Registry of Interpreters for the Deaf, Inc.

Journal of Interpretation

Volume 25 | Issue 1 Article 10

2016

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ABSTRACT

The grammar and vocabulary of higher mathematics are different from the grammar and vocabulary of conversational English and conversational American Sign Language (ASL). Consequently, mathematical language presents interpreters with a unique set of challenges. This article characterizes those aspects of mathematical grammar that are peculiar to the subject. (A discussion of mathematical vocabulary and its expression in ASL can be found elsewhere (Tabak, 2014).) An increased awareness of the grammar of mathematical language will prove useful to those interpreters for the deaf and deaf mathematics professionals seeking to express higher mathematics in ASL.

In this article one will, for the first time, find a model of mathematical language created for interpreters that identifies those aspects of the language that must be retained in any accurate interpretation. In particular, the article identifies the characteristic properties of propositions and propositional functions that must be preserved by the interpreter when propositions and propositional functions are expressed in ASL. It identifies the characteristic properties of mathematical language used in the definition of sets and in the statement of theorems that must be preserved by the interpreter when (mathematical) definitions and theorems are expressed in ASL. In addition, the article includes a method of symbolically representing mathematical language which is useful for analysis. The method enables the user to break down complicated-looking mathematical sentences into their simpler constituent parts in order to simplify the problem of interpretation. Numerous examples of mathematical language are included as are examples of how the diagramming technique can be used to clarify the structure of mathematical definitions and theorems.

The method by which this model of mathematical language should be implemented is left to groups of expert practitioners, deaf and hearing.

INTRODUCTION

Mathematical language has evolved to suit the needs of professional mathematicians but few others. Dense, logically precise, and distinct from conversational language in both vocabulary and grammar, it is often used to express ideas that have no readily identifiable counterparts in the real world—that is, the world as it is known through observation and experiment. As a consequence, mathematics is often opaque to the non-specialist, but even those fluent in American Sign Language (ASL) and with a strong background in mathematics can find it difficult to express mathematical language in ASL, either because so many common mathematical words have no counterparts in ASL or because the grammar of mathematics is so different from the grammar of ASL.

The characteristics of mathematical vocabulary and the difficulties associated with creating a mathematical vocabulary in ASL are addressed in Tabak (2014). The challenges associated with expressing the higher level aspects of mathematical English in ASL, as opposed to creating an adequate signed mathematical vocabulary, have not been previously examined.

Broadly speaking, the main difficulty with interpreting higher mathematics is that to some extent mathematical language is a “foreign” language, a language that requires years of study to master. In this article we identify those higher-level aspects of mathematical language to which interpreters, deaf mathematics professionals, and educators must direct their attention if mathematical language is to be faithfully rendered in ASL.

Our description of mathematical language is a formal one; it does not require knowledge of mathematics to appreciate. The primary goal of this article is to facilitate the analysis of mathematical language by interpreters so that mathematics can be expressed in ASL with greater precision and clarity, and precision and clarity are at the heart of mathematics no matter in which language mathematics is expressed.

This article is addressed primarily to academic interpreters, especially those at the college and university level, whose assignments call upon them to interpret classes and seminars in higher mathematics, but deaf mathematics professionals, deaf undergraduates enrolled in mathematics classes, educators for the deaf, and deaf graduate students studying higher mathematics should also find the information contained herein useful. Readers will, for the first time, find in this article a complete description of those aspects of grammar that characterize the language of higher mathematics from the point of view of the interpreter. It is also hoped that this information will be useful to those interested in creating a standard mathematical extension of ASL, one that would be uniform from interpreter to interpreter, educator to educator, and institution to institution; thereby enabling the deaf to enjoy the same linguistic advantages as their hearing counterparts.

In order to demonstrate the grammatical structure of mathematical language as clearly as possible and so facilitate its expression in ASL, this article contains a number of examples of mathematical sentences. To understand this article, which is concerned only with language, one need not understand the mathematical content of the sample sentences; only the grammar is of interest.

Some of the sample mathematical sentences to be found below are represented via algebraic symbols. The symbols and many of the formulae in which they appear will be familiar to readers who have interpreted courses that examine how theorems are proved. (These “proof courses” are now required in many undergraduate mathematics programs.) Algebraic symbolism—one need not remember any high school or college algebra to use it—allows one to deconstruct a mathematical sentence into its constituent parts without regard to either vocabulary or content. The symbols reveal the grammar of mathematics in a way that words do not. Each (algebraic) formula functions as a kind of schematic, the mathematical analogue of the sentence diagramming technique readers may remember from high school. The technique used here may be new to the reader, but with a little practice, the reader should be able to use it without difficulty. Several examples have been included with that goal in mind.

This article uses only nine standard logical symbols together with their most common verbalizations. Logical symbols are used because the grammar of mathematical language is taken from the grammar of logic, and one cannot help but encounter these logical symbols and their verbalizations when interpreting higher mathematics. The necessary symbols are identified below as S1 through S9. Smullyan (1995) completes his exposition of logic with less than nine symbols. Russell (1938) requires more than nine to complete his exposition. For the purposes of this paper, nine is sufficient.

Finally, the presentation of this information is a straightforward enumeration and description of those aspects of mathematical language that arise in practice. The goal is to characterize those higher-level aspects of mathematical language that must be preserved when expressed in ASL in order to ensure that each interpreted mathematical statement continues to convey mathematical meaning. A glossary of symbols is provided at the end of the article for ease of reference. No attempt is made within this article to suggest the particular signs or signed expressions needed to express mathematical language in ASL. Those decisions are better left to one or more groups of expert practitioners, deaf and hearing. For a discussion of the extent and nature of mathematical vocabulary and the nature of mathematical inquiry, see Tabak (2014).

Interpreting Propositions and Propositional Logic

An elementary proposition is a sentence that consists of a subject and a single assertion about the subject. “Euclid is a mathematician.” is an example of an elementary proposition, but “Euclid is a mathematician and soldier.” is not an elementary proposition because it makes two assertions about Euclid. Henceforth, when we discuss propositions, we mean elementary propositions. Smullyan (1995) calls elementary propositions “atomic.” They are the smallest information-bearing units in mathematical language and function as building blocks from which larger and more complex statements are formed. One cannot hope to successfully interpret mathematics if one cannot interpret propositions.

Each mathematical discipline has associated with it a very large set of propositions characteristic of that discipline. The propositions that arise in geometry, for example, are different from the propositions that arise in algebra. Mathematicians require an enormous and highly specialized vocabulary to state these propositions, and it is in the statement of these propositions that the ASL vocabulary deficit described by Tabak (2014) is almost entirely confined. In this article, we will have little to say about the deficit of mathematical vocabulary in ASL. Instead, in this section we concentrate on identifying the vocabulary-independent characteristics of propositions that must be preserved in any successful interpretation as well as the methods by which propositions are combined to produce more complex statements.

Interpreting Propositions

For the sake of definiteness, we begin with three examples of propositions:

Ex. 1: The set of prime numbers is infinite.

Ex. 2: Three is not divisible by two.

Ex. 3: Five is less than seven.

All three propositions listed above are true, but truth is not a defining characteristic of a proposition. The next three sentences are also propositions:

Ex. 4: The set of prime numbers is finite.

Ex. 5: Three is divisible by two.

Ex. 6: Seven is less than five.

Each of the six propositions listed above has a *truth value*—that is, each proposition is either true or false—and it is the existence of a truth value that is one of the defining characteristics of a proposition. A sentence that has no truth value is not a proposition. A faithful interpretation of a proposition must, therefore, retain this true/false property.

To be clear, during a lecture, the interpreter may encounter a proposition with an unknown truth value. Even the lecturer may not know whether the proposition in question is true or false. It is the unambiguous existence of a truth value, not a particular truth value, which is essential: If a proposition is not true, it must be false, and if it is not false, it must be true. Any interpretation of a proposition that fails to convey this true/false property is not meaningful. While propositions are usually grammatically simple, interpreting them can still be challenging, especially when, as is the case for every proposition listed above, the proposition incorporates the verb “is.” To see why, consider the problem of interpreting the proposition, “Three is divisible by two.”

As has been widely remarked, in signed conversation, native signers rarely find occasion to explicitly sign any form of the verb “is.” In ASL, “is” is a foreign word, and its absence from conversational language passes unnoticed. For this reason, one might be tempted to interpret the phrase “three is divisible by two” in the same way that one might interpret the phrase “three divided by two.” In other words, when interpreting the sentence “three is divisible by two,” one might suppress overt reference to the verb “is” and, by way of illustration, use the well-known signs for “three,” “divide,” and “two” in the following way: place “three” near the top of the signing space, "divide" in the middle, and "two" toward the bottom of the signing space. Vertical placement of the signs—as opposed to horizontal placement—rules out the other possible spoken phrase, "three divides two," which is, of course, a proposition with a meaning quite distinct from "three is divisible by two." The interpreter would then expect the client, and the teacher the student, to use contextual clues to distinguish between "three divided by two" and "three is divisible by two." This would be a mistake for the following reason:

One cannot rely on context to distinguish between a sentence that is a proposition and one that is not because often in mathematical language there are no contextual clues. In the example considered above, an accurate interpretation must reflect the fact that "three is divisible by two" is a proposition and "three divided by two" is not. Notice that the former has a truth value. It asserts something about the number three—namely, that the number three is even and so the truth value of the proposition is "false"—and just as important, because "three is divisible by two" is a proposition, it can be negated. ("Three is not divisible by two.") The ability to unambiguously assert and negate is crucial to mathematical discourse.

Now consider the expression "three divided by two." It asserts nothing. It has no truth value. It cannot be negated. It is simply an alternate way of expressing the number whose decimal representation is 1.5.

In higher mathematics, propositions are abstract, stand-alone entities comprised of a subject and an assertion about the subject—nothing more but nothing less. An unambiguous subject and an unambiguous assertion are always explicitly present in a spoken or written proposition, and they must be readily apparent in each signed proposition as well. The interplay between subject and assertion determines the truth value of the proposition, and, again, every mathematical proposition must be either true or false.

But while propositions must be either true or false, they need not "mean" anything in a larger context. Truth and falsity independent of meaning is part of what makes mathematical language different from ordinary language: Mathematicians seek mathematical truths; they are usually willing to skimp on extra-mathematical meaning. As Austin and Howson (1979) write in their article on language and mathematics, "The language of, say, Section 117 of *Principia Mathematica* on 'Greater and Less' achieves a precision and correctness to which ordinary language cannot aspire. Unfortunately, by itself it contains very little of the meaning of greater or less." (p. 178)

How can interpreters achieve this "precision and correctness to which ordinary language cannot aspire?" By way of illustration, the author often had the pleasure of engaging the late Oliver Blaylock, a native signer, in philosophical discussions. Blaylock, who was acutely aware of the need to state propositions unambiguously, would state each of his propositions with a stylized form of a common sign for "true." He would hold his index finger vertically against his lips and then extend his hand directly outward without bending the wrist. The motion is similar to what many Signed English dictionaries suggest for signing the word "be." The form of each of his propositions was always the same: subject-true-assertion. He formed the negation of a proposition by shaking his head as he asserted the proposition.

Each signed sentence of the form "subject-true-assertion" constituted a proposition because the sign "true" was used as a copula. In Standard English, a copula is a verb that links the subject of the sentence, which always appears before the copula, with the assertion that follows the copula. In a proposition that uses a copula, therefore, the function of each sign/word that appears in the sentence can be determined from its position within the sentence. As a consequence, the meaning of each sentence is intrinsic to the sentence—that is to say, the meaning of the sentence is independent of the context in which the sentence appears. In Blaylock's formulation, the sign or signs that preceded the sign "true" always constituted the subject of the sentence, and the sign or signs that followed "true" always constituted an assertion about the subject. No external referents were needed to identify the subject or the assertion. The meaning of each of Blaylock's propositions depended only on the order in which the signs appeared within the proposition. This is also a characteristic of mathematical English. To provide a nonmathematical example, "Dogs are mammals." is a (true) sentence that asserts something about dogs—namely, every dog is a mammal. It has a meaning quite distinct from the sentence, "Mammals are dogs." which is identical to the first proposition in all but word order and which makes the false assertion that every mammal is a dog.

Careful attention to the question of how best to express mathematical propositions in ASL is needed because mathematical language generally consists of abstract logical arguments devoid of the motivations that gave rise to them. External referents are almost always absent, and false propositions are part of the linguistic landscape. One must, therefore, have a way of signing mathematical propositions so that the subject of the sentence can be distinguished from the assertion without reference to a particular context or set of motivations. Each proposition must stand alone.

This is not to argue that interpreters should adopt the form used by Blaylock to express propositions. There are probably better ways to express mathematical propositions in ASL, and in particular, the use of the sign translated here as "true" would almost certainly be confusing in a mathematical context as mathematical propositions are as apt to be false as true.

The decision about how best to express mathematical propositions in ASL is a critical one: One cannot adequately express mathematical language in ASL without a satisfactory method of expressing propositions. Standardization is also important. A standardized method of signing propositions would mean that deaf students and deaf mathematics professionals need not develop a private language with each interpreter or educator with whom they work. This is, of course, an advantage already enjoyed by their hearing counterparts. We leave the decisions about how best to interpret propositions to one or more groups of expert practitioners, hearing and deaf.

Interpreting Propositional Logic

A large set of propositions is of little mathematical value in itself. Instead, propositions form the raw material from which mathematicians create new and more complex statements. The methods by which these compound statements are formed and the methods by which their truth values are determined are to be found within the field of propositional logic. An awareness of the grammar of propositional logic can be helpful because mathematical grammar is based in part on propositional logic. Statements in propositional logic—at least as it is used in mathematical language—are formed from only five logical connectives and a set of propositions. The symbols for these connectives, together with brief explanations and some common ways that they are expressed in English, are listed below.

S1: $\wedge$ Called "conjunction," this symbol is usually pronounced "and." If two arbitrary propositions are represented by the letters *p* and *q*, then $p \wedge q$ is a new statement and is usually expressed verbally as "*p* and *q*." The statement $p \wedge q$ has the value "true" provided *p* is true and *q* is true; the statement is false otherwise.

S2: $\vee$ Called "disjunction," this symbol is usually pronounced "or." If two arbitrary propositions are represented by the letters *p* and *q*, then $p \vee q$ is a new statement and is usually expressed verbally as "*p* or *q*." The statement $p \vee q$ has the value "true" provided *p* is true or *q* is true; the statement is false only when both *p* and *q* are false.

S3: (for propositions) $\Rightarrow$ Called "implication," this symbol is usually pronounced "implies" when it is used to connect two propositions. If two arbitrary propositions are represented by the letters *p* and *q*, then $p \Rightarrow q$ is usually expressed verbally as "*p* implies *q*." Russell (1938) calls the symbolic expression $p \Rightarrow q$ a "material implication." By this, Russell means that since *p* and *q* have definite truth values, so does the expression $p \Rightarrow q$. (The implication symbol is used, and sometimes verbally expressed, in a somewhat different way when used to connect propositional functions. See section 2.)

S4: (for propositions) $\Leftrightarrow$ Called "double implication," this symbol represents two applications of the implication symbol, and when used with propositions, the expression $p \Leftrightarrow q$ is often verbalized as "*p* is equivalent to *q*," meaning $p \Rightarrow q$ and $q \Rightarrow p$. By way of providing the reader with a little practice, the symbolic expression $p \Leftrightarrow q$ is equivalent to $(p \Rightarrow q) \wedge (q \Rightarrow p)$. (As with implication, the double implication symbol is used and sometimes pronounced in a somewhat different way when used in conjunction with propositional functions. See section 2.)

S5: $\neg$ Called "negation," this symbol is an example of what logicians call a "unary connective," because it "connects" only one proposition. If *p* is a true proposition then the expression $\neg p$ is false, and if *p* is false then $\neg p$ is true. By way of providing the reader with some additional practice, $\Rightarrow$ is often defined in terms of $\vee$ and $\neg$ in the following way: $(p \Rightarrow q) \Leftrightarrow ((\neg p) \vee q)$, that is, "'*p* implies *q'* is equivalent to 'not-*p* or *q'*." (The negation appearing in front of *p* applies only to the proposition *p*, and the parentheses appearing in the symbolic statement are used in just the same way as they are used in college algebra: for purposes of punctuation.)

There is no theoretical limit on the length or complexity of statements that can be formed from elementary propositions and logical connectives, but for the sake of clarity, such statements are usually kept fairly short. From an interpreter's perspective, then, the main challenge in interpreting statements involving propositional logic is producing faithful interpretations of the constituent propositions. Compound statements involving multiple propositions are simply individual propositions linked by the five connectives. Here, for purposes of illustration, are three statements involving two or more propositions together with their symbolic representations:

Ex. 7: "The set of prime numbers is infinite implies the set of integers is infinite." The word "implies" is what is new here. If we place the constituent propositions in square brackets, we can better see the form of the sentence:

[The set of prime numbers is infinite] implies [the set of integers is infinite].

This is a statement of the form $p \Rightarrow q$, where *p* represents the proposition "The set of prime numbers is infinite," and *q* represents the proposition "The set of integers is infinite."

Ex. 8: "Seven is less than eight and eight is less than nine implies seven is less than nine." Again, enclosing the propositions with square brackets and using a pair of parentheses to show which propositions are connected by the word "and" we obtain:

([Seven is less than eight] and [eight is less than nine]) implies [seven is less than nine].

This statement is of the form $(p \wedge q) \Rightarrow r$, where *p* represents the proposition "Seven is less than eight;" *q* represents the proposition "eight is less than nine;" and *r* represents the proposition "seven is less than nine."

Ex. 9: "100 is divisible by 4 implies 100 is divisible by 2 is equivalent to 100 is not divisible by 2 implies 100 is not divisible by 4." In this sentence, there are four propositions, and they are nested to form one principle implication and two subordinate ones. We continue to use square brackets to identify the individual propositions and parentheses to identify the subordinate propositions:

([100 is divisible by 4] implies [100 is divisible by 2]) is equivalent to ([100 is not divisible by 2] implies [100 is not divisible by 4]).

This statement is of the form $(p \Rightarrow q) \Leftrightarrow (\neg q \Rightarrow \neg p)$. We trust that by now the reader can identify the elementary propositions and the relevant connectives.

For purposes of emphasis, we repeat: To successfully interpret propositional logic, one need only decide upon a general method for interpreting propositions and a method for interpreting the five basic symbolic statements, $\neg p$, $p \wedge q$, $p \vee q$, $p \Rightarrow q$, and $p \Leftrightarrow q$, and the problem of interpreting propositional logic—at least as it is used in mathematical language—is solved since any other statements that arise in practice are combinations of these five.

We close this section with a few remarks on interpreting and parentheses. Mathematicians rarely pronounce parentheses when they are making a presentation. The resulting "parentheses-free" language is sometimes ambiguous. To see how this ambiguity arises, consider the phrase, "the square root of 2 plus 7." This phrase could mean $\sqrt{2+7}$, which is another way of writing "the square root of 9," or the phrase "the square root of 2 plus 7" could mean $\sqrt{2}+7$, which is a number a little larger than 8 and a little less than 9. Mathematicians eliminate this type of ambiguity by writing the expression in question on the board before, or sometimes after, they say (or sign) it. Interpreters cannot be expected to eliminate a mathematician's ambiguity in their interpretation of it. Rather, the interpretation should reflect the ambiguity of the mathematician's language and allow the client to compare the ambiguous interpretation with the written phrase just as everyone else must do. To expect more of the interpreter is to expect too much.

CONTEXT-FREE LANGUAGE, AN EXAMPLE

Many variants of English exist, and interpreters often encounter different variants during the course of their work. When a presenter on the BBC intones, "I am not unsympathetic." He or she has twice negated the elementary proposition, "I am sympathetic." to obtain a sentence with a meaning that is identical to the original, "I am sympathetic." A good interpretation will reflect this usage. When a country and western singer croons, "I don't have no money." he or she has twice negated the elementary proposition, "I have money." The first negation, "I don't have money." negates the elementary proposition, "I have money." The second negation is added for emphasis: "I have no money…really…none…none at all." A good interpretation must also reflect this use of the double negative.

The variant of English used in mathematical language is more closely related to the first example than the second, of course, but mathematical grammar extends Standard English to obtain new types of statements with definite truth values but no sensible meaning. Interpreting this kind of language requires a high-level of precision not so much to avoid producing nonsense but rather for producing just the right kind of nonsense, what mathematicians sometimes call, "formal nonsense." Consider the following nonmathematical example:

Let p denote the proposition, "The Golden Gate Bridge is a tomato." Let q denote the proposition, "Three equals five." We claim that the sentence, "The Golden Gate Bridge is a tomato implies three equals five," is a true statement.

To see why, first notice that "The Golden Gate Bridge is a tomato implies three equals five," is a statement of the form $p \Rightarrow q$. This is easier to see if we use square brackets to identify the component propositions: [The Golden Gate Bridge is a tomato] implies [three equals five]. Recall that in S5 (see above), where $\neg$, the symbol for negation, was introduced, we noted that $p \Rightarrow q$ is equivalent to the expression $(\neg p) \vee q$, and in S2 we noted that $(\neg p) \vee q$ is true when $\neg p$ is true or when q is true.

Now because the letter p represents the (false) proposition, "The Golden Gate Bridge is a tomato," p is false. Consequently, $\neg p$, which represents the proposition "The Golden Gate Bridge is not a tomato." is true. Because $\neg p$ is true, the expression $(\neg p) \vee q$ is also true (independently of the truth value of q). We can only conclude, therefore, that $p \Rightarrow q$, or "The Golden Gate Bridge is a tomato implies three equals five." is a true statement. It is, however, difficult to say in what context such a sentence would occur and what, if anything, this true implication reveals about the nature of tomatoes, the Golden Gate Bridge, or elementary arithmetic.

To summarize: every sentence arising in propositional logic is either true of false, but whether such sentences must be something more than true or false, whether they must also mean something in an extra-mathematical sense, is not a mathematical question, and most mathematicians have expressed little interest in addressing it. Instead, free from the constraints of the physical world, mathematicians construct new statements from old, and experience has shown that some of these new statements have found application outside of mathematics. It is also true that many have not. As the preceding example demonstrates, expressing formal, and apparently meaningless, sentences in ASL in such a way as to preserve their logical structure requires more than a good mathematical vocabulary. The question of how best to interpret propositions when interpreting mathematical language is an important one, and it remains an open question.

INTERPRETING PROPOSITIONAL FUNCTIONS AND FIRST ORDER LOGIC

Propositional logic is useful for expressing modern mathematical thought, but by itself it is insufficiently expressive. It is ill-suited for making the broad generalizations that mathematicians frequently seek.

The language of modern mathematics depends in an essential way on the theory of sets, and a great deal of mathematical language is concerned with the twin problems of defining sets and making meaningful statements about them. To accomplish these goals, mathematicians use sentences called "propositional functions." The study of propositional functions and the rules that govern their use are part of the discipline of first order logic. First order logic encompasses the grammar of modern mathematics. From the interpreter's point-of-view, the language of first order logic is only a modest extension of the language used to express propositional logic.

We begin with some examples of propositional functions, and invite the reader to compare these propositional functions with those examples of propositions given in Section 1.1 that they most closely resemble:

Ex. 10: The set X is infinite.

Ex. 11: X is divisible by two.

Ex. 12: X is less than Y.

While identical in form to propositions, propositional functions differ from propositions in that they have no truth value. Consider, for example, the sentence, "The set X is infinite." It is neither true nor false because the meaning of the letter X is not specified. It is a variable. For some values of X the sentence is true; for other values of X the sentence is false. The sentences in examples 10 through 12 are not, therefore, propositions despite the fact that grammatically they are identical with certain propositions.

As they appear in examples 10, 11, and 12, the letters X and Y are unrestricted in the sense that no limitations were imposed on what the variables might represent. In practice, however, variables in propositional functions are allowed to range only over some specified set of values, where the values might belong to a set of numbers or perhaps some other type of set. By way of illustration, in example 10, the lecturer might specify that X can represent certain subsets of the set of integers. In example 11, the lecturer might specify that X can represent a positive integer. In example 12, the letters X and Y might be restricted to the set of real numbers greater than zero and less than one. Other choices are possible. In general, the choices a mathematician makes with regard to formulating propositional functions and their associated sets depend on the goals of the mathematician.

Interpreters have, of course, goals that are quite different from those of the mathematician. Because we seek to faithfully represent the mathematician's language, a faithful interpretation of a propositional function need only satisfy the following test: Once a definite value for each variable appearing in the propositional function has been specified, the result is a proposition. A propositional function is, therefore, no harder or easier to sign than a proposition because they have exactly the same grammatical structure. The difference between propositional functions and propositions is the substitution within the former of variables for definite nouns.

Propositional functions can be combined using the five logical connectives—marked S1 through S5 above—to create "compound propositional functions," and this is accomplished in a way that is similar to the way propositions are combined to form more compound sentences. Details are given below. Interpreters should expect to encounter these types of compound propositional functions during lectures.

In what follows, and for ease of presentation, we will use functional notation to represent propositional functions, so, for example, $P(X)$, $Q(X)$, and $R(X)$ will represent propositional functions of one variable in just the same way that we used *p*, *q*, and *r* to represent elementary propositions, and $P(X,Y)$ will be used to represent a propositional function of two variables. (Propositional functions of more than two variables also exist. The statements made here about propositional functions of one and two variables carry over to propositional functions of many variables.)

Propositional functions are encountered in two contexts: in the definition of sets and in the statement and proof of theorems. We begin with the definition of sets.

Interpreting Sentences that Define Sets

When defining sets, propositional functions are usually used in conjunction with the expression "such that" and/or the symbol $\in$. "Such that" and $\in$ are integral to the language of mathematics, and Russell (1938) regards them as primitive in the sense that they cannot be defined within the context of set theory, which means that one should use specialized signs for these terms.

S6: $\in$ Membership. Often pronounced "is an element of," "belongs to," "is in," "in," or "is a member of," this symbol denotes membership in a set. For example, $x \in S$ might be pronounced "*x* is an element of the set *S*," or "*x* belongs to *S*," or "*x* is in the set *S*."

S7: "Such that." There is no generally agreed upon symbolism for the phrase "such that." Sometimes it is represented by a colon, sometimes by a vertical line, sometimes the two words are written out in full, and sometimes the expression is abbreviated "s.t." Other methods of notation exist. In what follows "such that" is represented by a colon.

To see how "such that" and $\in$ are used in conjunction with propositional functions to define sets, let the letter *S* represent the set to be defined. The speaker will usually begin the definition of *S* by identifying a propositional function, *P(X)*, together with a set of values over which the variable *X* may range. "The set of values over which the variable may range," is often represented with a single letter. So, by way of example, let the letter *D* represent the set of values over which *X* may range. The set *S* is then defined as the set of all *X* in *D* for which *P(X)* is true. The whole process often involves just one or two sentences. Example 13 demonstrates how this is done using one sentence; example 14 demonstrates how this is done using two sentences; the italics emphasize where "such that" and $\in$ arise:

Ex. 13: "*S* is the set of all *X in D such that* $P(X)$ (is true)." In symbols: $S = \{X \in D : P(X)\}$.

Ex. 14: "Let *X belong to D*. *S* is the set of all *X such that* $P(X)$ (is true)." In symbols: Let $X \in D$. $S = \{X : P(X)\}$.

The words "is true" are placed in parentheses in examples 13 and 14 because these words are almost always suppressed in the definition of a set. Henceforth, we will follow this convention as well. For the sake of completeness, we take note of one more way of defining *S*: "Let *X* belong to *D*. *X* belongs to *S* if and only if *P(X),"* a formulation about which we will have more to say in the next section. The "if and only if" formulation can be expressed symbolically in the following way: Let $X \in D$. $(X \in S) \Leftrightarrow P(X)$.

By way of illustration, the next two examples define the set *E* of positive even numbers in one and two sentences, respectively. The letter *N* is used to represent the set of positive integers, and the italicized words again indicate the use of $\in$ and "such that."

Ex. 15: "*E* denotes the set of those *X belonging to N such that X* is divisible by 2." Symbolically, this sentence can be written as $E = \{X \in N : P(X)\}$, where *P(X)* represents the propositional function "*X* is divisible by 2."

Ex. 16: "Let *X belong to N*. *E* denotes the set of all *X such that X* is divisible by 2.*"* In symbols, these two sentences can be written as follows: Let $X \in N$. $E = \{X : P(X)\}$, and (again) *P(X)* represents the propositional function "*X* is divisible by 2."

In summary, in order to interpret set definitions in ASL successfully, it is usually enough to know an unambiguous, and hopefully standardized, method of signing propositional functions and to have specialized signs for "such that" and $\in$.

Interpreting Theorems and Their Proofs

In mathematics, a theorem is a statement that has been logically deduced from previously established statements. In other words, a statement is not a theorem unless it has been proved. From the point of view of an interpreter, the language that one finds in the proof of a theorem is the same type of language one finds in the statement of the theorem. To facilitate the exposition, therefore, we will only discuss theorems and their interpretation.

Most theorems consist of combinations of propositional functions, logical connectives, the expression "such that," the term $\in$, and two more words/symbols, called logical quantifiers. This section is primarily devoted to the two quantifiers. First, however, it is necessary to revisit the implication and double implication symbols.

Implication and Double Implication Revisited.

When the symbols for implication and double implication are used to connect propositional functions, the meaning of the statements and often their verbalizations are different from the way that $\Rightarrow$ and $\Leftrightarrow$ are used with propositions. Interpreters must be aware of these differences because some, but not all, mathematicians are careful about making verbal distinctions between the logical connectives $\Rightarrow$ and $\Leftrightarrow$ when they are used to connect propositions and $\Rightarrow$ and $\Leftrightarrow$

when they are used to connect propositional functions. From a logical point of view, this variation in usage is not entirely satisfying. From a practical point of view, however, both client and interpreter quickly adjust to the quirks of the lecturer. One need only be aware of the issue.

Consider the symbolic statement $P(X) \Rightarrow Q(X)$, where *P(X)* and *Q(X)* denote propositional functions, and the letter *X* represents the same undetermined value. It is often pronounced, "If *P(X)* then *Q(X)*." Many theorems are of this form. Because propositional functions have no truth value, it cannot be the case that *P(X)* implies *Q(X),* because neither *P(X)* nor *Q(X)* assert anything—that is, they have no truth values—and so the symbolic statement $P(X) \Rightarrow Q(X)$ is neither true nor false. Instead, statements of the form $P(X) \Rightarrow Q(X)$ assert that *if P(X)* is true for some value of *X then Q(X)* must also be true for that same value of *X*. Of the conditions under which *P(X)* is true, the theorem asserts nothing.

Precisely the same sorts of statements can be made with regard to the symbolic statement $P(X) \Leftrightarrow Q(X)$, which is often pronounced, "*P(X)* if and only if *Q(X).*" As with single implication, the symbolic statement $P(X) \Leftrightarrow Q(X)$ provides no information about the conditions under which either *P(X)* or *Q(X)* might be true. Instead, $P(X) \Leftrightarrow Q(X)$ makes the following assertion about the relationship that exists between the two propositional functions: For any particular value of *X*, *P(X)* and *Q(X)* are either both true or both false.

Statements that use implication and double implication symbols to connect propositional functions are important parts of mathematical language, and one encounters them often. There are several different ways of verbalizing the two symbolic statements $P(X) \Rightarrow Q(X)$ and $P(X) \Leftrightarrow Q(X)$ besides those already listed. Here is a list of the ways that one is likely to hear these symbolic formulae expressed verbally:

S3: (for propositional functions) $\Rightarrow$ The symbolic expression $P(X) \Rightarrow Q(X)$ is usually pronounced as "If *P(X)* then *Q(X)*," but it is sometimes pronounced as

a.) "*P(X)* is sufficient for *Q(X)*."

b.) "*Q(X)* is necessary for *P(X)*."

c.) "*P(X)* only if *Q(X)*."

d.) "*P(X)* implies *Q(X)*."

Again, in each of these cases, the words "is true" are suppressed. The meaning of these sentences is, however, easier to understand if the words are explicitly stated, e.g. "If *P(X)* is true for some value of *X* then *Q(X)* is true for the same value of *X*."

S4: (for propositional functions) $\Leftrightarrow$ The symbolic expression $P(X) \Leftrightarrow Q(X)$ is (again) usually verbalized as "*P(X)* if and only if *Q(X)*," but it is sometimes pronounced as

a.) "*P(X)* is necessary and sufficient for *Q(X)*."

b.) "A necessary and sufficient condition for *P(X)* is *Q(X)*."

c.) "*P(X)* is equivalent to *Q(X)*."

d.) "*P(X)* implies *Q(X)* and conversely."

Just as is the case for single implication, for double implication the words "is true" are also implied but not explicitly expressed. By way of example, "*P(X)* if and only if *Q(X)*." could be written as "*P(X)* is true if and only if *Q(X)* is true for the same value of *X*."

Theorems that assert relationships between propositional functions while making no assertions regarding the conditions under which the functions themselves are true or false are examples of what Russell (1938) calls "formal implication." In "Mathematics and the Metaphysicians," he offers this provocative definition of mathematics in order to emphasize the nature of mathematical reasoning: "Mathematics may be defined as the subject in which we never know what we are talking about, nor whether what we are saying is true." (Russell, 1956, p. 1577) The quote is a bit of hyperbole, but it serves to illustrate what makes mathematical language so abstract: Mathematicians often discuss in detail the logical relationships that exist among propositional functions which are, as has already been pointed out, sentences that assert nothing. Interpreted mathematical language must, therefore, also have this same abstract property: It must be precise enough to reflect the relevant logical relationships among the propositional functions of interest, but it cannot express or imply anything about the truth of the functions themselves.

We close this section by noting that the decision to write most propositional functions appearing in the preceding paragraphs as functions of one variable was made only to avoid unnecessary notational complications. In the case of three variables, for example, a propositional function may be written as $P(X,Y,Z)$, and in the case of many variables, a propositional function might be written as $P(X_1,X_2,X_3,\ldots,X_n)$. The considerations described above apply to propositional functions of several variables in just the same way as they apply to propositional functions of one variable.

LOGICAL QUANTIFIERS.

Despite Russell's provocative "definition" of mathematics, quoted above, mathematicians sometimes do seek to determine whether what they are saying is true. Theorems of this sort may demonstrate that a given propositional function is true for every element in a particular set. Other times, theorems demonstrate that a particular propositional function is false for every member of a given set. In still other cases, mathematicians demonstrate that there exists at least one element of a given set such that the propositional function is true. All such theorems depend upon the use of "logical quantifiers," of which there are exactly two, each with its own symbol. To successfully express higher mathematics in ASL, one must be prepared to interpret statements that include these quantifiers because such statements appear often. Here are some examples of theorems in which the logical quantifiers appear (Verbalizations of the quantifiers are italicized):

Ex. 17: Let *R* denote the set of real numbers. *There exists* an *X* in *R* such that $X^2 - 2X + 1 = 0$.

Ex. 18: Let *R* denote the set of real numbers. *For every X* in *R*, *X+2* is greater than *X*.

Ex. 19: Let *N* denote the set of positive integers. *For every X* in *N there exists* a *Y* in *N* such that *Y* is greater than *X*.

Here are the symbols used to represent the logical quantifiers:

S8: $\exists$ Called the "existential quantifier," the symbol $\exists$ is used to assert that within a given set *D* there exists a value for *X* such that *P(X)* is true, where *P(X)* is a given propositional function. The symbol $\exists$ is usually pronounced "there exists," but sometimes it is pronounced "for some" as in, "For some *X* belonging to *D*, *P(X)* is true." We emphasize that the assertion "there exists" does not mean that *P(X)* is true for every *X* belonging to *D*, only that *P(X)* is true for at least one value in *D*. By way of illustration, refer to Example 17: It is false that $X^2 - 2X + 1 = 0$ for every real number *X*; it is, however, true that there exists at least one real number such that $X^2 - 2X + 1 = 0$, namely $X = 1$. Less formally, the existential quantifier is used to assert that the propositional function *P(X)* is at least occasionally true. (One also encounters statements that assert that there exists exactly one *X* such that *P(X)* is true; symbolically this is represented as $\exists$ followed by an exclamation mark. One also encounters statements that assert that *X* does not exist. Symbolically, this is written as $\exists$ with a backslash drawn through it.)

S9: $\forall$ Called the "universal quantifier," the symbol $\forall$ is used to assert that for a particular propositional function *P(X)* and a particular set *D*, *P(X)* is true for every *X* in *D*. The universal quantifier is usually verbalized as "for each," or "for every." Example 18, above, illustrates the use of the universal quantifier.

The sentences in examples 17 through 19 can now be expressed symbolically. In each formula appearing below we use parentheses to identify each phrase that includes a logical quantifier. To facilitate comparison with the sentence from which the formula is derived, we follow each formula with the original sentence, given above, written in parentheses:

Ex. 17.1: Let *R* denote the set of real numbers. $(\exists X \in R) : X^2 - 2X + 1 = 0$. (*There exists* an *X* in *R* such that $X^2 - 2X + 1 = 0$.)

Ex. 18.1: Let *R* denote the set of real numbers. $(\forall X \in R) X + 2 > X$. (*For every X* in *R*, *X+2* is greater than *X*.)

Ex. 19.1: Let *N* denote the set of positive integers. $(\forall X \in N)(\exists Y \in N) : Y > X$. (*For every X* in *N there exists* a *Y* in *N* such that *Y* is greater than *X*.)

Since the equation appearing in example 17.1 and the inequalities appearing in 18.1 and 19.1 are themselves propositional functions, these examples can also be represented in the following still more abstract forms:

Ex. 17.2: $(\exists X \in R) : P(X)$.

Ex. 18.2: $(\forall X \in R) Q(X)$.

Ex. 19.2: $(\forall X \in N)(\exists Y \in N) : P(X, Y)$

When written at this level of abstraction, example 17.2 simply asserts that there is at least one element in the set *R* such that the propositional function *P(X)* is true. Example 18.2 asserts that for each element in the set *R* the propositional function *Q(X)* is true, and example 19.2 asserts that for every element *X* in the set *N* there is at least one element *Y* in the set *N* such that the propositional function *P(X,Y)* is true. (Now, perhaps, Russell's definition of mathematics does not seem quite so outlandish.)

CONCLUSION

Our model of mathematical language is complete. It is grammatically simple in the sense that it contains only a few components. In mathematics, these components are used over and over again in the creation of new statements. We summarize the results: To create a mathematical extension of ASL sufficient to meet the needs of interpreters, deaf mathematics professionals, educators, and deaf students of higher mathematics one needs the following:

1.) a standardized, and hopefully elegant, method of interpreting propositions; one that unambiguously preserves their truth values and which does not depend on contextual clues to identify which part of the proposition is the subject and which part of the proposition is the assertion. This method of interpretation would apply directly to the interpretation of propositional functions.

2.) signs for the eight symbols, $\wedge$, $\vee$, $\Rightarrow$, $\Leftrightarrow$, $\neg$, $\in$, $\exists$, $\forall$, and the phrase "such that"

3.) a signed mathematical vocabulary sufficient to express the set of propositions characteristic of the mathematical discipline of interest.

It is from these three elements that the language of higher mathematics is constructed.

As indicated in Tabak (2014), the ASL vocabulary deficit with respect to higher mathematics is enormous and has been the subject of some attention. The vocabulary deficit can, however, be resolved incrementally, course-by-course and subject-by-subject. The higher level grammatical aspects of mathematical language discussed in this paper are of a single piece. A standard method for interpreting propositions, propositional functions, and the logical symbols listed in this paper should be developed now, once and for all.

The ability to interpret the structural elements of mathematical discourse unambiguously and elegantly is a necessary precondition for successfully expressing any lecture on higher mathematics in ASL. The approach described here, which emphasizes the structure of mathematical language and deemphasizes content, can be especially useful to interpreters because it enables them to exploit the grammatical simplicity of mathematical language to produce faithful interpretations of mathematical statements without a detailed knowledge of the underlying mathematics. It is further hoped that this article will also serve to direct the attention of interpreters, educators, and deaf mathematics professionals to the problem of developing an accurate and standardized method of expressing mathematical language in ASL.

FOUR ADDITIONAL EXAMPLES

To further illustrate the ideas described in this article, we close with four well-known theorems. They are written in three ways: first, as they would appear in a text or written on a board; second, we insert square brackets to isolate the propositional functions and parentheses to enhance readability; and third, the sentences are represented using algebraic symbols. Some theorems have simple representations; others are more complex.

The reader is cautioned that there is more than one way to write each expression symbolically. One could, for example, write $R(m,p)$ or $R(p,m)$ in place of $R(m^{p-1},p)$ in example 22. The choice is only a matter of personal preference.

Finally, each theorem is prefaced by a sentence or two that identifies the set(s) over which the variable(s) appearing within the statement of the theorem may range. Sometimes called "statements of assignment," these sentences are not given special attention because, except for vocabulary, they pose no special challenge to an interpreter. By way of illustration, in example 20, "Let *X* represent a partially ordered set." is a statement of assignment.

Ex. 20: **Zorn's Lemma**. Let *X* represent a partially ordered set.

a.) If every chain in *X* has an upper bound then *X* has a maximal element.

b.) If [every chain in *X* has an upper bound] then [*X* has a maximal element].

c.) $P(X) \Rightarrow Q(X)$

Ex. 21: **The Bolzano-Weierstrass Theorem**. Let *X* represent a subset of *n*-dimensional Euclidean space.

a.) *X* is compact if and only if *X* is closed and *X* is bounded.

b.) [*X* is compact] if and only if ([*X* is closed] and [*X* is bounded]).

c.) $P(X) \Leftrightarrow (Q(X) \wedge R(X))$

Ex. 22: **Fermat's Theorem**. Let *p* and *m* be positive integers.

a.) If *p* is prime and *m* and *p* are relatively prime then m^{p-1} is congruent to 1 modulo *p*.

b.) If ([*p* is prime] and [*m* and *p* are relatively prime]) then [m^{p-1} is congruent to 1 modulo *p*].

c.) $(P(p) \wedge Q(m,p)) \Rightarrow R(m^{p-1},p)$

Ex. 23: **Brouwer's Fixed Point Theorem**. Let *D* denote the set of all points in the plane whose distance from the origin does not exceed 1. Let *S* denote the set of all continuous functions that map *D* into *D*.

a.) For every *f* belonging to *S* there exists an *x* belonging to *D* such that $f(x) = x$.

b.) (For every *f* belonging to *S*) (there exists an *x* belonging to *D*) such that [$f(x) = x$].

c.) $(\forall f \in S)(\exists x \in D) : P(x, f(x))$

Glossary of Symbols

$\wedge$ conjunction, usually pronounced "and"

$\vee$ disjunction, usually pronounced "or"

$\neg$ negation, usually pronounced "not"

$\Rightarrow$ implication, usually pronounced "implies" or "If…then…"

$\Leftrightarrow$ double implication, usually pronounced "is equivalent to" or "if and only if."

: pronounced "such that"

$\in$ membership, usually pronounced "belongs to," "is in," or "is an element of"

$\exists$ existential quantifier, usually pronounced "there exists" or "for some"

$\forall$ universal quantifier, usually pronounced "for each" or "for every"

p, q, r used to denote propositions

$P(X), Q(X), R(X), P(X,Y), Q(X,Y), R(X,Y), P(X,Y,Z),$ etc., used to denote propositional functions

References

Austin, J.L. & Howson, A.G. (1979). Language and Mathematical Education. *Educational Studies in Mathematics, vol. 10, no. 2*, 161-197. URL: http://jstor.org/stable/3482013

Russell, B. (1956) Mathematics and the Metaphysicians. In J. Newman (ed.) *The World of Mathematics,* vol. 3 (p. 1577). Mineola, New York: Dover Publications.

Russell, B. (1938). *Principles of Mathematics*. New York, New York: W.W. Norton & Co.

Smullyan, R.M. (1995). *First-Order Logic*. Mineola, New York: Dover Publications.

Tabak, J. (2014) What is Higher Mathematics? Why Is It So Hard to Interpret? What Can Be Done? *Journal of Interpretation:* Vol. 23: Iss. 1, Article 5. Available at: http://digitalcommons.unf.edu/joi/vol23/iss1/5