Astronomy Laboratory Exercise 15

The Motions of Earth

A significant advantage of the heliocentric model of the Solar System is how it naturally explains the daily and annual motions of the Sun, Moon, and stars as resulting from the rotation and revolution of Earth. But other motions of Earth are required by current models of the Universe, which are less well known. The Earth, for example, moves as a result of the Moon's gravity, as evidenced by tides in the oceans. And Earth moves with the Solar System in an orbit around the disk of the Milky Way galaxy, while the Milky Way orbits in a cluster of galaxies, which are collectively falling toward a much larger cluster of galaxies. These motions of Earth are discussed and compared in this exercise.

Physicists distinguish velocity from speed by indicating velocity is a vector, a quantity having both a magnitude and a direction, while speed is a scalar, a quantity having only a magnitude. To illustrate, an automobile's speedometer gives only speed. A driver must know the direction from other information! Speed is measured in units of distance divided by time, such as m/s or km/hr.

Earth's rotation gives all locations on Earth, except at the poles, a velocity to the east. The speed depends on the distance from the equator and decreases with the cosine of latitude. So the speed due to rotation is zero at the poles.

Example 1: Calculate the speed of a person at the equator due to Earth's rotation. This speed is one Earth's circumference per day, where the circumference is $2\pi R$, $R$ being the radius of Earth (6,378 km). $2\pi R / 1 \text{ day} = 464 \text{ m/s or } 1670 \text{ km/hr}$. For comparison, cars on interstate highways typically travel 100 km/hr, while commercial jet aircraft usually cruise at a little over 1000 km/hr.

The speed from Earth's revolution about the Sun is Earth's orbital path length divided by one year. The direction of this velocity is along Earth's orbit. The direction of this velocity rotates slowly in the plane of the ecliptic, making one revolution per year. This velocity will appear to be changing direction continuously from a fixed location on Earth, because of Earth's rotation. It will be approximately east on Earth at solar midnight, while near the equator it will be approximately straight up at sunrise, and it will be close to west at solar noon. It is the tilt of Earth's rotational axis relative to the plane of the ecliptic that causes the eastward velocity from the Earth's rotation to swing daily from a maximum of $23.5^\circ$ above the plane of the ecliptic to $23.5^\circ$ below it. Figure 15-1 illustrates the relative directions of these velocities.

A globe showing the tilt of Earth's axis provides a useful display for these velocities. A pointer, such as a pencil, laid on the table by the base of the globe, pointing away from the globe, can show the direction of the velocity from revolution. A second pointer held against
Figure 15-1. The relative directions of velocities from the motions Earth from rotation and revolution, A. as seen from above the ecliptic, and B. as seen from the plane of the ecliptic at the time of the summer solstice in the northern hemisphere.

and parallel to the equator, while turning the globe and second pointer together, will show in three-dimensions the relative directions of these velocities.

Some other motions, considered below, have higher speeds than those just considered, and collectively all of these motions are identified as major motions of Earth. Still other motions, also to be considered below, have much lower speeds and are considered to be minor motions. Major motions result from Earth's rotation and revolution, from the Solar System moving within the Galaxy, and from movements of the Galaxy.

The most significant motion inside the Milky Way galaxy is the Sun moving toward Vega at about 20 km/s, and moving with Vega in a nearly circular orbit through the Galaxy's disk and around the Galaxy's center. The orbit around the Galaxy occurs with a speed of about 240 km/s, and is in the direction toward the constellation Cygnus and away from Orion. The center of the galaxy is in the direction of the constellation Sagittarius. The circuit time, sometimes called a galactic year, is about 230 million years. These motions are detected by several methods, including the Doppler shifting of 21 cm radiation from hydrogen located through out the galaxy disk.

The Milky Way galaxy is also in orbit about the center of mass of a cluster of about 30 galaxies called the Local Group. The Local Group contains the Milky Way, Andromeda galaxy, the Large and Small Magellanic Clouds, all of which can be seen with the unaided
eyes from Earth, and other galaxies. The Milky Way is moving with a velocity of about 80 km/s towards Andromeda galaxy. Additionally, the Local Group is moving collectively towards a local supercluster, the Virgo Cluster, which is in the direction towards the Hydra-Centaurus supercluster, beyond which lies a more massive supercluster, called the Great Attractor. Our motion in that direction was detected from variations in the cosmic background radiation, recorded by the COBE (Cosmic Background Explorer) satellite, and is believed to be due to a large mass concentration in the Great Attractor. The Local Group is moving with a speed of about 500 km/s towards the Great Attractor. A globe showing the locations of the constellations as they appear on the celestial sphere from Earth provides a useful model on which to show the relative directions of these velocities.

There are still other motions of Earth, with much smaller associated speeds than those just considered. These minor motions are included here because their effects are readily detected. One of these motions results from the Earth and Moon orbiting the center of mass of the Earth-Moon system. The period of this motion is about a month, and the path length is relatively short, so the speed is very low. Nevertheless, this motion helps create tides in Earth's oceans, it contributes to the precession of the Earth's axis, and also to slowing Earth's rate of rotation. The latter effect lengthens the day by about 2 milliseconds per century.

Another minor motion is associated with the precession of Earth's axis, which causes the celestial poles of Earth to move around a circle with an angular radius of 23.5°. The period of this motion is about 26,000 years. Because of this long period, the resulting speeds are extremely low, and are discussed here only because this motion causes the dates when the Sun enters different constellations of the zodiac to slowly change. The position of the Sun at the time of birth determines one's sign in the practice of natal Sun-sign astrology. Ancient astrology also considered the positions of the Moon and other then-known planets at the time of birth. Apparently this creates too many variations in horoscopes charts for today's tabloids. Even just using a simplified "Sun-only" system, modern tabloid astrology has not kept up with the slow precession of Earth's axis, as illustrated in the exercise on Star Wheels.

**Acceleration** of an object is defined as the rate its velocity changes. Acceleration, like velocity, is a vector. The units of acceleration are speed per time and are usually given as distance per second per second, such as m/s² or km/s². The acceleration due to gravity on the Earth's surface, usually called g, is 9.8 m/s² down. This means that an object dropped from rest will accelerate to a speed of 9.8 m/s after it has fallen for 1 second, ignoring any friction, such as with air. g provides a useful reference for other accelerations.

Dealing with gravity is an important part of every day life for people on Earth, and healthy adults are usually so accustomed to doing it that they don't often consciously think about it. But when a person loses consciousness for a moment, as when a boxer or other athlete is knocked out, the effects of gravity on that person, and their temporary inability to cope with it, are immediately apparent. Gravity continually pulls us down. When we think
about it, we can feel the forces we use to stop ourselves from being accelerated downward and into whatever is below us.

Accelerations from the various motions of Earth are all very small compared to \( g \), and so may be expected to go mostly unnoticed. The small values of these accelerations may also be surprising because of the high associated speeds described above. But small accelerations accumulated for long periods of time may result in high speeds.

Example 2: Find the velocity achieved by an object that is accelerated by \( g \) for six months.

\[
v = (9.8 \, \text{m/s}) \cdot (1.6 \cdot 10^6 \, \text{s}) = 1.6 \cdot 10^8 \, \text{m/s}.
\]

Note that this value is just over half the speed of light. \( g \) is a large acceleration.

When an object travels around a circle at constant speed, the velocity changes direction continuously. This change is defined as centripetal acceleration, and this acceleration results from the force that produces the circular motion. Many amusement park rides take advantage of centripetal acceleration. For example, one may ride inside a large rotating barrel. While spinning, the bottom of the barrel can be removed. The ride is exciting because it looks like one will fall, but no one falls because they are being pressed out against the barrel sides. The centripetal acceleration is toward the center of the barrel, and has the magnitude, \( a_c = \frac{v^2}{r} \), where \( v \) is the velocity and \( r \) is the radius of the circular motion. For orbital motion, the velocity is the orbit length divided by the orbit period \( T \), or \( v = \frac{2\pi r}{T} \). Inserting this value of \( v \) into the equation for \( a_c \), we have, \( a_c = \frac{4\pi^2 r}{T^2} \). This formula will be used below. Note that the units of \( a_c \) are just those of acceleration, \( \text{m/s}^2 \).

When a satellite orbits a massive body, the gravitational force of attraction from the massive body accelerates the satellite, creating the satellite's centripetal acceleration. The satellite is said to be weightless, or in free fall, as it is "falling around" the central massive body. The environment inside an orbiting space shuttle is described as being micro-\( g \), indicating that parts of the shuttle closer to Earth are attracted to Earth by slightly more force than parts further away. But the structure of the shuttle keeps all of its parts moving together as one body. These different forces of gravity acting on different parts of a body are classified as tidal forces, and their differences become larger as the orbiting and orbited objects become larger, more massive, and closer together. A prominent theory on the origin of planetary rings is that they result when tidal forces break apart satellites that stray too close to a planet. Comets have been observed to break into pieces near planets, verifying such effects.

Earth experiences centripetal acceleration from its revolution about the Sun, and also from co-orbiting with the Moon. If it were not for the Moon, the center of mass of the Earth would move in an elliptical orbit around the Sun. But the Moon's presence causes the Earth and Moon to co-orbit their common center of mass, as that center of mass orbits the Sun. Consider the motion of a hypothetical double planet consisting of two planets with
equal mass. The center of mass of this pair would be halfway between them, and that center of mass would travel in an ellipse about their star.

Example 3: Find the distance from the center of Earth to the center of mass of the Earth-Moon system.

The location of the center of mass away from the center of Earth is \( D_e = \frac{D_M M_M}{M_M + M_E} \), where \( D_M \) is the distance from the center of Earth to the center of the Moon, and \( M_M \) and \( M_E \) are the masses of the Moon and Earth, respectively. The usual values give, \( D_e = 4,674 \text{ km} \), which, since the radius of Earth is 6,378 km, is still inside Earth.

Example 4: Determine the velocity and acceleration that results from Earth co-orbiting the Moon, at the place on Earth's surface closest to the Moon, and also at the place furthest from the Moon.

On the side of Earth closest to the Moon, the distance from the Earth-Moon center of mass is \( 6378 - 4674 \text{ km} = 1704 \text{ km} \), while on the side farthest from the Moon the distance is \( 6378 + 4674 \text{ km} = 11,052 \text{ km} \). The centripetal accelerations at these locations are, \((2\pi)^2 \times 1704 \text{ km}/(1 \text{ month})^2 = 0.010 \text{ mm/s}^2 \), and \((2\pi)^2 \times 11,052 \text{ km}/(1 \text{ month})^2 = 2.3 \times 10^{-8} \text{ km/s}^2 = 0.023 \text{ mm/s}^2 \) respectively.

The effects of tidal forces from Earth on the Moon have caused the Moon's rotation and revolution about Earth to become synchronized, so the same side of the Moon faces Earth all of the time. This same process is slowing Earth's rotation. If it could proceed for long enough, it would ultimately synchronize the rotation of Earth and revolution of the Moon, causing the lunar month to equal the solar day, which would then be equal to about 47 of our present solar days. The time required for this to happen is estimated to be over 40 billion years, longer than the Sun is expected to last. However, Pluto and its moon, Charon, have already achieved this stable configuration.

The magnitude of the centripetal acceleration from the Earth-Moon system's revolution about the Sun is, \((2\pi)^2 \times 1.50 \times 10^8 \text{ km}/(1 \text{ year})^2 = 5.9 \times 10^{-6} \text{ km/s}^2 = 5.9 \text{ mm/s}^2 \). Notice that this is small compared to \( g \), 9.8 m/s^2. So one should not expect to feel this acceleration, although it does have observable effects. For example, it contributes to tides in Earth's oceans. The direction of this acceleration vector is towards the Sun.
Procedures

Apparatus
calculator, colored pencils, globe of Earth showing Earth's tilt, globe of celestial sphere showing the constellations, stick-on notepads, and scissors.

A. Comparing Speeds
1. Calculate the speed associated with the Earth's revolution in km/hr and m/s. Assume Earth's orbit is a circle with a radius 1 AU, where 1 AU = 150. x 10^6 km.

2. Verify the value given in the discussion above for the speed of the Sun moving about the center of the Milky Way, assuming the Galaxy's center is 35 kly distant, and the galactic year is 230 million years. Note that one kly is one thousand light-years. This can be converted to meters by multiplying one thousand times the speed of light in meters per second times the number of seconds in one year.

3. Make a table listing the speeds associated with all motions of Earth, those you calculated as well as those given in the text above, arranged from highest to lowest.

4. Draw a circle of about 10 cm radius to represent the Earth's orbit about the Sun. Use different colors to show the paths of the Earth and Moon, and greatly exaggerate their separation, to say 10% of the distance to the Sun. Show the paths followed by the Earth and Moon, as they execute sinuous trajectories, making 12 cycles (one per month) in one revolution.

5. Determine at what time(s) of day and what days of the year your velocity from Earth's rotation will be in the same direction as your velocity from the Earth's revolution. Hint: use pencil pointers to show these two velocities on a globe that shows Earth's tilt, and consider only special days, such as the days of the vernal and autumnal equinoxes and the summer and winter solstices.

6. Construct stick-on note pointers as shown in Figure 15-2. Then place these pointers on a globe of the celestial sphere showing the constellations, so the pointers show the directions of the velocity of the Solar System within the Galaxy, and the velocity of the Galaxy through space.

B. Comparing Accelerations
1. Verify the value reported above by calculating the value of the centripetal acceleration for the Earth in its revolution about the Sun.

2. Calculate the centripetal acceleration produced by the Solar System orbiting the center of the Milky Way galaxy.

3. Make a table of accelerations associated with all of the major motions of Earth, arranged from highest to lowest.
Figure 15-2. Construct stick-on note pointers by making four equally spaced cuts through the adhesive side of a stick-on note, as shown in A., and fold that note into a box, as shown in B., with the adhesive side in. Then, fold the flaps created by the cuts out, as shown, and stick the pointer created onto the celestial sphere, as described in the text.
Astronomy Laboratory Exercise 16

Kepler's Laws of Planetary Motion I

In struggling to understand the motions of the planets in the sky, Johannes Kepler empirically worked out three basic laws, now known as Kepler's Laws of Planetary Motion. The first law (Law of Ellipses) states that the orbit of a planet is an ellipse, with the Sun at one focus. The second law (Law of Equal Areas) states that a line drawn from a planet to the Sun sweeps out equal areas in equal amounts of time. The third law (Harmonic Law) states that the cube of the semi-major axis is proportional to the square of the sidereal period. Kepler developed these laws to apply to a single planet orbiting the Sun. However, Kepler's Laws provide good approximations of the motions of many orbiting bodies. This laboratory exercise concerns the first two laws.

The first law states that an orbiting body travels in an ellipse. An ellipse is a two-dimensional shape for which the sum of the lengths of a line drawn from one focus to a point on the ellipse and another line drawn from the other focus to that same point remains constant for every point along the ellipse. Usually, when describing an ellipse, one talks about its eccentricity. Referring to Figure 16-1, the eccentricity is defined as

\[ e = \frac{c}{a} = \frac{a-b}{a} \]  

where \( c \) is the distance from the center of the orbit to a focus, \( a \) is the semi-major axis, and \( b \) is the semi-minor axis. From this definition, the closest point of a planet from the Sun (perihelion distance) is

\[ R_p = a - c = a - ae = a(1 - e) \]  

Similarly, the aphelion (farthest distance from the Sun) is

\[ R_a = a + c = a + ae = a(1 + e) \]  

Figure 16-1: An ellipse. a is the semi-major axis, b is the semi-minor axis, c is the distance from the center to a focus, e is the eccentricity, \( R_p \) is the peri-distance, and \( R_a \) is the apo-distance.

Example 1: To test the Law of Ellipses, consider the Moon orbiting Earth. The Moon's orbital eccentricity is given as 0.055, and its semi-major axis is equal to 384,400 km. On 12 August 1994, the Moon is at perigee at a distance of 369,453 km. On 27 August 1994, the Moon is at apogee at a distance of 404,332 km. Check these values using the above formulae.

\[ R_p = a(1 - e) = (384400 \cdot \text{km})(1 - 0.0550) = 363000 \cdot \text{km} \]
\[ R_a = a(1 + e) = (384400 \cdot \text{km})(1 + 0.0550) = 406000 \cdot \text{km} \]

These predicted values are within 2 percent of the actual values.
The Law of Equal Areas deals with the speed of the orbiting object. A line connecting the Sun to a planet sweeps out equal areas in equal periods of time. Consider Figure 16-2. A line drawn between the Sun and the planet at point 1 moves around with the planet for, say, one month to point 2. The line is said to sweep out the pie-slice area \( A \). During a similar one month period, the line sweeps out the area \( B \) as the planet travels from point 3 to point 4. Kepler's second law states that these areas are equal. In order for this to be true, the distance traveled by the planet when farther from the Sun (point 3 to point 4) must be shorter than the distance traveled by the planet when closer to the Sun (point 1 to point 2). Since the amount of time is the same, the planet must be traveling faster when closer to the Sun than when farther away.

The speed of the planet at any point in its orbit may be determined from

\[
v = \sqrt{\frac{4\pi^2 \cdot a^3}{P^2 \left( \frac{2}{r} - \frac{1}{a} \right)}}
\]

where \( a \) is the semi-major axis of the planet's orbit, \( P \) is the sidereal period of revolution, and \( r \) is the current distance from the Sun. As with the Law of Ellipses, Kepler's Second Law can be applied to any body orbiting another.

![Figure 16-2: The planet sweeps out the equal areas, A and B, in equal amounts of time. The planet orbits from point 1 to point 2 in the same amount of time it takes it to orbit from point 3 to point 4.](image)

Example 2: The Moon's perigee and apogee distances were determined in Example 1. Using these distances and Equation 4, the Moon's maximum and minimum speeds can be computed. Thus,

\[
v_{\text{max}} = \sqrt{\frac{4\pi^2 \cdot (384400 \cdot \text{km})^3}{(2.36 \cdot 10^6 \cdot \text{sec})^2 \left( \frac{2}{363258 \cdot \text{km}} - \frac{1}{384400 \cdot \text{km}} \right)}} = 108 \text{ km/sec}
\]

\[
v_{\text{min}} = \sqrt{\frac{4\pi^2 \cdot (384400 \cdot \text{km})^3}{(2.36 \cdot 10^6 \cdot \text{sec})^2 \left( \frac{2}{405542 \cdot \text{km}} - \frac{1}{384400 \cdot \text{km}} \right)}} = 0.969 \text{ km/sec}
\]
Procedures

Apparatus
none.

Table 1: Some moons of Jupiter.

<table>
<thead>
<tr>
<th>moon</th>
<th>semi-major axis (km)</th>
<th>eccentricity</th>
<th>sidereal period (days)</th>
<th>diameter (km)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Amalthea</td>
<td>180,500</td>
<td>3.00*10^{-3}</td>
<td>0.498</td>
<td>270*</td>
</tr>
<tr>
<td>Io</td>
<td>422,000</td>
<td>0.000</td>
<td>1.769</td>
<td>3,630</td>
</tr>
<tr>
<td>Europa</td>
<td>671,000</td>
<td>0.000</td>
<td>3.551</td>
<td>3,138</td>
</tr>
<tr>
<td>Ganymede</td>
<td>1,070,000</td>
<td>2.00*10^{-3}</td>
<td>7.155</td>
<td>5,262</td>
</tr>
<tr>
<td>Callisto</td>
<td>1,883,000</td>
<td>8.00*10^{-3}</td>
<td>16.689</td>
<td>4,800</td>
</tr>
<tr>
<td>Himalia</td>
<td>11,480,000</td>
<td>0.158</td>
<td>250.57</td>
<td>180*</td>
</tr>
<tr>
<td>Sinope</td>
<td>23,700,000</td>
<td>0.275</td>
<td>758.0</td>
<td>40*</td>
</tr>
</tbody>
</table>

* Some moons are not spherical, the diameter given is for the longest axis.

1. Using Table 1 and Equations 2 and 3, construct a table of perijovian and apojovian distances for these seven moons of Jupiter.

2. Of the above seven moons, which have circular orbits? Which have the most eccentric orbits?

3. Use the data constructed in part 1 to construct a table of minimum and maximum speeds for the seven jovian moons listed in Table 1. Express these speeds in kilometers per second.

4. On average, the Moon subtends an angle of 0.5° (or 30') in our sky. Add to the table created above two columns containing the angular size of each moon at perijovian and apojovian as seen from the cloud-tops of Jupiter. Figure 16-3 shows a right triangle formed by a line from the observer's position in the cloud-tops of Jupiter to one side of the moon, across the diameter of the moon, then back to the observer. The subtended angle is the angular size of the moon as seen from the jovian cloud-tops. The radius of Jupiter is 71,400 km. Record your answers in minutes-of-arc.
Figure 16-3: From the surface of a planet of radius $r$, the angular size $\theta$ of a moon of diameter $D$ at a distance of $R$ from the center of the planet is given by

$$\theta = \arctan\left(\frac{D}{R - r}\right).$$
Astronomy Laboratory Exercise 17

Kepler's Laws of Planetary Motion II

In struggling to understand the motions of the planets in the sky, Johannes Kepler empirically worked out three basic laws, now known as Kepler's Laws of Planetary Motion. The first law (Law of Ellipses) states that the orbit of a planet is an ellipse, with the Sun at one focus. The second law (Law of Equal Areas) states that a line drawn from a planet to the Sun sweeps out equal areas in equal amounts of time. The third law (Harmonic Law) states that the cube of the semi-major axis is proportional to the square of the orbital period. Although Kepler developed these laws to apply to a single planet orbiting the Sun, they provide excellent approximations to many orbital scenarios. This laboratory exercise will explore the Jovian System with the third law.

The Harmonic Law may be stated as

\[ a^3 = kP^2, \quad (1) \]

where \( a \) is the semi-major axis of the orbit, \( P \) is the sidereal period of revolution, and \( k \) is a constant of proportionality called the Kepler constant. This constant depends only on the body being orbited and not the orbiting body. So, once the Kepler constant is known for the central body, all objects in orbit about it will use that constant. One can solve for the Kepler constant if one knows the orbital characteristics of at least one body.

Example 1: The Kepler constant for the Sun can be determined from the orbital characteristics of Earth.

Earth orbits the Sun at an average distance of 1 astronomical unit (AU) in 1 year (y). So, the Kepler constant for the Sun is

\[ k = \frac{a^3}{P^2} = \frac{(1 \cdot \text{AU})^3}{(1 \cdot \text{y})^2} = 1 \cdot \frac{\text{AU}^3}{\text{y}^2}. \]

Example 2: Jupiter can be observed to take about 11.86 years to complete one orbit of the Sun. What is the average distance of Jupiter from the Sun?

The average distance of a planet from the Sun is the same as its semi-major axis, so we apply the Harmonic Law to this problem. Thus,

\[ a = \sqrt[3]{k \cdot P^2} = \sqrt[3]{1 \cdot \frac{\text{AU}^3}{\text{y}^2}} (11.86 \cdot \text{y})^2 = 5.20 \cdot \text{AU}. \]

On average, Jupiter orbits the Sun at a distance of 5.20 times the average distance of Earth from the Sun.

Example 3: The Kepler constant of Earth may be found using the Moon's orbit.

The Moon orbits at an average distance of 384,400 kilometers (km) in 27.3 days (d). Thus, the Kepler constant of Earth is
Now, since a communications satellite also orbits Earth, we can find the $a$ or $P$ for a satellite using this value of $k$. A geosynchronous satellite revolves at the same rate that Earth rotates (i.e., one revolution per day). The distance of this satellite from the center of Earth is

$$a = \sqrt[k]{\frac{kP^2}{2}} = \sqrt[3]{\frac{7.62 \cdot 10^{13} \text{ km}^3}{d^2}} = 42400 \text{ km}.$$

Hence, a body in orbit at a distance of 42,400 km from the center of Earth will revolve once a day. Since Earth has a radius of 6378 km, the satellite is roughly 36,000 km above the surface. This is the altitude of most communications satellites.

The Harmonic Law was later derived by Sir Isaac Newton from the basic principles of rotary motion and gravitation. The gravitational force, $F$, between two masses is given by

$$F = G \cdot \frac{Mm}{r^2}, \quad (2)$$

where $G$ is the Universal Gravitational constant ($6.672 \cdot 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2$), $M$ and $m$ are the masses of the two bodies, and $r$ is the distance between the centers of the two bodies. The unit of force in the MKS system is the newton (N) and is equivalent to the force needed to accelerate a 1 kg mass at 1 m/s$^2$.

Example 4: An 80.0 kg being standing on Earth's surface is attracted toward the center of Earth by the force of gravity. Earth has a mass of $5.98 \cdot 10^{24}$ kg and a radius of 6378000 m. From Equation 2, the force of gravity acting on the being is

$$F = \left(6.672 \cdot 10^{-11} \text{ N} \cdot \text{m}^2/\text{kg}^2 \right) \left(\frac{5.98 \cdot 10^{24} \text{ kg}}{6378000 \text{ m}^2} \right) 80.0 \text{ kg} = 785 \cdot \text{N}.$$

Thus, the weight of the being is 785 newtons on Earth's surface. On the Moon, whose mass and radius are $7.35 \cdot 10^{22}$ kg and 1738 km, respectively, an 80.0 kg being would weigh only 130. N.

Newton's second law of motion states that the force on a mass, $m$, undergoing acceleration, $a$, is

$$F = ma. \quad (3)$$

Since Equations 2 and 3 both give the force acting on the orbiting body, one can solve for the acceleration due to gravity ($g$). Thus,

$$g = \frac{GM}{r^2}. \quad (4)$$
Notice that the mass of the orbiting body has been eliminated from the formula.

Example 5: The acceleration due to the gravity of Earth at its surface (g) is given by Equation 4,

\[ g = \frac{GM}{r^2} = \frac{G \cdot (5.98 \cdot 10^{24} \text{ kg})}{(6378000 \text{ m})^2} = 9.81 \text{ m/s}^2. \]

Example 6: Another way to determine the weight of a being is to use Newton's second law (Equation 3). If we again consider an 80.0 kg being on the surface of Earth, its weight would be

\[ F = mg = (80.0 \cdot \text{ kg}) \left( 9.81 \cdot \frac{\text{m}}{\text{s}^2} \right) = 785 \cdot \text{N}. \]

This is the same weight computed in Example 4.

The centripetal acceleration, \( a \), on an orbiting body is given by

\[ a = \frac{v^2}{r}, \quad (5) \]

where \( v \) is the average tangential velocity of the orbiting body, and \( r \) is the average distance from the body to the center of the orbit. Since the planet stays in orbit, the centripetal acceleration must equal the acceleration due to gravity. Setting the right-hand side of Equation 4 to the right-hand side of Equation 5, one has

\[ \frac{v^2}{r} = \frac{GM}{r}. \quad (6) \]

This average tangential velocity of an orbiting mass is given by the length of the orbit divided by the amount of time it takes to go around the orbit (its period of revolution), or

\[ v = \frac{2\pi \cdot r}{P}, \quad (7) \]

where \( P \) is the sidereal period of the orbiting mass and \( r \) is the average radius of the orbit.

Now, substituting Equation 7 into Equation 6 and rearranging, gives

\[ r^3 = \left( \frac{GM}{4\pi^2} \right) \cdot P^2. \quad (8) \]

This is Newton's restatement of Kepler's third law of planetary motion. The Kepler constant (k) is, thus,

\[ k = \frac{GM}{4\pi^2}. \quad (9) \]
Example 6: Equation 9 can be used to compute the mass of the central body if the Kepler constant is known. The Kepler constant of Earth \((7.62\cdot10^{13} \text{ km}^3/\text{d}^2)\) was computed in Example 3. This value must be converted to base MKS units before it can be used in Equation 9.

Thus,

\[
k = 7.62\cdot10^{13} \frac{\text{km}^3}{\text{d}^2} \cdot \left( \frac{1000 \cdot \text{m}}{1 \cdot \text{km}} \right)^3 \cdot \left( \frac{1 \cdot \text{d}}{86400 \cdot \text{s}} \right)^2 = 1.02\cdot10^{13} \frac{\text{m}^3}{\text{s}^2}.
\]

Solving Equation 9 for the mass of the central body \((M)\),

\[
M = \frac{4\pi^2 \cdot k}{G} \tag{10}
\]

Now, the mass of Earth is

\[
M = \frac{4\pi^2 \cdot \left(1.02\cdot10^{13} \frac{\text{m}^3}{\text{s}^2}\right)}{\left(6.672\cdot10^{-11} \frac{\text{N} \cdot \text{m}^2}{\text{kg}^2}\right)} = 6.04\cdot10^{24} \text{ kg}
\]

If this method is used with more accurate data, the mass becomes \(5.98\cdot10^{24} \text{ kg}\), which is only 1 percent from the rough calculation performed above.

Most astronomy magazines include graphs representing the positions of the largest moons of Jupiter and Saturn during the month. These 2-D plots of the orbits appear as sine waves. Figure 17-1 is just such a plot of the Galilean satellites, Io, Europa, Ganymede, and Callisto. The vertical axis of the graph represents time, with each horizontal line marking a day. The horizontal axis of the graph represents distance from the center of Jupiter. The two vertical lines in the center of the graph represent the eastern and western limbs of Jupiter. The distance between the two vertical lines gives the diameter of Jupiter on the scale of the graph.

The period of a moon can be determined by counting the number of days between consecutive peaks on the wave (its wavelength). The amplitude of the wave, measured from halfway between the two vertical lines to the peak of a moon's wave, gives the semi-major axis of the moon.
Figure 17-1: A plot showing the relative positions of the Galilean Moons of Jupiter, as seen from Earth on the days indicated.
Procedures

Apparatus

current 2-D graphical representation of orbital data for the Galilean satellites, millimeter rulers.

1. Determine the scale along the vertical axis of the 2-D plot by measuring the distance between consecutive day marks in millimeters. This will allow you to determine the period of a moon to the hour. Such accuracy is necessary.

2. Determine the scale along the horizontal axis of the 2-D plot. Here, we are going to use the radius of Jupiter as our unit of distance. Measure the distance between the two vertical lines in the center (which is to scale with the diameter of Jupiter), and divide this by two. Whenever you measure the amplitude of a wave, this scale value will convert that number to jovian radii (j.r.).

3. Make a table containing the periods (in days) and semi-major axes (in jovian radii) of the four Galilean moons. To measure the period, simply measure the distance (in mm) between two consecutive peaks along the vertical axis. Multiply this measurement by the vertical scale. To measure the semi-major axis distance for a moon, measure from the peak of the wave to the center of the double lines, then multiply by the horizontal scale factor.

4. Using the data collected in part 3 and Equation 1, calculate the Kepler constant of Jupiter, k_J. What are the units of this constant of proportionality? Include these units in the column header in your table. The Kepler constant should be the same for each moon, since they all orbit Jupiter, but printing and measuring errors may cause slight differences.

5. Compute the arithmetic mean of these values for the jovian Kepler constant, k_1. This mean value should be used for all bodies orbiting Jupiter.

6. Sinope has a period of 758 days. Find the semi-major axis of this moon's orbit in both jovian radii and kilometers using k_1 and Equation 1. (Note: 1 j.r. = 71,400 km.) Compare the semi-major axis of the orbit of Sinope, calculated above, to the published value of 2.37⋅10^7 km (i.e., compute the percentage error of the calculated semi-major axis to the published value).

7. Using k_1 and Equation 10, determine the mass of Jupiter. Hint: you must convert k_1 to standard MKS units first. Why is this so? The published value is 1.90⋅10^{27} kg. Compare the calculated value to the published value by computing the percentage error.
Astronomy Laboratory Exercise 18

**Orbiting Earth**

Astronomy is progressing at an astonishing rate due in part to a variety of robot spacecraft and orbiting observatories that provide images of celestial objects in the x-ray, ultraviolet, infrared, and microwave portions of the electromagnetic spectrum. Even in the visible region of the spectrum, images from telescopes in space are used in making significant discoveries at an unprecedented rate. It is clear that astronomy, as it exists now, depends on the successful operation of robot spacecraft and observatories orbiting Earth. Thus, the geometry, mechanics, and technology of space transportation has become an important part of astronomy.

Isaac Newton (1642-1727) was aware that it is possible to orbit Earth if sufficient speed can be achieved. Just as an object dropped near Earth is accelerated toward Earth by gravity, so too an object on orbit near Earth is accelerated toward Earth. One can think of a body on a circular orbit as a body that is falling toward Earth while it travels above the surface at such a rate that it is in effect falling around Earth, but never getting closer to it. The speed required to achieve circular orbit is,

$$V = \left(\frac{G \cdot M}{R^2}\right)^{\frac{1}{2}}$$

where $G$ is the universal gravitational constant, $M$ is the mass of Earth, and $R$ is the radius of the orbit. Rockets capable of achieving orbital speeds were first developed during the second half of the 20th century, making it possible to conduct astronomical observations from above Earth's atmosphere.

**Example 1:** Calculate the speed required to orbit Earth just above the atmosphere. The value of the universal gravitational constant is $6.67 \cdot 10^{-11}$ Nm$^2$/kg$^2$, the mass and radius of Earth are, $5.98 \cdot 10^{24}$ kg and $6.37 \cdot 10^6$ m. The radius of the orbit required is about 130 km plus the radius of Earth. To three significant figures, this is $6.50 \cdot 10^6$ m. The above formula and values yield, 7830 m/sec or 28,200 km/hr.

**Example 2:** Calculate the Moon's speed in its orbit about Earth. Assume the Moon's orbit is circular with a radius of 384,000 km. The above formula and values yield, 1020 m/sec or 3670 km/hr.

For a machine to orbit Earth, it is necessary for it to first get above the dense part of the atmosphere, then acquire "orbital" velocity horizontally. The primary reason rockets take off vertically is to use their powerful engines to lift the crafts and carry them through the dense, lower part of the atmosphere at low speed. Traveling at high speeds in the atmosphere would require strong, and therefore heavy, airframes and also would be accompanied by the continual loss of significant energy to friction with the atmosphere. Thus, it is possible to maintain orbital speeds only above most of the atmosphere.
Satellites on orbit about Earth obey Kepler's laws (see Exercises 16 and 17, Kepler's Laws of Planetary Motion I and II) and move in elliptical orbits at prescribed speeds with Earth's center of mass at one focus. A satellite's speed, direction, and distance from Earth at the time it is released or its rocket engines are turned off, will determine the size, shape, and orientation of the orbit. The main force acting on a satellite orbiting above the atmosphere is gravity, and gravity will cause the satellite to pass through the point where it was released (or its engine shut off) on each circuit. Satellites on orbit are often referred to as being in free fall, and astronauts and other object accompanying them experience weightlessness.

The closest point of an orbit to Earth is its perigee and the farthest point is its apogee. The perigee and apogee are on opposite sides of the orbit's major axis. The orbit is fixed relative to Earth's center of mass in space, independent of Earth's rotation. The plane of the orbit must always include Earth's center of mass. If an orbit is circular, the perigee and apogee are at the same distance from Earth's center of mass, and the satellite's speed is constant. Orbits just above the atmosphere are called low orbits, low Earth orbits, or LEO's.

Inclination is the angle an orbit makes with the plane of the equator, where this angle is measured counter-clockwise from the equator where the satellite crosses the equator heading north. Thus, it is possible to have an orbit with inclination greater than 90 degrees. An orbit directly over Earth's equator has an inclination of zero degrees, and is called an equatorial orbit. An orbit that passes over both poles has an inclination of 90 degrees, and is called a polar orbit. A satellite on polar orbit above the terminator (the line dividing the sunlit hemisphere from the dark hemisphere of a planet or moon) would be in sunlight all of the time. This is the only orbit which can provide continuous solar power, and is one type of sun-synchronized orbit.

Another type of sun-synchronized orbit is a polar orbit in which a satellite crosses the equator at the same local solar time on each orbit. Landsats used this orbit so the images of Earth they provided were obtained with essentially the same angle of illumination on each orbit, and therefore could be directly compared. The images from these satellites have been very useful in studying the planet Earth. Satellites that require large amounts of electric power when they operate out of the sunlight must carry rechargeable batteries or nuclear (i.e., radioactive) generators.

The ground trace of a satellite is the track it makes over the surface of Earth. The highest northern and southern latitudes reached by a satellite's ground trace is equal to the satellite's orbital inclination (or 180 degrees minus that angle, if it is greater than 90 degrees). Ground traces often make unusual patterns when drawn on a flat map of Earth, but are revealed to be "great circles" when shown on a globe. The great circles are where the plane of the orbit intercepts the surface of Earth. The ground trace for a satellite in polar orbit when it comes down over the north pole, heading due south, is a line tilted toward the west because of Earth's rotation. The same is true as the satellite heads north from the south pole.
The period of an orbit is the time it takes the satellite to complete one circuit, and can be calculated for circular orbits with the equation,

\[ P = \left( \frac{4 \cdot \pi^2 \cdot R^3}{G \cdot M} \right)^{\frac{1}{2}} \]  

(2)

if the radius, R, is known. Note that the shortest period occurs with the lowest orbit.

Example 3: Find the minimum period for an orbiting satellite.
The minimum period would occur for an orbit just above Earth's atmosphere. Using the values for G, M, and R given in Example 1 above, with Equation 2 yields, 5210 seconds, 86.9 minutes or 1.45 hours.

This equation may be manipulated to give the radius of a circular orbit for a particular period, P, as,

\[ R = \left( \frac{P^2 \cdot G \cdot M}{4 \cdot \pi^2} \right)^{\frac{1}{3}} \]  

(3)

Example 4: Find the radius of an orbit that would cause a satellite to orbit Earth with a period of one sidereal day, 23 hours and 56 minutes.
Equation 3 and this period yields a radius of 42,200 km. A satellite on orbit at that radius would be 35,800 km above Earth's surface.

A satellite on an equatorial orbit at 35,800 km above Earth traveling east, is in a geostationary or geosynchronous orbit, often called geosych. Many communications and weather satellites are on geosynch. From vantage points on Earth, these satellites appear fixed in the sky. "Satellite" dishes aimed at these points may receive microwave signals from the satellites 24 hours a day. From the satellite's viewpoint, its transmitter can send signals to, and its cameras can see, nearly one-third of Earth's surface, although it cannot effectively reach high (north or south) latitudes. The ground trace for such an orbit is a spot on the equator. Geosynch is an example of a high orbit.

Satellites on geosynch and other orbits are subjected to small gravitational forces, perturbations, from the Moon, Sun, and the oblate shape of Earth, which produce small accelerations that accumulate over time and cause the satellite to rotate and also to move out of their desired positions. "Station keeping" is required to keep satellites oriented properly, so cameras, telescopes and radio antennae are pointed in the correct directions, and also to maintain the velocity required by the orbit. Electrically powered gyroscopes rotating around different axes on a satellite can be sped up or slowed down to maintain rotational control of the satellite, and small "thruster" rockets can be momentarily fired to correct the satellite's velocity. The useful lives of many satellites on high orbits end when their gyroscopes fail, or their thrusters run out of fuel. The space shuttle can provide
service calls to satellites on low orbits, but currently there is no capacity for service calls to high orbits.

An ideal location for a space port would be the top of the highest mountain on Earth's equator. This would allow satellites to achieve orbit with the minimum expenditure of rocket fuel. Launching eastward takes advantage of Earth's rotation, which provides a velocity of 1670 km/hr eastward at the equator. See Exercise 15, Motions of Earth. The Kennedy Space Center, at Cape Canaveral, Florida, is the primary launch facility of the United States space program. The location was chosen to take advantage of year-round warm weather, the availability of then undeveloped land, of being relatively close to Earth's equator in then "secure" US territory, and also to be on the eastern seaboard. Being on the eastern seaboard places rockets that fail early during their launches over the Atlantic ocean, where there are no cities to be damaged by the falling debris. The facilities used by the European Space Agency in French Guiana, are well situated by being near the equator, while the Russian Cosmodrome at Biakonur is not. Japan, China and India also have launched Earth orbiting satellites from their own territories.

To achieve orbit, a rocket must change its velocity from whatever it is to the orbital speed described above. Firing a rocket engine for a measured time while the rocket is in flight will produce a change in velocity, often referred to as $\Delta V$ (pronounced delta vee). Larger $\Delta V$'s require more fuel. All of the fuel consumed by a rocket in flight must be lifted by the rocket engines at take-off.

Example 5: Calculate the $\Delta V$ required to put a satellite in LEO by launching eastward, southward, and westward from the equator.

Orbital speed for LEO, calculated in example 1, is 28,200 km/hr, but when launching eastward from the equator the $\Delta V$ needed is lessened by the rotational speed of Earth, 1670 km/hr. Thus, 28,200 - 1,670 = 26,530 km/hr is the required speed for an eastward launch. Launching southward, the $\Delta V$ needed is just 28,200 km/hr. When launching westward, the $\Delta V$ needed is the 28,200 + 1670 = 29,870 km/hr.

Acquiring another orbit in the same orbital plane (i.e., a higher or lower orbit) requires changing speeds ($\Delta V$'s). While on orbit this is accomplished by firing the rocket engines. To move to a higher orbit, the rocket engines are fired so as to speed up the spacecraft, which causes the spacecraft to move higher. This, in turn, causes it to slow down, so it ends up higher but traveling slower than before the rocket engines were fired. To move to a lower orbit, the rocket engines are fired to slow the spacecraft down, which causes it to fall to a lower orbit. This, in turn, causes it to speed up. Orbiting spacecraft could not carry enough fuel to slow down sufficiently to land on Earth, if they could not use friction from traveling through Earth's atmosphere. Since the Moon has no atmosphere, the Apollo spacecraft had to carry substantial fuel from Earth to use to slow down sufficiently to land on the Moon. Changing the inclination of the orbit of an Earth orbiting spacecraft requires so much rocket fuel that it is usually not possible to change the inclination significantly.
Rocket scientists and engineers are continuing to develop engines that provide increased values of $\Delta V$ per kilogram of consumed fuel and oxidizer. Scientific and technical equipment are being made smaller and of lower mass, which reduces the mass that needs to be lifted. This has made it possible to achieved orbit by launching in any direction, but launching eastward still allows lower cost orbits, and so are usually the orbits of choice.

Satellites are often observed passing overhead just after sunset or before sunrise, when the sky is rather dark, but the Sun is not too far below the horizon. At this time, the satellites are illuminated by sunlight, while the observer below is in darkness. If the directions are known to the observer, it is fairly easy to distinguish polar from inclined and equatorial orbits. The National Aeronautics and Space Administration (NASA) maintains a site on the Internet which provides up-to-date information on their launches from Kennedy Space Center, and many people observe Space Shuttle launches. It is not possible to get very close to the launch sites, so telescopes are useful for viewing and photographing the launches.

Earth turns under the orbit, which alters the orbital period of a satellite as measured from a fixed point on the ground. In the 87 minutes calculated for the minimum orbital period, Earth turns about 22 degrees, so a satellite on a low Earth orbit of zero degrees inclination will require longer than 87 minutes to pass overhead again. The period of the orbit calculated above will not be the period of the orbit observed on Earth, because of the Earth's motion. The synodic period, $P'$, can be calculated from,

$$\frac{1}{P'} = \frac{1}{P} - \frac{1}{E}, \quad (4)$$

where $P$ is the sidereal period and $E$ is the period of Earth's rotation. This equation can be rearranged to give,

$$P' = \frac{(E \cdot P)}{(E - P)} \quad (5)$$

Example 6: Calculate the period that would be observed by a fixed observer for the orbit just considered.

The orbital time can be calculated from the Equation 5,

$$P' = \frac{(1440 \cdot \text{min})(87 \cdot \text{min})}{(1440 \cdot \text{min} - 87 \cdot \text{min})} = 93 \cdot \text{min}$$
Procedures

Apparatus
globe of Earth, rolls of removable tape, drawing compass, millimeter scale, meter stick.

A. Drawing Orbits to Scale
1. Use a drawing compass and draw as accurately as possible a circle to represent Earth at the scale of 1.00 cm = 1000 km. Then, using the same center and scale, draw a circle to represent the orbit of a satellite 130 km above Earth.

2. Use a drawing compass to draw another circle to represent Earth at the scale of 1.00 cm = 5000 km. Then, using the same center and scale, draw a circle to represent the orbit of a satellite at geosynch.

3. Draw free hand, as accurately as possible, a circle to represent Earth at the scale of 1.00 cm = 50,000 km. Do this by placing two small marks a distance apart to represent the diameter of Earth, and then sketch a circle of that diameter. Then, using the center of that circle, and the same scale, draw a circle with a drawing compass to represent the orbit of the Moon.

4. On the drawing showing the orbit of the Moon around Earth, add to scale as accurately as possible a low Earth orbit and a geostationary orbit.

5. Comment on the difficulty of accurately showing these three orbits on the same drawing.

B. Showing Orbits on a Globe
1. Measure and record the diameter of your globe in cm. Compute the scale factor for your globe (a ratio which will convert cm to km for your globe). Then calculate the number of centimeters from the globe's surface a satellite would be on geosynch. Record that number in your report.

Position a meter stick perpendicular to the surface with one end on the equator of your globe, and place your head against the meter stick so one eye is at the equivalent of the geosynch distance from the globe. Look at the globe from that point, and make a list of which continents can be seen as the globe is rotated. Can all continents be seen? Which can not? What are the highest and lowest latitudes that can be seen?

What longitude (on the equator) would be the best location for a communications satellite on geosynch to serve both USA and Europe? The USA and Japan? Japan and Australia? Is it possible for a communications satellite to serve USA, Japan and Australia at the same time?
2. Are there markings on your globe that show the ground traces for equatorial and polar orbits? If so, identify them in your report. If not, use a roll of removable tape to mark them on the globe.

3. Use a roll of removable tape to mark the ground trace of an orbit inclined by 45 degrees. Start by using a protractor or 45° triangle and place the tape on the globe so it crosses the equator at 45°. Stick the tape on the globe to complete an orbit, making sure the tape is straight and not stretched. This ground trace should reach a maximum of latitude of 45° north and 45° south, and it should close on itself. Ground traces for satellites in such orbits do not actually close on themselves because of Earth's rotation, as discussed above.

C. Questions and Problems
1. Determine the number of sunrises and sunsets that astronauts would observe during a 24 hour day from a crewed satellite on LEO.

2. Calculate the velocity and period of a low lunar orbit, given that the Moon's mass and radius are \(7.35 \times 10^{22}\) kg and 1740 km, respectively.

3. Given the idea that a smaller \(\Delta V\), and therefore less fuel, is required for launching from the equator of Earth, why did the Apollo astronauts land close to the lunar equator? Consider the \(\Delta V\) required for both the landings and subsequent take-offs.
Astronomy Laboratory Exercise 19
The Elliptical Orbit of Mercury

Planetary orbits were discovered to be elliptical by Johannes Kepler in 1609 using the most accurate data then available on the angular positions of the planets. This data was recorded at Uraniborg, Tycho Brahe's observatory on the island of Hveen, Denmark, during the last few decades before telescopes were used in astronomy. Consequently, the accuracy of the data was limited by the approximate 1 minute-of-arc resolution of the human eye. Kepler acquired Tycho's data after his death in 1601. This exercise will do graphically what Kepler did with mathematical calculations. How can the changing positions of a planet in the night sky be used to show the planet's orbit is an ellipse? Pondering this question should help one appreciate Kepler's contribution to science. This exercise also provides an endorsement of accurate data, for with less precise data Kepler may not have been able to distinguishing between circular and elliptical orbits.

The basic properties of an ellipse are shown in Figure 19-1, where the ratio of $c$ to $a$ gives the eccentricity, $e$. Note that when $c$ equals zero, the two foci merge into a single point, the eccentricity goes to zero, and the figure becomes a circle.

Mercury is moderately difficult to observe, because it is relatively small, fairly dark, always close to the Sun, and, except during some total solar eclipses, is never in a dark sky. Normally, Mercury is seen either in the eastern sky in the morning twilight just before sunrise, or in the evening twilight of the western sky just after sunset. The best time to observe Mercury on each passage is when it is near its greatest angular distance from the Sun. This angle is known as the maximum elongation.

Tycho's records contained right ascension (RA) and declination (DEC) values for the Sun, Mercury, and many other objects. Kepler calculated the angular distance between the Sun and Mercury, which gave the values of maximum elongation for each passage of Mercury. A list of dates and values of maximum elongations are given in Table 1. Plotting these data using the procedure described below will show that the orbit of Mercury is elliptical. The eccentricity of the orbits of all other planets, excluding Pluto, are much less than that of Mercury. The graphical procedures used here would make the orbits of all other planets except Pluto appear as circles. When Kepler studied the orbits of the planets, he did not depend on diagrams, but rather used more accurate mathematical calculations which
showed that the orbits of the other planets were also ellipses. See Exercise 20, Motions of Mars. His analysis of planetary orbits led to a general statement that is now known as Kepler's first law: The orbits of planets are ellipses with the Sun at one focus. Also see Exercise 16, Kepler's Laws of Planetary Motion I.

The ancient astronomers of Greece were familiar with the planet Mercury, but they also believed they knew Apollo, another planet. Apollo appeared only during the morning in the eastern sky. It was actually Mercury, but they did not recognize it as the same object that appeared in the evening sky. It is easy to understand how the ancients could have made this mistake; after all, east and west are opposites.

The Moon's orbit is also an ellipse. It can be easily verified that the Moon's orbit about Earth is not a circle: just photograph the Moon every few days or so over a month using the same camera and lens, and compare the image sizes. Many introductory astronomy texts include such a series of pictures to show the Moon's phases. The disk of the Moon appears its smallest each month at apogee (its farthest point from Earth) and its largest at perigee.

There are special numerical relationships between the periods of revolution and rotation of both Mercury and the Moon. The ratio for the Moon is one-to-one, as it rotates once during each revolution around Earth. Consequently, the same side of the Moon is always seen from Earth. If this relation existed between a planet and the Sun, then on that hypothetical planet there would be a year, the period of its revolution, but there would be no day, or solar day, as the Sun would never rise or set. The Sun would remain fixed at the same place in the sky all year, just as Earth remains fixed in the lunar sky when observed from the surface of the Moon.

Features on Mercury's surface are hard to observe from Earth, because they are not very large or distinctive and because Mercury is close to the Sun. The angular size of Mercury at its closest approach to Earth is only about 13 seconds-of-arc, so even with high magnification, only a few surface features are faintly discernible. For comparison, the angular size of the Moon is about half a degree, or 1800 seconds-of-arc. In the mid-nineteenth century, Giovanni Schiaparelli concluded from observations that Mercury kept one side facing the Sun. It was reasonable that this would be the case, as the Sun causes large tidal forces on Mercury that could have locked its rotation and revolution into a one-to-one resonance, just like the Moon. However, in 1965, astronomers reflected radio waves off Mercury from the Arecibo radio telescope in Puerto Rico, and noticed that some of the waves were Doppler shifted towards longer wavelengths while others were Doppler shifted towards shorter wavelengths. The details of these observations were consistent with Mercury having a 58.65 day rotational period, which is not equal to its 87.97 day period of revolution. Observations of Mercury from Mariner space craft verified the 58.65 day rotation in 1974. This was clearly a triumph for radio astronomy.
Example 1: What is the ratio of Mercury's year to its sidereal day?

The ratio of Mercury's year to its sidereal day is $87.97 / 58.65$ or 1.500. Thus, they are locked in at three-to-two resonance. An exercise below shows that this means a solar day on Mercury is actually longer than a year on Mercury.
Procedures

Apparatus

millimeter ruler, and protractor.

A. Plotting Mercury's Orbit

1. Figure 19-2 gives Earth's orbit with the dates marked. Because of the small eccentricity of Earth's orbit, it appears as a circle with the Sun at one focus. Plot each of the maximum elongation values given in Table 1 as follows:
   a) Draw a light and straight pencil line from the position of Earth to the Sun for each date.
   b) Center a protractor at the position of Earth on each date given in Table 1, then measure, mark, and draw a heavier line so the angle from the Earth-Sun line to this new line is equal to the maximum elongation angle for that date. This new line represents the line of sight from Earth to Mercury for that date. Mercury must have been somewhere along the Earth-to-Mercury lines on the given dates, so make sure these lines extend well past the Sun. Keep your pencil sharp as you draw.

2. After lines are drawn for all of the data from Table 1, sketch a smooth curve around the Sun that just touches the heavier lines. Sketch this curve lightly at first, and then as it takes shape, make it darker, erasing and redrawing as needed. This smooth curve should contact each of the Earth-to-Mercury lines, but not cross any of them. This curve is your plot of the orbit of Mercury.

3. Determine the shape of Mercury's orbit by drawing a straight line through the largest "diameter" of Mercury's orbit that also passes through the Sun. This should be the major axis of an ellipse. Bisect this major axis to find the semi-major axis and the center of the ellipse. Draw the minor axis through the center, perpendicular to the major axis. Measure the length of each in mm and record them in a table. Also, measure the radius of Earth's orbit in mm, which is 1 AU on this diagram, and include that in your table.

4. Calculate the scale factor which will convert from mm to AU for your diagram. Convert the semi-major axis, a, of Mercury's orbit to AU. Does the result agree with 0.39 AU? Calculate the percentage error.

5. Use your measured value of a to calculate the sidereal period, P, of Mercury in years, using Kepler's third law (see Exercise 17, Kepler's Laws of Planetary Motion II). To convert to days, multiply the sidereal period in years by 365.25 days per year. Compare this value to the value of Mercury's 87.97 day, and calculate the percentage error.

6. The Sun lies at one of the foci of the ellipse which is Mercury's orbit. Measure the distance from the Sun to the center of the ellipse and call it c. Divide c by the semi-major axis, a, to get the eccentricity, e, of the ellipse. Compare your value of c for Mercury with the published value of 0.2056, and compute the percentage error.
7. Determine from your diagram the maximum and minimum possible distances between Earth and Mercury in mm. Calculate the distances these values represent in both AU and km, where 1 AU is $1.50 \cdot 10^8$ km.

8. Given that Mercury's diameter is 4880 km, calculate the maximum and minimum size Mercury will appear from Earth in radians and degrees. (Hint: the values in radians will be Mercury's diameter divided by its distance in the same units. Radians are converted to degrees by multiplying by 57.3 degrees per radian.)

B. The Relative Length of a Solar Day and Year on Mercury
Note: for this exercise, a new unit of time is defined, a Mercury minute, abbreviated Mm. One Mm is defined as one ninetieth of a Mercury year, so there are 90 Mms in a Mercury year. Notice that there are 60 Mms per sidereal Mercury day (because 58.65 Earth days times (90 Mms / 87.97 Earth days) = 60 Mms). Also, note that one quarter of a sidereal Mercury day is 15 Mms.

1. A greatly enlarged Mercury with a tower pointing toward the Sun is shown in Figure 19-3. At the base of the tower, it is solar noon on Mercury in the position shown. Sketch Mercury again each quarter of a sidereal day, which should show the tower pointing down at 15 Mms, right at 30 Mms, up at 45 Mms, left again at 60 Mms (the end of the first sidereal day), etc. Continue adding more sketches of Mercury and its tower at 15 Mms intervals until the Sun shines down the tower again. This will occur when it is again solar noon at the tower base, which will be the end of the first solar day on Mercury.

2. Explain how the length of a solar day on Mercury compares to a year on Mercury. Find and record the length of a solar day on Mercury in Earth days.

C. Additional Questions
1. Explain how the maximum elongation values would differ from those given in Table 1 if Mercury's orbit were a circle.

2. If each value of maximum elongation were uncertain by 4 degrees, what could be concluded from this data about the shape Mercury's orbit? Explain.

3. In what phase will Mercury appear at the times of maximum elongation given in Table 1?

4. In what phase will Mercury appear at its closest to Earth? In what phase will Mercury appear at its furthest from Earth?
Figure 19-2: The orbit of Earth with the Sun near the center. The marks give the position of Earth in its orbit for various dates. The size of the spot representing the Sun is approximately to scale, however the line representing Earth's orbit is so thick that it would cover both the Earth and Moon in their orbits. An oversized Earth is shown for February 1, so the direction of EE (eastern elongation) and WE (western elongation) can be shown in comparison with the directions of Earth's revolution and rotation.
Figure 19-3: A sketch of Mercury's orbit, not to scale, showing a greatly exaggerated Mercury with a tower pointing toward the Sun. This orbit is divided into sixths, and will be used in the procedures to determine the relative lengths of a solar day and year on Mercury.
Table 1: Maximum Elongations for Mercury.

<table>
<thead>
<tr>
<th>Year</th>
<th>Date</th>
<th>Angle and Direction</th>
</tr>
</thead>
<tbody>
<tr>
<td>1967</td>
<td>16 Feb</td>
<td>18° east</td>
</tr>
<tr>
<td></td>
<td>31 Mar</td>
<td>28° west</td>
</tr>
<tr>
<td></td>
<td>12 Jun</td>
<td>25° east</td>
</tr>
<tr>
<td></td>
<td>30 Jul</td>
<td>20° west</td>
</tr>
<tr>
<td></td>
<td>09 Oct</td>
<td>25° east</td>
</tr>
<tr>
<td></td>
<td>18 Nov</td>
<td>19° west</td>
</tr>
<tr>
<td>1968</td>
<td>31 Jan</td>
<td>18° east</td>
</tr>
<tr>
<td></td>
<td>13 Mar</td>
<td>27° west</td>
</tr>
<tr>
<td></td>
<td>24 May</td>
<td>23° east</td>
</tr>
<tr>
<td></td>
<td>11 Jul</td>
<td>21° west</td>
</tr>
<tr>
<td></td>
<td>20 Sep</td>
<td>26° east</td>
</tr>
<tr>
<td></td>
<td>31 Oct</td>
<td>18° west</td>
</tr>
<tr>
<td>1969</td>
<td>13 Jan</td>
<td>18° east</td>
</tr>
<tr>
<td></td>
<td>23 Feb</td>
<td>26° west</td>
</tr>
<tr>
<td></td>
<td>06 May</td>
<td>21° east</td>
</tr>
<tr>
<td></td>
<td>23 Jun</td>
<td>23° west</td>
</tr>
<tr>
<td></td>
<td>03 Sep</td>
<td>27° east</td>
</tr>
<tr>
<td></td>
<td>15 Oct</td>
<td>18° west</td>
</tr>
<tr>
<td></td>
<td>28 Dec</td>
<td>19° east</td>
</tr>
</tbody>
</table>
Astronomy Laboratory Exercise 20

The Motions of Mars

The daily rising, crossing the sky, and setting of the Sun, Moon, planets, and stars has been observed for all of human history. The brightness of some of these objects vary, and some exhibit curious motions relative to patterns created by others. Explaining these observations has challenged shamans, priests, and teachers for thousands of years. In the 1590's, Johannes Kepler wrote that he believed that understanding the motions of Mars was the key to understanding astronomy. This exercise describes how Kepler came to the conclusion that Mars has an elliptical orbit about the Sun (and not a circular orbit about the Earth or Sun).

Kepler found encouragement from the Copernican heliocentric model. Copernicus had learned of a Sun-centered model from the ancient Greeks and published his ideas in *De Revolutionibus*, in the year 1543 CE. Kepler was the first professional astronomer to openly uphold Copernicus' heliocentric theory. Although Kepler's work provided only a modest improvement in astronomers' ability to predict the positions of planets, it gave a rational explanation for the curious motions of planets. The exercises below show how the apparent motion of Mars results from the combination of its own and Earth's orbital motions. Kepler's accomplishments set the stage for the important work of Galileo and Newton. Also see Exercises 16 and 17, Kepler's Laws of Planetary Motion I and II.

We live in a rational age, even if many people today still do not seem rational. Many current traditions are derived from previous times when superstitions were much more influential. To illustrate, there are seven days in a week now because there were seven celestial bodies, the Sun, Moon, Mars, Mercury, Jupiter, Venus, and Saturn, seen in ancient times moving among the stars. These seven objects were all known as planets and they bear the names of mythical gods in different traditions. The names given the days of the week in European languages, listed in Table 1, illustrate connections between those names and the celestial bodies.

<table>
<thead>
<tr>
<th>Modern Day</th>
<th>&quot;Planet&quot;</th>
<th>Anglo-Saxon God</th>
<th>Day in Other Tongues</th>
</tr>
</thead>
<tbody>
<tr>
<td>Sunday</td>
<td>Sun</td>
<td>Sunne</td>
<td>Sonntag (German)</td>
</tr>
<tr>
<td>Monday</td>
<td>Moon</td>
<td>Mona</td>
<td>Lundo (Esperanto)</td>
</tr>
<tr>
<td>Tuesday</td>
<td>Mars</td>
<td>Tiw</td>
<td>Martes (Spanish)</td>
</tr>
<tr>
<td>Wednesday</td>
<td>Mercury</td>
<td>Woden</td>
<td>Mercoledi (Italian)</td>
</tr>
<tr>
<td>Thursday</td>
<td>Jupiter</td>
<td>Thunor</td>
<td>Jeudi (French)</td>
</tr>
<tr>
<td>Friday</td>
<td>Venus</td>
<td>Frig</td>
<td>Venerdi (Italian)</td>
</tr>
<tr>
<td>Saturday</td>
<td>Saturn</td>
<td>Saturn (Roman)</td>
<td>Dé Sathairn (Gaelic)</td>
</tr>
</tbody>
</table>

The discovery of the remaining objects we know today as planets had to await the development of telescopes. Understanding that the Sun and Moon were not planets followed naturally from acceptance of the heliocentric model of the Solar System.
The planets constitute a tiny fraction of the thousands of observable sky objects. The extraordinary attention given to planets may seem reasonable when considering that many ancient peoples, lacking bright lights and spending more time outside, would have been quite familiar with the night sky. And that the human eye and brain often pay particular attention to things that move and otherwise appear unusual. The peculiar motions of the planets relative to the background stars, as can be seen on a graphical almanac and will be plotted in an exercise below, would seem to demand attention from those who noticed them.

A first-time observer would probably not notice the planets, unless the planets were brighter than other objects in the sky. And even then, he or she would not notice them moving relative to the stars. However, thoughtful observers watching the nighttime sky over many weeks would notice that the planets slowly move relative to the starry background. The usual motion of the superior planets, those with orbits beyond Earth's, is to move eastward in the sky relative to the stars. This is called prograde motion. Periodically, this prograde motion stops and a westward motion, called retrograde motion, occurs, followed in a few weeks by a return to prograde motion. Mars illustrates this in a clear and dramatic fashion, as indicated by its ancient Egyptian name, "sekded-ef em khetkhet," which means "one who travels backwards."

The inferior planets, Mercury and Venus, have their own characteristic motions and are not ordinarily seen to cross the meridian. They do cross the meridian, of course, but only when the Sun is high in the sky so they can not be seen, except when a total solar eclipse occurs near the meridian. For additional information, see Exercise 19, Elliptical Orbit of Mercury, and Exercise 24, Solar Eclipses.

Kepler worked as Tycho Brahe's assistant, starting in the year 1600. Large sextants were used in Tycho's observatory, Uraniborg, to obtain the best observational data then recorded on the angular positions of the planets. This was "pre-telescopic" data, and so was limited in accuracy by the 1 minute-of-arc resolution of unaided eyes. Kepler acquired Tycho's data upon his death, and used this data to test his ideas about the geometry of planetary orbits. Kepler knew that it took 687 days for Mars to complete one orbit about the Sun (one heliocentric circuit), while Earth requires 730 days to complete two heliocentric circuits. Kepler performed parallax measurements to determine the distance from Earth to Mars by noting the positions of Mars every 687 days, when Mars is at the same place in its orbit but Earth is not. Figure 20-1 illustrates the arrangement, and shows how it is possible to triangulate the position of Mars graphically from different positions along Earth's orbit. See Exercise 21, Parallax, for more information on Parallax. Kepler used numerical methods to determine the distance to Mars, since that could be done more accurately than graphical methods. Kepler had to interpolate to get data on some dates, as Tycho's observatory was on the island of Hveen, and data would, for example, be missing for cloudy days. By observing how Mars' distance from Earth changed, Kepler was able to show that the orbit of Mars is not circular but is close to an ellipse. Errors and uncertainties in the data limited the accuracy to which Kepler could fit Mars' orbit to an ellipse.
Figure 20-1: The distance to Mars from Earth at positions $E_1$ and $E_2$ is determined by triangulation. The heliocentric longitudes, $\theta_1$ and $\theta_2$, fix the positions of Earth along its orbit at $E_1$ and $E_2$. The geocentric longitudes of Mars, $\phi_1$ and $\phi_2$, determine the position of Mars when Earth is at $E_1$ and $E_2$. The orbit of Mars can then be determined by repeating this process many times. The symbols representing the Sun and planets are not to scale.
Procedures

Apparatus

drawing compass, graphical almanac and/or an observer's handbook, protractor, and straight edge.

A. The Retrograde Motion of Mars

1. Graph the trajectory of Mars in Figure 20-2 using data from Table 2. Make a point on the chart for each of the data set values and connect the points chronologically with a smooth curve. Notice that each day Mars and stars in the background will rise and set together, but the relative position of Mars changes from day to day.

2. Record the dates when the motion of Mars plotted in Figure 20-2 reverses itself.

3. Graphically explain the retrograde motion of Mars by connecting the corresponding points in Figure 20-3 with the same numbers by lines drawn to intersect the vertical line at the right. Number these intersection points on the vertical line as shown by the example. Both planets revolve about the Sun in the same direction, but Earth has a higher orbital speed and smaller orbit so that it periodically overtakes Mars. As Earth passes Mars at 5, Mars appears to move backward (retrograde), to the west in the sky, even though it is actually moving to the east in its own orbit. As Earth approaches position 7, Mars again resumes its eastward (prograde) motion in the sky.

4. Using a graphical almanac or current observer's handbook, identify the other planets that undergo retrograde motion during the current calendar year. Make a table listing these planets and the dates when the retrograde motions begin and end.

B. The Elliptical Orbit of Mars

1. Use data from Table 3 to plot the orbit of Mars relative to the orbit of Earth shown in Figure 20-4. Begin with the first data set. Measure and mark the angles of heliocentric longitude of Earth for the two dates in this data set, by using a protractor to measure the angle counterclockwise from the vernal equinox line. Mark the positions on Earth's orbit, which give the positions of Earth on the given dates. Then, construct lines parallel to the vernal equinox line which run through those positions of Earth just given. Use the protractor to measure the geocentric longitude of Mars, again in the counterclockwise direction, from the lines parallel to the vernal equinox lines. Then draw two lines from the positions of Earth in these directions (as indicated by the geocentric longitudes of Mars). These lines cross at the position Mars had on those dates, and therefore locate a point on the orbit of Mars.

2. Repeat the above for each set of data given in Table 3. Then, sketch an ellipse to provide the best fit for all of the positions determined on the orbit of Mars.
3. Draw a straight line from the apparent perihelion of Mars through the Sun to the aphelion. Bisect this line to find the center of the orbit. Measure and record the distance from the Sun to the center of the orbit (often called \( c \)). Measure and record the semi-major axis, the distance from the center of the orbit to either the perihelion position or the aphelion position. Determine the eccentricity, \( e \), of the orbit of Mars by dividing \( c \) by the semi-major axis. Compare this value to the current value for Mars, 0.093.

Table 2: Data on the position of Mars in 1595 and 1596, at 0000 Universal Time. The date is given by month (mm) and day (dd), RA by hours (hh) and minutes (mm), and DEC by degrees (dd) and minutes of arc (mm).

<table>
<thead>
<tr>
<th>Date (mmdd)</th>
<th>RA / DEC (hhmm (+/-)ddmm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>- 15 95 -</td>
<td></td>
</tr>
<tr>
<td>0620</td>
<td>0030 +0047</td>
</tr>
<tr>
<td>0701</td>
<td>0058 +0337</td>
</tr>
<tr>
<td>0710</td>
<td>0120 +0549</td>
</tr>
<tr>
<td>0720</td>
<td>0144 +0806</td>
</tr>
<tr>
<td>0801</td>
<td>0211 +1034</td>
</tr>
<tr>
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<td>0230 +1212</td>
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<tr>
<td>0820</td>
<td>0250 +1347</td>
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<td>0338 +1740</td>
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<tr>
<td>1010</td>
<td>0337 +1756</td>
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<td>- 15 96 -</td>
<td></td>
</tr>
<tr>
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</tr>
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<td>0201</td>
<td>0318 +2013</td>
</tr>
<tr>
<td>0210</td>
<td>0334 +2112</td>
</tr>
<tr>
<td>0220</td>
<td>0353 +2213</td>
</tr>
</tbody>
</table>
Table 3: From Tycho's data, which can be used for parallax measurements of the distance between Earth and Mars.

<table>
<thead>
<tr>
<th>Date</th>
<th>Heliocentric Long.</th>
<th>Geocentric Long. of Mars</th>
</tr>
</thead>
<tbody>
<tr>
<td>1585 Feb. 17</td>
<td>159</td>
<td>135</td>
</tr>
<tr>
<td>1587 Jan. 5</td>
<td>115</td>
<td>182</td>
</tr>
<tr>
<td>1591 Sept. 19</td>
<td>005</td>
<td>284</td>
</tr>
<tr>
<td>1583 Aug. 6</td>
<td>323</td>
<td>346</td>
</tr>
<tr>
<td>1593 Dec. 7</td>
<td>086</td>
<td>003</td>
</tr>
<tr>
<td>1595 Oct. 25</td>
<td>042</td>
<td>050</td>
</tr>
<tr>
<td>1587 Mar. 28</td>
<td>197</td>
<td>168</td>
</tr>
<tr>
<td>1589 Feb. 12</td>
<td>154</td>
<td>219</td>
</tr>
<tr>
<td>1585 Mar. 10</td>
<td>180</td>
<td>132</td>
</tr>
<tr>
<td>1587 Jan. 26</td>
<td>136</td>
<td>185</td>
</tr>
</tbody>
</table>
Figure 20-2: A section of sky on which to plot the motion of Mars in right ascension and declination using the data presented in Table 2.
Figure 20-3: Orbits of Earth and Mars that can be used to illustrate how the relative motions of these planets result in the retrograde motion of Mars as observed from Earth.
Figure 20-4: A plot of Earth's orbit about the Sun, to be used for triangulation to determine the orbit of Mars using data from Table 3.
The distance to a star needs to be known before many of its properties can be determined. Most stars are so far away that even in the most powerful telescopes they appear only as points of light. Fortunately, there are some stars that are close enough to the Solar System that their distances can be determined by a simple geometrical method called parallax. Parallax is easy to observe: note the apparent change in position of a finger held at arm's length against the background when viewed first with one eye and then the other. This apparent movement or displacement is also detectable in stars as Earth revolves about the Sun, but only by measuring very small angles. Parallax provides a firm basis for measuring distances to nearby stars.

The word "parallax" first appears in English writings in 1594, and is defined in Webster's Ninth Collegiate Dictionary, 1983, as, "the apparent displacement or difference in apparent direction of an object as seen from two different points not on a straight line with the object." Even the parallactic shift of a nearby star due to the motion of Earth around the Sun is so small that it was not until 1838 that sufficiently large telescopes and high-quality, photographic films became available for Friedrich Wilhelm Bessel (1784 - 1846) to measure it.

This exercise demonstrates how the distance to nearby terrestrial objects can be measured using the parallax that occurs as a result of the separation of the eyes. The different view given by each eye provides important information to the brain to help produce depth perception and estimate distances, as may be noticed by using only one eye for a while. It used to be thought that humans determined the distances to objects by measuring how far the eyes had to turn inward to see the objects. This turns out to be false. If it were true, one could only determine the distance of one object at a time. A single, brief glance with both eyes open is all that is needed to estimate the distances to many objects. This is much too fast for a process that requires one to focus on each object sequentially.

Part of the ability to judge distance comes from comparing the relative sizes of familiar objects. An oak leaf, for example, looks only half as big (subtends only half the angle) when it is twice as far away. Even the blurring effect of atmospheric haze assists in distance determination and depth perception. The absence of familiar objects and an atmosphere can make it difficult to navigate on the Moon.

Parallax measurements on stars are often made by photographing the relative positions of stars about three months apart, then measuring the angle of the apparent shift. This is called heliocentric parallax with a baseline of 1 astronomical unit (1 AU), because Earth moves 1 AU perpendicular to the line of sight to the star in about three months. The baseline, b, the parallax angle, p, and the distance to the star, d, shown in Figure 21-1, are related by the parallax formula,

$$\tan(p) = \frac{b}{d}.$$  (1)
Notice that $b$ and $d$ are measured in the same units. It might be helpful if the tangent need not be determined. For small angles, $p$, measured in radians, the tangent of $p$ is approximately equal to $p$. That is,

$$p \approx \tan(p) = \frac{b}{d}.$$  \hspace{1cm} (2)

This is known as the small angle approximation.

---

Figure 21-1: Heliocentric parallax using a baseline of 1 astronomical unit.

A new unit of distance came into usage in the early 1900's, called a parsec (pc), from "parallax of one second-of-arc." It is the distance a star must be from Earth in order to show a parallactic shift of 1 second-of-arc (1") when a baseline of 1 AU is used.

Example 1: Find the equivalent distance in astronomical units and meters for a parsec. Use Equation 1 with a parallax angle of 1 second-of-arc (1").

Thus,

$$d = 1 \cdot \text{pc} = \frac{1 \cdot \text{AU}}{\tan(1\text{"})} = \frac{1 \cdot \text{AU}}{\tan\left(1\text{o} \cdot \frac{1\text{o}}{3600}\right)} = 206,265 \cdot \text{AU}.$$  

Since $1 \text{ AU} = 1.496 \cdot 10^{11}$ m, a parsec is also

$$(206,265 \cdot \text{AU})\left(\frac{1.496 \cdot 10^{11} \cdot \text{m}}{1 \cdot \text{AU}}\right) = 3.086 \cdot 10^{16} \text{ m}.$$  

There is an inverse relationship between the parallax angle and the distance. That is, a closer object has a larger parallax angle than an object that is farther away. Since, by definition, a star that exhibits a heliocentric parallax of 1" is at a distance of 1 pc, Equation 1 can be simplified to

$$d = \frac{1 \cdot \text{pc}}{p}.$$  \hspace{1cm} (3)

Note that this equation is true only if the parallax angle is measured in seconds-of-arc and the distance is measured in parsecs.
Example 2: What will be the heliocentric parallax angle of a star that is 2 pc away.
\[ p = \frac{1 \cdot \text{pc}''}{d} = \frac{1 \cdot \text{pc}''}{2 \cdot \text{pc}} = 0.5'' . \]

Example 3: A star has a parallax angle of 0.1''. How far away is it?
\[ d = \frac{1 \cdot \text{pc}''}{0.1''} = 10 \cdot \text{pc} . \]

Astronomers often use another unit, the light-year (ly). This is a unit of distance, not time, and is the distance light travels in a vacuum in one year. The exact speed of light in a vacuum is 299792458 meters per second or, to three significant figures, \( 3.00 \times 10^8 \text{ m/s} \). The number of seconds in a year is
\[ 1 \cdot \text{yr} = \frac{365.25 \cdot \text{d}}{24 \cdot \text{hr}} \cdot \frac{3600 \cdot \text{sec}}{1 \cdot \text{sec}} = 31557600 \cdot \text{sec} . \]
So, to three significant figures, a light-year is \( 9.46 \times 10^{15} \text{ km} \). A parsec is about \( 3.26 \text{ light-years} \).

Example 4: The distance to Alpha Centauri was measured by Henderson in 1839 using parallax. Modern measurements give a parallax angle of 0.76 seconds-of-arc (4700 times smaller than one degree!). What is the distance to Alpha Centauri?
\[ d = \frac{1 \cdot \text{pc}''}{0.76''} = 13 \cdot \text{pc} = 43 \cdot \text{ly} = 270,000 \cdot \text{AU} . \]
This angle, 0.76'', is about the same angle subtended by an American quarter dollar coin at a distance of 8 km!

A meter cross-staff may be used to determine the distances of nearby objects using parallax. A meter cross-staff is made by taping a metric ruler (with millimeter marks) to one end of a meter stick. As in Figure 21-2, the meter cross-staff is held under the right eye so the ruler is one meter from the eye along the line of sight. An object at a distance, \( d \), away is viewed past the ruler and is aligned with a mark on the ruler. Without moving the meter cross-staff from the right eye, vision is switched to the left eye (a baseline of \( y \)). The object appears to move against the ruler a distance of \( x \) from the original mark. Using trigonometry and similar triangles, one discovers that
\[ \tan(p) = \frac{x}{d-1 \cdot \text{m}} = \frac{y-x}{d} . \]
Rearranging these equations shows that \( d \) (in meters) is,
\[ d = \frac{y \cdot (1 \cdot \text{m})}{y-x} , \]
where the 1 meter in the numerator comes from the length of the staff.
Figure 21-2: Parallax using a meter cross-staff.

The small angle approximation can be used to simplify the calculation of the parallax angle using Equation 4. If $p$ is allowed to be measured in radians, then

$$p \approx \frac{y}{d},$$

where the distance between the observer's eyes, $y$, and the distance to the object, $d$, must be measured in the same distance units.

Example 5: Using a meter cross-staff, an object appears to shift 5.4 cm along the ruler. The distance between the observer's eyes is 6.1 cm. How far is the observer from the object? What is the parallax angle in radians?

$$d = \frac{(6.1 \text{ cm}) (1 \text{ m})}{(6.1 \text{ cm} - 5.4 \text{ cm})} = 8.7 \text{ m}$$

$$p \approx \frac{6.1 \text{ cm}}{870 \text{ cm}} = 0.0070 \text{ radians}.$$
Procedures

Apparatus
ruler, meter stick, and mirror (at least 15 cm long by 5 cm wide).

Note: Part A must precede part B, but part C may be done at any time.

A. Inter-eye Distance
1. Place a mirror near the edge of the lab table. Place a millimeter ruler on top of the mirror, and look at your face in the mirror. Arrange the ruler so its top edge runs from the middle of one eye to the middle of the other in your image. Close your right eye, look into the mirror and move your head (or the ruler) so you see the image of your left pupil bisected by a mark on the ruler (this may be any mark you choose). Then switch eyes without moving your head, and read the ruler mark at which you see the center of your open, right eye. The difference between these two is the distance between your eyes. It is important to measure this distance accurately and precisely (to the millimeter). Record this value.

2. Repeat the inter-eye distance measurement several times.

3. Compute the arithmetic mean of these inter-eye distance measurements. Use this mean value in the calculations of part B.

B. Parallax
Caution! Move about the lab and halls with care so as not to poke anyone in the eye with a meter cross-staff.

1. Use the meter cross-staff by holding one end close to one eye, with the ruler forming a cross-beam on the opposite end. Look at an object with one eye, noting the object's apparent position on the ruler, then switch to the other eye and again note its position. It is important that the meter cross-staff is not moved during this step. The difference between the two readings is the apparent displacement, x, due to parallax. Measure this value to the millimeter. Construct a table of the name of the object, its parallactic displacement (x), its calculated parallax angle (p in radians), its calculated distance (d_{calc} in meters), its actual distance (d_{act} in meters), and the percentage error between these two distance measurements.

2. Repeat the steps of B.1 for four more objects, as close as 2 meters and as far as 30 meters. Choose objects at distances throughout this range so you can determine for what range of distances this device is useful.

3. Discuss your results, considering the important sources of error. For what range of distances (minimum to maximum) is this method useful? Would you get more accurate results if your eyes were further apart? Why or why not?
C. Units of distance
Use a baseline of 1 AU for heliocentric parallax calculations.

1. If a star is 25 parsecs away, what is its heliocentric parallax angle in seconds-of-arc?

2. How far is the star of C.1 in light-years?

3. How long does it take light from Arcturus (Alpha Boötes) with a heliocentric parallax angle of 0.0909" to reach us? Your answer should be in years.

4. Two stars have different parallax angles. The first star has a parallax angle of 0.196", and the second star has a parallax angle of 0.0625". Which star is closer?

5. The semi-major axis of Ceres is 2.77 AU. On average, how many meters is Ceres from the Sun?

6. How long is an astronomical unit (AU) in light-minutes (lmin)? Construct a table listing the names of the planets (in increasing distance from the Sun), their mean distances in AU and in lmin.

7. Add another column to the above table that contains the semi-major axis distances of the planets in gigameters (Gm). 1 Gm is $10^9$ m.

8. Approximately how long does it take a radio message to travel from Earth to Jupiter at their closest? Give your answer in minutes.

9. If Antares (Alpha Scorpii) is 120 pc away, how long would it take a spacecraft traveling at 10% of the speed of light to go from Earth to Antares? Give your answer in years.

10. Approximately how many centimeters is equivalent to a light-nanosecond (lns)? A nanosecond is $10^{-9}$ s.