})$ *such that the restriction of* ∂ *on* H_1 *coincides with* φ.

Proof. Define a linear map $\partial \in \mathfrak{gl}(\mathcal{A})$ by

$$\partial(y^r z^s) = ry^{r-1}z^s\varphi(y) \ + \ sy^r z^{s-1}\varphi(z),$$

where $r, s = 0, 1, 2 \ldots$. One routinely checks that ∂ is a derivation of $\mathcal{A}$. Moreover, by construction we find $\partial(y) = \varphi(y)$ and $\partial(z) = \varphi(z)$. By this comment, the restriction of ∂ on H_1 coincides with φ. Therefore, the existence of ∂ has been shown. We now discuss the uniqueness of ∂. Suppose ∂' is another element of $Der(\mathcal{A})$ such that the restriction of ∂' on H_1 coincides with φ. Then $\partial - \partial'$ vanishes on H_1. By Corollary 6.4, it follows that $\partial - \partial' = 0$. The result follows. $\square$

Next, we will show that the algebra $\mathcal{A}$ has an L-module structure. We first recall some preliminaries. Consider the two-dimensional complex column vector space $\mathbb{C}^2$. Then the Lie algebra L acts by left multiplication on $\mathbb{C}^2$. So, we view $\mathbb{C}^2$ as a L-module. Recall that $H_1 = Span\{y^{1-i}z^i\}_{i=0}^{1}$ has a basis y, z. Consider the vector space isomorphism $H_1 \longrightarrow \mathbb{C}^2$ that sends

$$y \longmapsto \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad z \longmapsto \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{32}$$

By these comments, we have the following lemma.

Lemma 6.6. *With reference to above discussion, there exists an L-module structure on H_1 such that*

$$e.y = 0, \qquad h.y = y, \qquad f.y = z, \tag{33}$$

$$e.z = y, \qquad h.z = -z, \qquad f.z = 0. \tag{34}$$

Proof. Since $\mathbb{C}^2$ is an L-module, the isomorphism (32) induces an L-module structure on H_1 such that for any $x \in L$

$$x.y = x \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \qquad \text{and} \qquad x.z = x \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Thus, we have

$$e.y = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0, \qquad ez = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = y,$$

$$h.y = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} = y, \qquad h.z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -z,$$

$$f.y = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = z, \qquad f.z = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0.$$

This lemma has been proven. □

By Lemma 6.6 we can view an element of L as a linear map on H_1. For each $\varphi \in L$, Lemma 6.5 tells us that there exists a unique derivation $\partial_\varphi \in Der(\mathcal{A})$ such that the restriction of ∂ on H_1 coincides with φ. So, we have a map $L \longrightarrow Der(\mathcal{A})$ that sends $\varphi \longmapsto \partial_\varphi$.

Lemma 6.7. *The map $L \longrightarrow Der(\mathcal{A})$, $\varphi \longmapsto \partial_\varphi$ is an injective homomorphism of Lie algebras.*

Proof. Let f denote such a map. First, we show f is a Lie algebra homomorphism, that is,

$$f([\varphi, \phi]) = [f(\varphi), f(\phi)],$$

for $\varphi, \phi \in L$. So, we show that

$$\partial_{[\varphi,\phi]} = [\partial_\varphi, \partial_\phi]. \tag{35}$$

Since $Der(\mathcal{A})$ is the Lie algebra, we find that both sides in (35) are in $Der(\mathcal{A})$. Now, consider $\partial_{[\varphi,\phi]} - [\partial_\varphi, \partial_\phi] \in Der(\mathcal{A})$. Recall the basis $\{y, z\}$ for H_1. Then,

$$\begin{aligned}
&(\partial_{\varphi\phi-\phi\varphi} - (\partial_\varphi\partial_\phi - \partial_\phi\partial_\varphi))(y) \\
&= \partial_{\varphi\phi-\phi\varphi}(y) - \partial_\varphi\partial_\phi(y) + \partial_\phi\partial_\varphi(y) \\
&= \varphi\phi(y) - \phi\varphi(y) - \varphi\phi(y) + \phi\varphi(y) \\
&= 0.
\end{aligned}$$

Similarly, we have

$$(\partial_{\varphi\phi-\phi\varphi} - \partial_\varphi\partial_\phi + \partial_\phi\partial_\varphi)(z) = 0.$$

By these comments, $\partial_{[\varphi,\phi]} - [\partial_\varphi, \partial_\phi]$ vanishes on H_1. By Corollary 6.4, we have $\partial_{[\varphi,\phi]} = [\partial_\varphi, \partial_\phi]$.

To verify one-to-one, suppose $f(\varphi) = \partial_\varphi = 0$. Then, ∂_φ vanishes on H_1. This implies that φ vanishes on H_1. By Corollary 6.4 we have $\varphi = 0$. Therefore, $ker f = 0$. $\square$

Theorem 6.8. *The algebra $\mathcal{A}$ has an L-module structure such that each element of L acts on $\mathcal{A}$ as a derivation and* (33), (34) *holds.*

Proof. By Lemmas 6.6 and 6.7. $\square$

For the rest of this section we fix an integer N and consider the subspace H_N of $\mathcal{A}$. Note that the subspace H_N has a basis

$$\{y^{N-i}z^i\}_{i=0}^N = \{y^N, y^{N-1}z, y^{N-2}z^2, \ldots, yz^{N-1}, z^N\}.$$

Lemma 6.9. *The elements e, f, h act on the basis $\{y^{N-i}z^i\}_{i=0}^N$ as follows:*

$$\begin{aligned}
&e.(y^{N-i}z^i) = iy^{N-i+1}z^{i-1} && \text{for} \quad 1 \le i \le N, && e.y^N = 0, \\
&h.(y^{N-i}z^i) = (N-2i)y^{N-i}z^i && \text{for} \quad 0 \le i \le N, && \\
&f.(y^{N-i}z^i) = (N-i)y^{N-i-1}z^{i+1} && \text{for} \quad 0 \le i \le N-1, && f.z^N = 0.
\end{aligned}$$

Proof. The element e acts on $\mathcal{A}$ as a derivation. So, for $i = 0, 1, \ldots, N$, we have

$$e.(y^{N-i}z^i) = (N-i)y^{N-i-1}z^i(e.y) + iy^{N-i}z^{i-1}(e.z).$$

Since $e.y = 0$ and $e.z = y$, we have $e(y^{N-i}z^i) = iy^{N-i+1}z^{i-1}$. We obtain the other cases by similar argument. $\square$

Using Lemma 6.9 we describe the matrix representations of e, h, f with respect to the basis $\{y^{n-i}z^i\}_{i=0}^N$.

Lemma 6.10. *With respect to the basis $\{y^{n-i}z^i\}_{i=0}^N$, the matrix representing e, h, and f are*

$$e : \begin{bmatrix} 0 & 1 & & & & & \mathbf{0} \\ & 0 & 2 & & & & \\ & & 0 & 3 & & & \\ & & & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \\ & & & & & 0 & N \\ \mathbf{0} & & & & & & 0 \end{bmatrix}, \qquad f : \begin{bmatrix} 0 & & & & & & \mathbf{0} \\ N & 0 & & & & & \\ & \cdot & \cdot & & & & \\ & & \cdot & \cdot & & & \\ & & & 3 & 0 & & \\ & & & & 2 & 0 & \\ \mathbf{0} & & & & & 1 & 0 \end{bmatrix}$$

$$h : \begin{bmatrix} N & & & & & & \mathbf{0} \\ & N-2 & & & & & \\ & & N-4 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & 2-N & \\ \mathbf{0} & & & & & & -N \end{bmatrix}.$$

Proof. Lemma 6.9. $\square$

By Lemmas 6.9 and 6.10, we find that H_N is an (irreducible) L-module of $\mathcal{A}$. We finish this section with a comment.

Lemma 6.11. *Let V be an irreducible L-module with dimension $N+1$. Then the following (i) and (ii) are equivalent:*

(i) *The L-module V is isomorphic to H_N.*

(ii) *V has a basis $\{v_i\}_{i=0}^N$ such that*

$$h.v_i = (N-2i)vi \qquad \textit{for} \quad i = 0,1,\dots,N$$

and both $e.v_0 = 0$ and $f.v_N = 0$.

Proof. (i) $\implies$ (ii): It follows from Lemma 6.9.
(ii) $\implies$ (i): For $i = 0,1,2,\dots,$ the vector v_i is an eigenvector for h with eigenvalue $N-2i$. Start by recognizing that $\{N-2i\}_{i=0}^N$ are mutually distinct by construction. We then select an integer i, where $1 \leq i \leq N$. Recalling that $[h,e] = 2e$, we observe that $e.v_i \in \mathbb{C}_{v_i-1}$. Similarly, recalling that $[h,f] = 2f$, we see that $f.v_{i-1} \in \mathbb{C}_v i$. Now, let α_i and $\beta_i \in \mathbb{C}$ such that $ev_i = a_i v_{i-1}$ and $f_{v_{i-1}} = \beta_i v_i$. Let $\gamma_i = \alpha_i\beta_i$. For $i = 0,1,2,\dots,N$, we apply each side of $[e,f] = h$ to v_i and find $\gamma_{i+1} - \gamma_i = N-2i$, where $y_0 = 0$ and $\gamma_{N+1} = 0$. When we solve the recursion, we acquire $y_i = i(N-i+1)$ for $i = 0,1,2,\dots,N$. Renormalizing the basis $\{v_i\}_{i=0}^N$, we assume that $\alpha_i = i$ and $\beta_i = N-i+1$ for $1 \leq i \leq N$. With respect to $\{v_i\}_{i=0}^N$, the matrices representing e, f, and h are equivalent to the matrices from Lemma 6.10. To conclude, there exists an isomorphism of L-modules $V \to H_1(\mathcal{A})$ sending $v_i \mapsto y^{N-i}z^i$ for $i = 0,1,2,\dots,N$. $\square$

7. The L-module V

Throughout this section we fix an integer N. We denote by $V = H_N$. In Lemma 4.2 we discussed the dual standard basis e^*, f^*, h^* for L. In this section we describe the action of e^*, f^*, h^* on the L-module V. For $i = 0,1,\dots,N$, define

$$V_i = \mathbb{C}y^{N-i}z^i.$$

Observe that $\dim V_i = 1$ and $V = \sum_{i=0}^N V_i$, which is the direct sum. For $i = 0,1,\dots,N$, V_i is the eigenspace of h associated with eigenvalue $N-2i$. We call V_i the h-weight space for the eigenvalue $N-2i$. We call $V = \sum_{i=0}^N V_i$ the h-weight space decomposition of V.

Let us recall the dual standard basis e^*, h^*, and f^* for L from Lemma 4.2. We describe the action of e^*, f^*, h^* on the L-module V. Recall the basis $\{y, z\}$ for H_1. Define

$$y^* = (1-p)y + pz, \qquad z^* = (1-p)y + (p-1)z \tag{36}$$

Observe that y^*, z^* form a basis for H_1. Thus, the transition matrix from y, z to y^*, z^* is

$$\begin{bmatrix} 1-p & 1-p \\ p & p-1 \end{bmatrix}.$$

Note that this transition matrix is the same as R from (19). Using this matrix we have

$$y = y^* + \frac{p}{1-p} z^*, \qquad z = y^* - z^*. \tag{37}$$

Lemma 7.1. *The elements e^*, f^*, h^* act on the basis y^*, z^* for H_1 as follows.*

$$e^*.y^* = 0, \qquad h^*.y^* = y^*, \qquad f^*.y^* = z^*, \tag{38}$$

$$e^*.z^* = y^*, \qquad h^*.z^* = -z^*, \qquad f^*.z^* = 0. \tag{39}$$

Proof. We compute $e^*.y^*$. Using Lemmas 4.2, 6.6 and (36) we have

$$\begin{aligned} e^*.y^* &= \left((p-1)e + ph + \frac{p^2}{1-p} f\right)\big((1-p)y + pz\big) \\ &= (p-1)(1-p)ey + p(1-p)hy + p^2 fy + (p-1)pez + p^2 hz + \frac{p^3}{1-p} fz \\ &= (p-1)(1-p)(0) + p(1-p)y + p^2 z + (p-1)py + p^2(-z) + \frac{p^3}{1-p}(0) \\ &= 0. \end{aligned}$$

The other cases are similar. $\square$

Observe that by (36) the set $\{y^{*N-i} z^{*i}\}_{i=0}^N$ forms a basis for V.

Lemma 7.2. *The elements e^*, h^*, f^* act on $\{y^{*N-i} z^{*i}\}_{i=0}^N$ as follows:*

$$e^*.(y^{*N-i}z^{*i}) = iy^{*N-i+1}z^{*i-1} \qquad \textit{for} \quad 1 \le i \le N, \qquad e^*.y^{*N} = 0,$$

$$h^*.(y^{*N-i}z^{*i}) = (N-2i)y^{*N-i}z^{*i} \qquad \textit{for} \quad 0 \le i \le N,$$

$$f^*.(y^{*N-i}z^{*i}) = (N-i)y^{*N-i-1}z^{*i+1} \qquad \textit{for} \quad 0 \le i \le N-1, \qquad f^*.z^{*N} = 0.$$

Proof. We compute the action of e^* on $y^{*N-i}z^{*i}$. Using Lemma 7.1, we have

$$\begin{aligned} & e^*.(y^{*N-i}z^{*i}) \\ &= (N-i)y^{*N-i-1}z^{*i}(e^*y^*) + iy^{*N-i}z^{*i-1}(e^*z^*) \\ &= (N-i)y^{*N-i-1}z^{*i}(0) + iy^{*N-i}z^{*i-1}(y) \\ &= iy^{*N-i+1}z^{*i-1}. \end{aligned}$$

The other cases are similar. ◻

Lemma 7.3. *With respect to the basis $\{y^{*N-i}z^{*i}\}_{i=0}^{N}$, the matrices representing e^*, h^*, f^* are as follows:*

$$e^* \ : \begin{bmatrix} 0 & 1 & & & & & \mathbf{0} \\ & 0 & 2 & & & & \\ & & 0 & 3 & & & \\ & & & \cdot & \cdot & & \\ & & & & \cdot & \cdot & \\ & & & & & 0 & N \\ \mathbf{0} & & & & & & 0 \end{bmatrix}, \qquad f^* \ : \begin{bmatrix} 0 & & & & & & \mathbf{0} \\ N & 0 & & & & & \\ & \cdot & \cdot & & & & \\ & & \cdot & \cdot & & & \\ & & & 3 & 0 & & \\ & & & & 2 & 0 & \\ \mathbf{0} & & & & & 1 & 0 \end{bmatrix}$$

$$h^* \ : \begin{bmatrix} N & & & & & & \mathbf{0} \\ & N-2 & & & & & \\ & & N-4 & & & & \\ & & & \cdot & & & \\ & & & & \cdot & & \\ & & & & & 2-N & \\ \mathbf{0} & & & & & & -N \end{bmatrix}.$$

Proof. By Lemma 7.2. ◻

For $i = 0, 1, \ldots, N$, define

$$V_i^* = \mathbb{C}y^{*N-i}z^{*i}. \tag{40}$$

Observe that $\dim V_i^* = 1$ and $V = \sum_{i=0}^N V_i^*$, which is the direct sum. We call V_i^* the h^*-weight space for the eigenvalue $N - 2i$ and call $V = \sum_{i=0}^N V_i^*$ the h^*-weight space decomposition of V.

For notational convenience, define

$$k_i = \binom{N}{i}\left(\frac{p}{1-p}\right)^i \qquad i = 0, 1, \ldots, N. \tag{41}$$

Note that $k_0 = 1$.

Definition 7.4. Define a bilinear form $\langle\ ,\ \rangle : V \times V \longrightarrow \mathbb{C}$ by

$$\langle y^{N-i}z^i, y^{N-j}z^j\rangle = \delta_{ij}\frac{1}{k_i(1-p)^N}, \qquad \text{for} \quad i, j = 0, 1, \ldots, N, \tag{42}$$

where the scalars k_i $(0 \le i \le N)$ are from (41).

We note that the bilinear form (42) is symmetric since

$$\langle y^{N-j}z^j, y^{N-i}z^i\rangle = \delta_{ji}\frac{1}{k_j(1-p)^N} = \langle y^{N-i}z^i, y^{N-j}z^j\rangle,$$

for $i, j = 0, 1, \ldots, N$. We also note that the bilinear form (42) is nondegenerate and satisfies

$$\langle V_i, V_j\rangle = 0 \qquad \text{if} \quad i \neq j \quad \text{for} \quad i, j \in 0, 1, \ldots, N. \tag{43}$$

Lemma 7.5. *For $\varphi \in L$, and $u, v \in V$*

$$\langle \varphi.u, v\rangle = \langle u, \varphi^\dagger.v\rangle, \tag{44}$$

where $\dagger$ is the antiautomorphism of L from Lemma 5.4.

Proof. Without loss of generality, we may assume φ is in the basis e, h, f and u, v are in the basis $\{y^{N-i}z^i\}_{i=0}^N$. Set $u = y^{N-i}z^i$ and $v = y^{N-j}z^j$. First, we assume

that $\varphi = e$. Using (42), we have

$$\begin{aligned}\langle e.u, v\rangle &= \langle e.y^{N-i}z^i, y^{N-j}z^j\rangle \\ &= \langle iy^{N-i+1}z^{i-1}, y^{N-j}z^j\rangle \qquad \text{(by Lemma 6.9)} \\ &= i\delta_{i-1,j}\frac{1}{k_j(1-p)^N}.\end{aligned}$$

In Lemma 5.6 we saw that $e^\dagger = \frac{p}{1-p}f$. So,

$$\begin{aligned}\langle u,\ e^\dagger.v\rangle &= \langle y^{N-i}z^i,\ \frac{p}{1-p}f.y^{N-j}z^j\rangle \\ &= \langle y^{N-i}z^i,\ \frac{p}{1-p}(N-j)y^{N-j-1}z^{j+1}\rangle \\ &= \frac{p(N-j)}{1-p}\langle y^{N-i}z^i,\ y^{N-j-1}z^{j+1}\rangle \\ &= \frac{p(N-j)}{1-p}\ \delta_{i,j+1}\ \frac{1}{k_i(1-p)^N}.\end{aligned}$$

By (41), $(1-p)ik_i = p(N-j)k_j$, provided that $i-1=j$. That is, $(1-p)(j+1)k_{j+1} = p(N-j)k_j$. Thus, we have

$$\langle e.u, v\rangle = \langle u, e^\dagger.v\rangle.$$

The other cases for $\varphi = h$ and $\varphi = f$ are similar. □

Lemma 7.6. *Recall the spaces $\{V_i^*\}_{i=0}^N$ from* (40). *The bilinear form $\langle\ ,\ \rangle$ satisfies*

$$\langle V_i^*, V_j^*\rangle = 0 \qquad \textit{if} \quad i \neq j. \tag{45}$$

Proof. For $0 \le i, j \le N$ with $i \neq j$, pick $u \in V_i^*$ and $v \in V_j^*$ so that

$$h^*.u = (N-2i)u \qquad \text{and} \qquad h^*.v = (N-2j)v.$$

Now observe,

$$\begin{aligned}(N-2i)\langle u, v\rangle = \langle h^*.u, v\rangle &= \langle u, h^{*\dagger}.v\rangle \qquad \text{(by (44))} \\ &= \langle u, h^*.v\rangle \\ &= (N-2j)\langle u, v\rangle.\end{aligned}$$

Thus, we have

$$((N-2i)-(N-2j))\langle u,v\rangle = 0 \implies 2(j-i)\langle u,v\rangle = 0.$$

Since $i \neq j$, it follows $\langle u,v\rangle = 0$. The result follows. $\square$

In linear algebra, for a given basis $\{u_i\}_{i=0}^{n}$ for V, there exists a unique basis $\{v_i\}_{i=0}^{n}$ for V such that $\langle u_i, v_j\rangle = \delta_{ij}$. The bases $\{u_i\}_{i=0}^{n}$ and $\{v_i\}_{i=0}^{n}$ are said to be *dual* with respect to $\langle\ ,\ \rangle$.

Lemma 7.7. *With respect to $\langle\ ,\ \rangle$, the bases*

$$\{y^{N-i}z^i\}_{i=0}^{N} \qquad \text{and} \qquad \{k_i(1-p)^N y^{N-i}z^i\}_{i=0}^{N}$$

for V are dual.

Proof. Since

$$\begin{aligned}\langle y^{N-i}z^i,\ k_j(1-p)^N y^{N-j}z^j\rangle &= k_j(1-p)^N\langle y^{N-i}z^i, y^{N-j}z^j\rangle \\ &= k_j(1-p)^N \delta \frac{1}{k_j(1-p)^N} \\ &= \delta_{ij},\end{aligned}$$

the result follows. $\square$

Lemma 7.8. *For the dual basis $\{k_i(1-p)^N y^{N-i}z^i\}_{i=0}^{N}$ from Lemma 7.7, the sum of the basis vectors is y^{*N}.*

Proof. We have

$$\begin{aligned}\sum_{i=0}^{N} k_i(1-p)^N y^{N-i}z^i &= \sum_{i=0}^{N}\binom{N}{i}(1-p)^N p^i(1-p)^{-i}y^{N-i}z^i \\ &= \sum_{i=0}^{N}\binom{N}{i}((1-p)y)^{N-i}(pz)^i \\ &= ((1-p)y+pz)^N \\ &= y^{*N},\end{aligned}$$

where the last equality holds by (36). $\square$

Lemma 7.9. *For $i, j = 0, 1, 2, \ldots, N$,*

$$\langle y^{*N-i} z^{*i}, y^{*N-j} z^{*j} \rangle = \delta_{ij} k_i^{-1}. \tag{46}$$

Proof. Suppose $i \neq j$. Since $y^{*N-i} z^{*i} \in V_i{}^*$, $y^{*N-j} z^{*j} \in V_j{}^*$, and $\langle V_i{}^*, V_j{}^* \rangle = 0$ by Lemma 7.6, it follows that

$$\langle y^{*N-i} z^{*i}, y^{*N-j} z^{*j} \rangle = 0.$$

Assume $i = j$. We will use induction on i. First, assume $i = 0$. We see that

$$\begin{aligned}
\langle y^{*N}, y^{*N} \rangle &= \| y^{*N} \|^2 \\
&= \left\| \sum_{\ell=0}^{N} k_\ell (1-p)^N y^{N-\ell} z^\ell \right\|^2 \\
&= \sum_{\ell=0}^{N} \left\| k_\ell (1-p)^N y^{N-\ell} z^\ell \right\|^2 \\
&= \sum_{\ell=0}^{N} k_\ell^2 (1-p)^{2N} \left\| y^{N-\ell} z^\ell \right\|^2 \\
&= \sum_{\ell=0}^{N} k_\ell^2 (1-p)^{2N} \frac{1}{k_\ell (1-p)^N} \\
&= \sum_{\ell=0}^{N} k_\ell (1-p)^N \\
&= \sum_{\ell=0}^{N} \binom{N}{\ell} (1-p)^N \frac{p^\ell}{(1-p)^\ell} \\
&= \sum_{\ell=0}^{N} \binom{N}{\ell} (1-p)^{N-\ell} p^\ell \\
&= \Big((1-p) + p \Big)^N \\
&= 1.
\end{aligned}$$

Therefore, (46) holds for $i = 0$. Next, assume that $i \geq 1$. By Lemma 7.5,

$$\left\langle e^*.(y^{*N-i}z^{*i}), y^{*N-i+1}z^{*i-1}\right\rangle = \left\langle y^{*N-i}z^{*i}, e^{*\dagger}.(y^{*N-i+1}z^{*i-1})\right\rangle. \tag{47}$$

Looking at the left-hand side of (47), we have

$$\begin{aligned}\left\langle e^*.(y^{*N-i}z^{*i}), y^{*N-i+1}z^{*i-1}\right\rangle &= \langle iy^{*N-i+1}z^{*i-1}, y^{*N-i+1}z^{*i-1}\rangle \\ &= i\left\|y^{*N-i+1}z^{*i-1}\right\|^2 \\ &= i(k_{i-1})^{-1},\end{aligned}$$

where the last equality holds by our induction hypothesis. Recall from Lemma 5.6 that $e^{*\dagger} = \frac{p}{1-p}f^*$. Using this, we compute the right-hand side of (47) to get

$$\begin{aligned}\left\langle y^{*N-i}z^{*i}, e^{*\dagger}.(y^{*N-i+1}z^{*i-1})\right\rangle &= \left\langle y^{*N-i}z^{*i}, \frac{p}{1-p}f^*(y^{*N-i+1}z^{*i-1})\right\rangle \\ &= \frac{p}{1-p}\left\langle y^{*N-i}z^{*i}, (N-i+1)y^{*N-i}z^{*i}\right\rangle \\ &= \frac{p}{1-p}(N-i+1)\left\|y^{*N-i}z^{*i}\right\|^2.\end{aligned}$$

By these comments, we have

$$i(k_{i-1})^{-1} = \frac{p}{1-p}(N-i+1)\left\|y^{*N-i}z^{*i}\right\|^2,$$

which implies

$$\begin{aligned}\left\|y^{*N-i}z^{*i}\right\|^2 &= i(k_{i-1})^{-1}\frac{1-p}{p(N-i+1)} \\ &= i\frac{(i-1)!(N-i+1)!}{N!}\frac{(1-p)^{i-1}}{p^{i-1}}\frac{1-p}{p(N-i+1)} \\ &= \frac{i!(N-i)!}{N!}\frac{(1-p)^2}{p^i} \\ &= k_i^{-1}.\end{aligned}$$

Therefore, (47) holds at i. This lemma has been proven. $\square$

Lemma 7.10. *With respect to* $\langle\ ,\ \rangle$, *the bases*

$$\{y^{*N-i}z^{*i}\}_{i=0}^{N} \qquad \textit{and} \qquad \{k_i y^{*N-i}z^{*i}\}_{i=0}^{N}$$

for V *are dual.*

Proof. We have

$$\begin{aligned}
&\langle y^{*N-i}z^{*i}, k_j y^{*N-j}z^{*j}\rangle \\
&= k_j\langle y^{*N-i}z^{*i}, y^{*N-j}z^{*j}\rangle \\
&= k_j\delta_{ij}k_i^{-1} \\
&= \delta_{ij}.
\end{aligned}$$

□

Lemma 7.11. *For the dual basis* $\{k_i y^{*N-i}z^{*i}\}_{i=0}^{N}$ *from Lemma 7.10, the sum of the basis vectors is* y^N.

Proof. We have

$$\begin{aligned}
\sum_{i=0}^{N} k_i y^{*N-i}z^{*i} &= \sum_{i=0}^{N}\binom{N}{i}\left(\frac{p}{1-p}\right)^{i} y^{*N-i}z^{*i} && \text{(by (41))}\\
&= \sum_{i=0}^{N}\binom{N}{i} y^{*N-i}\left(\frac{p}{1-p}z^*\right)^{i}\\
&= \left(y^* + \frac{p}{1-p}z^*\right)^{N}\\
&= y^N,
\end{aligned}$$

where the last equality holds by (37). □

We have discussed two bases $\{y^{N-i}z^i\}_{i=0}^{N}$ and $\{y^{*N-i}z^{*i}\}_{i=0}^{N}$ for V. We now describe each element of one as a linear combination of the other basis elements. We will use Krawtchouk polynomials $\{K_i(x)\}_{i=0}^{N}$:

$$K_i(x) = K_i(x;p,N) = \sum_{n=0}^{N}\frac{(-i)_n(-x)_n}{(-N)_n}\frac{1}{n!}\left(\frac{1}{p}\right)^{n}, \tag{48}$$

where $(\alpha)_n = \alpha \cdot (\alpha+1) \cdot (\alpha+2) \dots (\alpha+n-1)$.

Lemma 7.12. *For $j = 0, 1, 2, \dots, N$, we have*

(i) $y^{*N-j} z^{*j} = \sum_{i=0}^{N} \binom{N}{i} (1-p)^{N-i} p^i K_i(j) y^{N-i} z^i$.

(ii) $y^{N-j} z^j = \sum_{i=0}^{N} \binom{N}{i} \left(\frac{p}{1-p}\right)^i K_i(j) y^{*N-i} z^{*i}$.

Proof. (i): Recall from (37) that $z = y^* - z^*$. So, $z^* = y^* - z$. Using this, the left-hand side of the equation (i) becomes

$$
\begin{aligned}
y^{*N-j} z^{*j} &= y^{*N-j} (y^* - z)^j \\
&= y^{*N-j} \sum_{\ell=0}^{j} \binom{j}{\ell} (-1)^\ell y^{*j-\ell} z^\ell \\
&= \sum_{\ell=0}^{j} \binom{j}{\ell} (-1)^\ell y^{*N-\ell} z^\ell \\
&= \sum_{\ell=0}^{j} \frac{j!}{\ell!(j-\ell)!} (-1)^\ell y^{*N-\ell} z^\ell \\
&= \sum_{\ell=0}^{j} \frac{j(j-1)\cdots(j-\ell+1)}{\ell!} (-1)^\ell y^{*N-\ell} z^\ell \\
&= \sum_{\ell=0}^{j} \frac{(-j)_\ell}{\ell!} y^{*N-\ell} z^\ell \\
&= \sum_{\ell=0}^{N} \frac{(-j)_\ell}{\ell!} y^{*N-\ell} z^\ell \qquad \text{(since } (-j)_\ell = 0 \text{ if } \ell > j\text{)}. \qquad (49)
\end{aligned}
$$

Next, using (48) the right-hand side of the equation (i) becomes

$$
\begin{aligned}
&\sum_{i=0}^{N} \binom{N}{i} (1-p)^{N-i} p^i K_i(j) y^{N-i} z^i \\
&= \sum_{i=0}^{N} \binom{N}{i} (1-p)^{N-i} p^i \left(\sum_{\ell=0}^{N} \frac{(-i)_\ell (-j)_\ell}{(-N)_\ell} \frac{1}{\ell!} \frac{1}{p^\ell} \right) y^{N-i} z^i \\
&= \sum_{\ell=0}^{N} \frac{(-j)_\ell}{\ell!} \frac{1}{(-N)_\ell p^\ell} \sum_{i=0}^{N} \binom{N}{i} (1-p)^{N-i} p^i (-i)_\ell y^{N-i} z^i. \qquad (50)
\end{aligned}
$$

Comparing (49) and (50), it suffices to show that for $\ell = 0, 1, 2, \ldots, N$

$$y^{*N-\ell} z^{\ell} = \frac{1}{(-N)_\ell p^\ell} \sum_{i=0}^{N} \binom{N}{i} (1-p)^{N-i} p^i (-i)_\ell y^{N-i} z^i. \tag{51}$$

On the right-hand side of (51), the ith term vanishes for $i < \ell$ since

$$(-i)_\ell = (-i)(-i+1)\ldots(-i+\ell-1).$$

So, by setting $r = i - \ell$, the right-hand side of (51) is

$$\begin{aligned}
&\frac{1}{(-N)_\ell p^\ell} \sum_{i=0}^{N} \binom{N}{i} (1-p)^{N-i} p^i (-i)_\ell y^{N-i} z^i \\
&= \frac{1}{(-N)_\ell p^\ell} \sum_{r=0}^{N-\ell} \binom{N}{r+\ell} (1-p)^{N-r-\ell} p^{r+\ell} (-r-\ell)_\ell y^{N-r-\ell} z^{r+\ell} \\
&= (N-\ell)! \sum_{r=0}^{N-\ell} \frac{1}{(N-r-\ell)! r!} (1-p)^{N-r-\ell} p^r y^{N-r-\ell} z^{r+\ell} \\
&= z^\ell \sum_{r=0}^{N-\ell} \frac{(N-\ell)!}{(N-r-\ell)! r!} (1-p)^{N-r-\ell} p^r y^{N-r-\ell} z^r \\
&= z^\ell \sum_{r=0}^{N-\ell} \binom{N-\ell}{r} ((1-p)y)^{N-\ell-r} (zp)^r \\
&= z^\ell ((1-p)y + pz)^{N-\ell} \\
&= z^\ell y^{*N-\ell} \qquad\qquad \text{(by (36))}.
\end{aligned}$$

Therefore, we showed that(51) holds and proved (i).

(ii): Similar to (i). $\square$

8. Main Theorem

In the previous section, we discussed two bases $\{y^{N-i}z^i\}_{i=0}^N$ and $\{y^{*N-i}z^{*i}\}_{i=0}^N$. In this section, we find the inner products between these two bases. We recall the Krawtchouk polynomials $K_i(x)$ from (48).

Theorem 8.1. *For* $i, j = 0, 1, \ldots, N$,

$$\langle y^{N-i} z^i, y^{*N-j} z^{*j} \rangle = K_i(j) \tag{52}$$

Proof. Evaluate the left-hand side of (52) using (41), (42), and Lemma 7.12(i) to get

$$\begin{aligned}
\langle y^{N-i} z^i, y^{*N-j} z^{*j} \rangle &= \left\langle y^{N-i} z^i, \sum_{\ell=0}^{N} \binom{N}{\ell} ((1-p))^{N-\ell} p^\ell K_\ell(j) y^{N-\ell} z^\ell \right\rangle \\
&= \sum_{\ell=0}^{N} \binom{N}{\ell} ((1-p))^{N-\ell} p^\ell K_\ell(j) \langle y^{N-i} z^i, y^{N-\ell} z^\ell \rangle \\
&= \sum_{\ell=0}^{N} \binom{N}{\ell} ((1-p))^{N-\ell} p^\ell K_\ell(j) \delta_{i\ell} \frac{1}{k_i (1-p)^N} \\
&= \binom{N}{i} ((1-p))^{N-i} p^i K_i(j) \frac{1}{\binom{N}{i} \frac{p^i}{(1-p)^i} (1-p)^N} \\
&= K_i(j),
\end{aligned}$$

which is the right-hand side of (52). □

Now, we define $\mathbb{C}$-linear transformations $A : V \longrightarrow V$ and $A^* : V \longrightarrow V$ by

$$A = \frac{NI - a}{2}, \qquad A^* = \frac{NI - a^*}{2}, \tag{53}$$

where a, a^* are from (16):

$$a = 2(1-p)e + (1-2p)h + 2pf, \qquad a^* = h. \tag{54}$$

Note that on V,

$$a = NI - 2A, \qquad a^* = N - 2A^*. \tag{55}$$

This brings us to the following theorem.

Theorem 8.2. *For* $j = 0, 1, 2, \ldots, N$, *the following* (i), (ii) *hold.*

(i) $K_j(A) y^N = y^{N-j} z^j$

(ii) $K_j(A^*) y^{*N} = y^{*N-j} z^{*j}$

Proof. (i): By Lemma 4.2, we have

$$h^* = 2(1-p)e + (1-2p)h + 2pf = a.$$

By Lemma 7.2, for $i = 0, 1, 2, \ldots, N$, the vector $y^{*N-i}z^{*i}$ is an eigenvector for a with eigenvalue $N - 2i$. That is,

$$a.y^{*N-i}z^{*i} = (N-2i)y^{*N-i}z^{*i}.$$

Since $a = NI - 2A$, we have

$$\begin{aligned} & (NI - 2A)y^{*N-i}z^{*i} = (N-2i)y^{*N-i}z^{*i} \\ \Longrightarrow\quad & Ny^{*N-i}z^{*i} - 2Ay^{*N-i}z^{*i} = Ny^{*N-i}z^{*i} - 2iy^{*N-i}z^{*i} \\ \Longrightarrow\quad & Ay^{*N-i}z^{*i} = iy^{*N-i}z^{*i}. \end{aligned}$$

So, $y^{*N-i}z^{*i}$ is an eigenvector for A with eigenvalue i. Therefore,

$$\begin{aligned} K_j(A)y^N &= K_j(A)\sum_{i=0}^{N} y^{*N-i}z^{*i}k_i && \text{(by Lemma 7.11)} \\ &= \sum_{i=0}^{N} k_i K_j(i) y^{*N-i}z^{*i} && \text{(since } Ay^{*N-i}z^{*i} = iy^{*N-i}z^{*i}\text{)} \\ &= y^{N-j}z^j && \text{(by Lemma 7.12(ii)).} \end{aligned}$$

Therefore, we have shown (i).

(ii) The proof is similar. □

9. Applications

In this section, we use the results discussed in the previous section to obtain some well-known facts of Krawtchouk polynomials, including orthogonality relations, three-term recurrence, difference equation, and generating function.

Theorem 9.1 (Orthogonality relations)**.** *Krawtchouk polynomials satisfy the following orthogonality relations.*

(i) *For* $i, j = 0, 1, 2, \ldots, N$,

$$\sum_{n=0}^{N} K_n(i)K_n(j)\binom{N}{n}p^n(1-p)^{N-n} = \delta_{i,j}\binom{N}{i}^{-1}\left(\frac{1-p}{p}\right)^i. \tag{56}$$

(ii) *For* $m, n = 0, 1, 2, \ldots, N$,

$$\sum_{i=0}^{N} K_m(i)K_n(j)\binom{N}{i}p^i(1-p)^{N-i} = \delta_{m,n}\binom{N}{n}^{-1}\left(\frac{1-p}{p}\right)^n. \tag{57}$$

Proof. (i): Compute $\langle y^{*N-i}z^{*i}, y^{*N-j}z^{*j}\rangle$ in two ways. On one hand, using (46), (41) we have

$$\langle y^{*N-i}z^{*i}, y^{*N-j}z^{*j}\rangle = \delta_{ij}k_i^{-1} = \delta_{ij}\binom{N}{i}^{-1}\left(\frac{1-p}{p}\right)^i. \tag{58}$$

On the other hand, by Lemma 7.12(i) we write

$$y^{*N-i}z^{*i} = \sum_{n=0}^{N}\binom{N}{n}(1-p)^{N-n}p^nK_n(i)y^{N-n}z^n, \tag{59}$$

$$y^{*N-j}z^{*j} = \sum_{m=0}^{N}\binom{N}{m}(1-p)^{N-m}p^mK_m(j)y^{N-m}z^m. \tag{60}$$

Compute $\langle y^{*N-i}z^{*i}, y^{*N-j}z^{*j}\rangle$ using the equations (59) and (60) to get

$$\begin{aligned}&\left(\sum_{n=0}^{N}\binom{N}{n}(1-p)^{N-n}p^nK_n(i)\right)\left(\sum_{m=0}^{N}\binom{N}{m}(1-p)^{N-m}p^mK_m(j)\right)\\ &\qquad\times\left\langle y^{N-n}z^n, y^{N-m}z^m\right\rangle.\end{aligned} \tag{61}$$

Using (42), we have

$$\left\langle y^{N-n}z^n, y^{N-m}z^m\right\rangle = \delta_{n,m}\binom{N}{n}^{-1}\left(\frac{p}{1-p}\right)^{-n}(1-p)^{-N} \tag{62}$$

Evaluate (61) using (62) to get

$$\sum_{n=0}^{N}\binom{N}{n}K_n(i)K_n(j)(1-p)^{N-n}p^n. \tag{63}$$

Therefore, by (58) and (63) the result follows.

(ii): The proof is similar to (i). Use the fact $K_i(j) = K_j(i)$. $\square$

Theorem 9.2 (Three-term recurrence)**.** *Krawtchouk polynomials satisfy the following three-term recurrence. For $i, x = 0, 1, 2, \ldots, N$,*

$$xK_i(x) = i(p-1)K_{i-1}(x) - \big(i(p-1) + (i-N)p\big)K_i(x) + (i-N)pK_{i+1}(x). \quad (64)$$

Proof. Since $h^\dagger = h$, by Lemma 7.5

$$\left\langle h.(y^{N-x}z^x), y^{*N-i}z^{*i}\right\rangle = \left\langle y^{N-x}z^x, h.(y^{*N-i}z^{*i})\right\rangle. \quad (65)$$

Evaluating the left-hand side of (65), we have

$$\begin{aligned}
\left\langle h.(y^{N-x}z^x), y^{*N-i}z^{*i}\right\rangle &= \left\langle (N-2x)y^{N-x}z^x, y^{*N-i}z^{*i}\right\rangle && \text{(by Lemma 6.9)}\\
&= (N-2x)\left\langle (y^{N-x}z^x, y^{*N-i}z^{*i}\right\rangle \\
&= (N-2x)K_x(i) && \text{(by Theorem 8.1)}\\
&= (N-2x)K_i(x).
\end{aligned}$$

Next, we evaluate the right-hand side of (65). To do this, we recall the transition matrix (26). By this matrix, we have

$$h = 2(1-p)e^* + (1-2p)h^* + 2pf^*. \quad (66)$$

Evaluating the right-hand side of (65) and simplifying the result using Lemma 7.2 and Theorem 8.1, we have

$$\begin{aligned}
&\left\langle y^{N-x}z^x, (2(1-p)e^* + (1-2p)h^* + 2pf^*).y^{*N-i}z^{*i}\right\rangle \\
&= \left\langle y^{N-x}z^x, 2(1-p)e^*.(y^{*N-i}z^{*i}) + (1-2p)h^*.(y^{*N-i}z^{*i}) + 2pf^*.(y^{*N-i}z^{*i})\right\rangle \\
&= \left\langle y^{N-x}z^x, 2(1-p)iy^{*N-i+1}z^{*i-1}\right\rangle + \left\langle y^{N-x}z^x, (1-2p)(N-2i)y^{*N-i}z^{*i}\right\rangle \\
&\qquad + \left\langle y^{N-x}z^x, 2p(N-i)y^{*N-i-1}z^{*i+1}\right\rangle \\
&= 2(1-p)_iK_x(i-1) + (1-2p)(N-2i)K_x(i) + 2p(N-i)K_x(i+1) \\
&= 2(1-p)_iK_{i-1}(x) + (1-2p)(N-2i)K_i(x) + 2p(N-i)K_{i+1}(x).
\end{aligned}$$

We have evaluated both sides of the equation (65). By these comments we have

$$(N-2x)K_i(x) = 2(1-p)iK_{i-1}(x) + (1-2p)(N-2i)K_i(x) + 2p(N-i)K_{i+1}(x).$$

Rearranging the above equation, we obtain (64). The proof has been completed. □

Theorem 9.3 (Difference equation)**.** *Krawtchouk polynomials satisfy the following difference equation. For* $i, x = 0, 1, 2, \ldots, N$*,*

$$iK_i(x) = x(p-1)K_i(x-1) - (x(p-1)+(x-N)p)K_i(x) + (x-N)pK_i(x+1). \quad (67)$$

Proof. Exchange i on x in the three-term recurrence (64):

$$iK_x(i) = x(p-1)K_{x-1}(i) - \big(x(p-1)+(x-N)p\big)K_x(i) + (x-N)pK_{x+1}(i).$$

Apply $K_i(j) = K_j(i)$ to the above equation to get (67). □

Theorem 9.4 (Generating function)**.** *Let* t *be an indeterminate. Then for* $x = 0, 1, 2, \ldots, N$*,*

$$\left(1 - \frac{1-p}{p}t\right)^x (1+t)^{N-x} = \sum_{i=0}^{N} \binom{N}{i} K_i(x) t^i. \quad (68)$$

Proof. In Lemma 7.12(i), substitute y with $\frac{1}{1-p}$ and z with $\frac{t}{p}$ and simplify the result to get

$$\begin{aligned} y^{*N-j} z^{*j} &= \sum_{i=0}^{N} \binom{N}{i} (1-p)^{N-i} p^i K_i(j) \left(\frac{1}{1-p}\right)^{N-i} \left(\frac{t}{p}\right)^i \\ &= \sum_{i=0}^{N} \binom{N}{i} K_i(j) t^i. \end{aligned} \quad (69)$$

By (36) together with $y = \frac{1}{1-p}$ and $z = \frac{t}{p}$, we have

$$y^* = (1-p)y + pz = (1-p)\frac{1}{1-p} + p\frac{t}{p} = 1 + t$$

and

$$z^* = (1-p)y + (p-1)z = (1-p)\frac{1}{1-p} + (p-1)\frac{t}{p} = 1 + \frac{p-1}{p}t.$$

Thus, we have

$$y^{*N-j}z^{*j} = (1+t)^{N-j}\left(1+\frac{p-1}{p}t\right)^{j}. \tag{70}$$

Combining (69) with (70) and by setting $j = x$, we have (68). $\square$

References

[1] J. E. Humphreys. *Introduction to Lie Algebras and Representation Theory. Grad. Texts Math. 9, Springer*, New York and Berlin, 1972.

[2] K. Nomura, P. Terwilliger. *Krawtchouk polynomials, the Lie algebra sl2, and Leonard pairs. Linear Algebra Appl.* 437 (1) (2012), pp. 345-375.

[3] P. Feinsilver, J. Kocik, Krawtchouk polynomials and Krawtchouk matrices, in: Recent Advances in Applied Probability, Edited by Ricardo Baeza-Yates, etc. Springer Science+Business Media, Inc. 2005, pp. 115–141; arXiv:quant-ph/0702073.

[4] P. Feinsilver, Lie algebras and recurrence relations I, Acta Appl. Math. 13 (1988) 291–333.

[5] P. Feinsilver, Representations of sl(2) in the Boolean lattice, and the Hamming and Johnson schemes, preprint, arXiv:1102.0368.

[6] V. Mazorchuk. *Lectures on sl2 (C)-Modules. Imperial College Press.* London 2010.